Singular arcs in optimal control of continuous-time bimodal switched linear systems

Singular Arcs in Optimal Control of Continuous-Time Bimodal Switched Linear Systems

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***Abstract*—This paper considers a singular problem in optimal control of continuous-time bimodal switched linear systems. A relaxed switched system with a continuous-valued switching signal is considered and the representations of singular control and singular arcs are derived. The similarity in the structure between the singular control and a stabilizing switching law is revealed and an approximation of the singular control by a well-defined switching signal is addressed. The results are demonstrated by numerical simulations.**

***Index Terms*—Singular arcs, Optimal control, Switched linear systems.**

I. Introduction

Switched systems are a class of hybrid dynamical systems [1], [2]. A switched system consists of several subsystems and a switching signal that chooses an active subsystem. The behavior of switched systems depends not only on the properties of each subsystem but also on the switching signal. For example, a switched system consisting of asymptotically stable subsystems may be destabilized by some switching signals and vice versa.

Optimal control of switched systems has been one of the research topics [3]. The source of the difficulty arising in the optimal control is that the control input involves a *discrete-valued* switching signal. As it is difficult to deal with optimal control problems without any assumptions regarding the switching signal, various such assumptions are typically made, for instance prespecifying the order and/or number of switches. In [4], a two-stage optimization method was proposed to compute switching times and continuous-valued input signals under the assumption that the order and number of the switches among the subsystems are given. Infinite-horizon optimal control was considered by making the assumption that the number of switches is finite and one or more subsystems are asymptotically stable [5], [6], [7].

In addition to the above difficulty, an obscurity in the optimal control of continuous-time switched systems is that an optimal switching signal does not necessarily exist in the set of switching signals whose switching intervals have a positive lower bound. To deal with optimal control of switched systems, a discrete input set representing admissible discrete values for a switching signal is often relaxed to its convex hull. The system with the relaxed input set is called a relaxed system. When the minimum principle is applied to an optimal control problem for the relaxed system, there is a case when a control (relaxed switching signal) cannot be directly determined from the minimization condition of the associated Hamiltonian—this is the singular case [8], [9]. When the singular case happens, an optimal input does not take a value in the discrete set, and no discrete-valued switching signals whose switching intervals have a positive lower bound give an optimal cost value for the optimal control problem for the original switched system. A few authors have considered optimal control problems of switched systems involving the singular case [10], [11]. In [11], a method for identifying singular arcs was considered for nonlinear systems with limited state dimension. Here a singular arc refers to an extremal (a triplet of admissible state, co-state, and input satisfying state and co-state differential equations) in the singular case (see Def. 3.1, [11]). The singular case should be properly taken into account if an optimal control problem of interest possibly entails it. This point is important, especially when one resorts to numerical computation to solve such an optimal control problem [12], [13]. Even for linear switched systems with a standard quadratic cost functional, whether or not the singular case occurs in the optimal control problem has not been fully understood, e.g., general conditions that subsystems and cost function satisfy for the existence or no-existence of the singular case have not been provided so far (note that for systems with state dimension up to three, algebraic conditions have been proposed [11]). Another problem in practice associated with the singular case is that we need to find a well-defined switching signal (i.e., a switching signal taking discrete values is defined and its switching intervals have a positive lower bound, see 1.3.3 in [2] for its detailed definition) that approximates the singular control, possibly by a form of state feedback.

In this paper, we focus on a simple switched system, a continuous-time bimodal switched linear system, and give a partial solution to the issues mentioned above. We provide a method for identifying singular arcs where the singular control is restricted to a constant, using the Lyapunov equation for a convex combination of subsystems. We first relax a discrete-valued switching signal to a continuous-valued signal and find singular arcs for a system with the relaxed switching signal. Since singular controls cannot be realized by a well-defined switching signal, we describe how they can be approximated by a well-defined stabilizing switching law, which has a form of state feedback.

The contributions of the paper are twofold:

1) Explicit characterization of singular arcs with constant

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singular control (Theorems 1 and 2, Section III);

2) Clarify the relationship between the singular control and a well-defined stabilizing switching law (Section IV).

The paper is organized as follows. In Section II, a continuous-time bimodal switched linear system is introduced and singular control problems are discussed. In Section III, we present the main results of this paper. In Section IV, an approximation of the singular control is presented. In Section V, a numerical example is provided.

A preliminary version of this paper was presented at a conference [14]. The major differences lie in the significantly improved proof of Theorem 1 (Lemma 1 in the conference version), a newly added algorithm (Algorithm 1), and a new illustrative example.

II. Singular Arcs in the Optimal Control Problem of Continuous-Time Bimodal Switched Linear Systems

We consider a bimodal switched linear system;

$$\dot{x}(t) = A_{\kappa(t)}x(t),\ x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $\kappa(t) \in \{1,2\}$. The variable $\kappa(t)$ functions as a control input for (1), which chooses the current active subsystem from $\{A_1, A_2\}$. We make the following assumptions for the system (1).

(A1) There exists a stable convex combination of A_1 and A_2, i.e., $\exists \alpha \in [0,1]$ such that

$$A(\alpha) := \alpha A_1 + (1-\alpha)A_2$$

is stable.

(A2) For any $\alpha \in [0,1]$ that makes $A(\alpha)$ stable, $A(\alpha)$ has real eigenvalues.

Assumption (A1) is a necessary and sufficient condition for the quadratic stabilizability of bimodal switched linear systems (Theorem 14, [15]). (A2) is in practice easily checked numerically because the eigenvalues are continuous with respect to α.

The system (1) has an equivalent representation,

$$\begin{aligned}\dot{x}(t) &= (A_1 u(t) + (1-u(t))A_2)x(t)\\ &= A(u(t))x(t),\end{aligned} \tag{2}$$

where $u(t)$ takes values in the binary set $\{0,1\}$. We here relax the binary set to the interval $[0,1]$ and consider the relaxed system (2) where $u(t)$ is allowed to take values in $[0,1]$.

Let us consider the optimal control that minimizes the following cost functional:

$$\begin{aligned}J(x_0, u(\cdot)) := \frac{1}{2}\int_0^{T_f} x^T(t)Qx(t)dt + \frac{1}{2}x^T(T_f)Q_f x(T_f),\\ Q,\ Q_f > 0,\ T_f \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.\end{aligned} \tag{3}$$

Define the Hamiltonian function

$$H(x(t), \lambda(t), u(t)) := \frac{1}{2}x^T(t)Qx(t) + \lambda^T(t)A(u(t))x(t), \tag{4}$$

where $\lambda(t) \in \mathbb{R}^n$ is the adjoint variable. Let $u(t)$ and $x(t)$ be the optimal control and the corresponding trajectory, respectively. Then the minimum principle states that there exists an absolutely continuous function $\lambda(t)$ and the following conditions are satisfied.

(a) $x(t)$ and $\lambda(t)$ are the solution to the following differential equations:

$$\begin{aligned}\dot{x}(t) &= \frac{\partial}{\partial \lambda}H(x(t), \lambda(t), u(t))\\ &= A(u(t))x(t),\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{\lambda}(t) &= -\frac{\partial}{\partial x}H(x(t), \lambda(t), u(t))\\ &= -Qx(t) - A^T(u(t))\lambda(t).\end{aligned} \tag{6}$$

(b) $H(x(t), \lambda(t), u(t)) = \min_{\bar{u}(\cdot)} H(x(t), \lambda(t), \bar{u}(t))$.

(c) The adjoint variable λ satisfies the boundary condition

$$\lambda(T_f) = Q_f x(T_f). \tag{7}$$

The Hamiltonian function (4) is rewritten as

$$\begin{aligned}&u(t)\lambda^T(t)A_{12}x(t) + \lambda^T(t)A_2 x(t) + \frac{1}{2}x^T(t)Qx(t),\\ &A_{12} := A_1 - A_2,\end{aligned}$$

and it follows from the condition (b) that the optimal control $u(t) \in [0,1]$ is given by the bang–bang control:

$$u(t) = \begin{cases} 0, & \lambda^T(t)A_{12}x(t) > 0,\\ 1, & \lambda^T(t)A_{12}x(t) < 0.\end{cases} \tag{8}$$

This will be the case unless $\lambda^T(t)A_{12}x(t)$ is identically zero:

$$\lambda^T(t)A_{12}x(t) = 0,\ \forall t \in [a,b],\ a < b. \tag{9}$$

An extremal (x, λ, u) that satisfies (9) is called a singular arc (see Def. 3.1 in [11]) and the corresponding x and u are called a singular trajectory and a singular control, respectively. Note that the optimal control for the system with dimension $n = 1$ is easily found to be $u(t) = 2 - \arg\min_{i \in \{1,2\}} A_i$, $0 \leq t \leq T_f$, where A_1 or A_2 is at least negative by the assumption (A1). Thus, in the remainder of the paper, we assume $n \geq 2$.

The relaxation of the binary input set to the interval $[0,1]$ implies that the switched system is embedded in a larger family of systems, and is often used to consider optimal control problems for switched systems [10], [11], [12]. The reasons for considering the relaxation are summarized as follows.

1) For this class of systems, an optimal control takes the form of bang–bang control unless a singular control appears. In the bang–bang control case, an optimal control for the relaxed system is also a solution to the switched system.

2) An optimal switching input for the switched system does not exist if a singular control appears, i.e., no optimal switching signals whose switching intervals have a positive lower bound exist. Even if this situation occurs, a singular trajectory can be approximated by a switching control. It should be mentioned that this situation often arises in many randomly generated systems [16] as well as in practical control problems such as the control of electrical converters [11].

In this paper, we consider an infinite-horizon control problem,

$$\min_{u(\cdot)} \frac{1}{2}\int_0^\infty x^T(t)Qx(t)dt,\ x(0)=x_0, \quad \text{s.t. } (2), \tag{10}$$

and consider a way to identify singular arcs on the interval $[0,\infty)$. Note that considering an optimal control problem for the relaxed system (2) means that we try to find a Filippov solution in optimal control for the switched system (1).

Note that we can obtain an optimal switching signal numerically by using the computation methods reported by [7], [12] when the singular case does not occur. A method for creating a lookup table for the switching has been reported [7], and a method for computing an optimal switching signal over a finite horizon has been proposed [12].

III. Main Results

In this section, we present a lemma and two theorems. The first theorem (Theorem 1) describes a property of the Lyapunov equation for the convex combination system, which plays a key role in the proof of the second theorem.

Lemma 1: Define the set

$$\mathscr{A} := \{\alpha \in [0,1] \mid \exists P_\alpha > 0 : A^T(\alpha)P_\alpha + P_\alpha A(\alpha) + Q = 0\}. \tag{11}$$

Assume that there exists a solution to the optimization problem

$$\mathscr{O}_P : \min_{\alpha\in\mathscr{A}}\ g(\alpha,x_0) \tag{12}$$

$$g(\alpha,x_0) := x_0^T P_\alpha x_0, \tag{13}$$

and let α^* be the solution of $\mathscr{O}_P$. Then $u(t)=\alpha^*$ minimizes (10) among the set of constant inputs.

Proof: As the matrix $A(\alpha)$ is stable for any $\alpha\in\mathscr{A}$, $\int_0^\infty x^T(t)Qx(t)dt$ with a constant input can be computed as follows:

$$\begin{aligned}\int_0^\infty x^T(t)Qx(t)dt &= -\int_0^\infty x^T(t)\big(A^T(\alpha)P_\alpha + P_\alpha A(\alpha)\big)x(t)dt\\ &= -\int_0^\infty \frac{d}{dt}\left(x^T(t)P_\alpha x(t)\right)dt\\ &= x^T(0)P_\alpha x(0)\\ &= x_0^T P_\alpha x_0.\end{aligned} \tag{14}$$

Since the objective function of (12) is equivalent to (14) and α^* is the minimizer for (12) by the assumption, $u(t)=\alpha^*$ minimizes (10) among the set of constant inputs. ■

Note that $u(t)=\alpha^*$ is also the solution for initial states given by scalar multiples of x_0, i.e., βx_0 $(\forall\beta\in\mathbb{R})$.

The following theorem shows a property of the Lyapunov equation for the convex combination system.

Theorem 1: Let α^* be the solution of $\mathscr{O}_p$ and assume that $\alpha^*\notin\{0,1\}$. Let v^* be the eigenvector corresponding to an eigenvalue s^* of $A(\alpha^*)=A_2+\alpha^*A_{12}$ and let $P_{\alpha^*}>0$ be the solution to the Lyapunov equation:

$$A^T(\alpha^*)P_{\alpha^*} + P_{\alpha^*}A(\alpha^*) + Q = 0. \tag{15}$$

Assume further that v^* and x_0 are linearly dependent. Then the equality

$$\begin{aligned}&v^{*T}P_{\alpha^*}A_{12}v^* = 0\\ &(A_{12}=A_1-A_2)\end{aligned} \tag{16}$$

holds.

Proof: Since $g(\alpha,x_0)$, defined by (13), takes its minimum value in the interior of $\mathscr{A}$ by the assumption, the equality

$$\left.\frac{\partial g}{\partial\alpha}\right|_{(\alpha^*,x_0)} = 0$$

holds. Further,

$$\left.\frac{\partial g}{\partial\alpha}\right|_{(\alpha^*,v^*)} = 0$$

holds, because x_0 and v^* are assumed to be linearly dependent, i.e., $v^*=\beta x_0$ for some $\beta\in\mathbb{R}$. The partial derivative of g can be calculated as follows:

$$\begin{aligned}\frac{\partial g}{\partial\alpha} =& x_0^T\left(\frac{\partial}{\partial\alpha}P_\alpha\right)x_0\\ =& x_0^T\left(\frac{\partial}{\partial\alpha}\int_0^\infty e^{A^T(\alpha)t}Qe^{A(\alpha)t}dt\right)x_0\\ =& x_0^T\left\{\int_0^\infty\left(\frac{\partial}{\partial\alpha}e^{A^T(\alpha)t}\right)Qe^{A(\alpha)t}dt\right.\\ &\left.+\int_0^\infty e^{A^T(\alpha)t}Q\left(\frac{\partial}{\partial\alpha}e^{A(\alpha)t}\right)dt\right\}x_0\\ =& 2x_0^T\left\{\int_0^\infty e^{A^T(\alpha)t}Q\left(\frac{\partial}{\partial\alpha}e^{A(\alpha)t}\right)dt\right\}x_0.\end{aligned} \tag{17}$$

The integral term in (17) is further computed as

$$\begin{aligned}&\int_0^\infty e^{A^T(\alpha)t}Q\left(\frac{\partial}{\partial\alpha}e^{A(\alpha)t}\right)dt\\ &= \int_0^\infty e^{A^T(\alpha)t}Q\int_0^1 e^{\tau A(\alpha)t}A_{12}te^{(1-\tau)A(\alpha)t}d\tau dt\end{aligned} \tag{18}$$

by using the formula (Fact 11.14.3, [17]):

$$\frac{d}{d\alpha}e^{M+\alpha N} = \int_0^1 e^{\tau(M+\alpha N)}Ne^{(1-\tau)(M+\alpha N)}d\tau, \quad \forall M,N\in\mathbb{R}^{n\times n}.$$

We pre- and post-multiply the equation (18) with $\alpha=\alpha^*$ by v^{*T} and v^*, respectively. Then the equation

$$\begin{aligned}&v^{*T}\int_0^\infty e^{A^T(\alpha^*)t}Q\int_0^1 e^{\tau A(\alpha^*)t}A_{12}te^{(1-\tau)A(\alpha^*)t}d\tau dtv^*\\ &= \int_0^\infty e^{s^*t}v^{*T}Q\int_0^1 e^{\tau A(\alpha^*)t}A_{12}te^{s^*(1-\tau)t}d\tau dtv^*\\ &= \int_0^\infty e^{2s^*t}v^{*T}Q\int_0^1 e^{(A(\alpha^*)-s^*I)t\tau}A_{12}td\tau dtv^*\end{aligned} \tag{19}$$

is obtained. By using the relation $v^{*T}Q = -v^{*T}P_{\alpha^*}(A(\alpha^*)+s^*I)$, (19) is written as

$$\begin{aligned}&-\int_0^\infty e^{2s^*t}v^{*T}P_{\alpha^*}(A(\alpha^*)+s^*I)t\int_0^1 e^{(A(\alpha^*)-s^*I)t\tau}d\tau dtA_{12}v^*\\ &= -v^{*T}P_{\alpha^*}(A(\alpha^*)+s^*I)\int_0^\infty\int_0^1 e^{2s^*t}te^{(A(\alpha^*)-s^*I)t\tau}d\tau dtA_{12}v^*.\end{aligned} \tag{20}$$

Now we compute the double integral in (20). The change of variables, $t\tau = \sigma$, gives

$$\int_0^\infty \int_0^t e^{2s^*t} e^{(A(\alpha^*)-s^*I)\sigma} d\sigma dt. \tag{21}$$

By reversing the order of integration, (21) is computed as follows:

$$\begin{aligned}\int_0^\infty \int_\sigma^\infty e^{2s^*t} e^{(A(\alpha^*)-s^*I)\sigma} dt d\sigma &= \int_0^\infty \int_\sigma^\infty e^{2s^*t} dt e^{(A(\alpha^*)-s^*I)\sigma} d\sigma \\ &= -\frac{1}{2s^*} \int_0^\infty e^{2s^*\sigma} e^{(A(\alpha^*)-s^*I)\sigma} d\sigma \\ &= -\frac{1}{2s^*} \int_0^\infty e^{(A(\alpha^*)+s^*I)\sigma} d\sigma \\ &= \frac{1}{2s^*}(A(\alpha^*)+s^*I)^{-1}.\end{aligned}$$

Thus the equation (20) is equal to

$$\begin{aligned}&-v^{*T} P_{\alpha^*}(A(\alpha^*)+s^*I) \cdot \frac{1}{2s^*}(A(\alpha^*)+s^*I)^{-1} A_{12} v^* \\ &= -\frac{1}{2s^*} v^{*T} P_{\alpha^*} A_{12} v^*,\end{aligned}$$

and we have

$$\begin{aligned}0 &= \left.\frac{\partial g}{\partial \alpha}\right|_{(\alpha^*, v^*)} \\ &= 2\left(-\frac{1}{2s^*} v^{*T} P_{\alpha^*} A_{12} v^*\right).\end{aligned}$$

We hence obtain the equality (16). ■

Note that Theorem 1 holds only for a real eigenvalue s^* (see the assumption (A2)), and never holds for a complex eigenvalue: this is because the requirement of linear dependence of v^* and x_0, the latter of which is the state and always a real vector. Theorem 2 below shows that a singular control occurs if the initial state x_0 satisfies the conditions given in Lemma 1 and Theorem 1. It also provides an explicit representation of the singular control and singular arcs.

Theorem 2: Assume that the conditions given in Lemma 1 and Theorem 1 hold. Then,

$$(x^*(t), \lambda^*(t), u^*(t)) = (e^{s^*t} x_0, P_{\alpha^*} x^*(t), \alpha^*),\ t \geq 0 \tag{22}$$

is a singular arc, and the quadratic cost in (10) is given by

$$\frac{1}{2} x_0^T P_{\alpha^*} x_0. \tag{23}$$

Further, when the condition

$$X \neq 0 \tag{24}$$

$$X := -v^{*T}\{(A_{12}^T Q + QA_{12})A_{12} - P_{\alpha^*}[A_{12}, [A_2, A_{12}]]\} v^*$$

holds, where $[\cdot,\cdot]$ is defined by $[M,N] := MN - NM$ for any square matrices $M, N \in \mathbb{R}^{n\times n}$, the necessary condition for the optimality of the singular control is given by

$$X < 0. \tag{25}$$

Proof: The partial derivative of the Hamiltonian (4) is

$$\frac{\partial H}{\partial u} = \lambda^T A_{12} x. \tag{26}$$

Its first and second time derivatives are calculated as follows:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) &= \dot{\lambda}^T A_{12} x + \lambda^T A_{12} \dot{x} \\ &= (-Qx - A^T(u)\lambda)^T A_{12} x + \lambda^T A_{12} A(u) x \\ &= -x^T Q A_{12} x - \lambda^T [A(u), A_{12}] x \\ &= -x^T Q A_{12} x - \lambda^T [A_2, A_{12}] x\end{aligned} \tag{27}$$

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) \\ &= -x^T(A^T(u)QA_{12} + QA_{12}A(u))x \\ &\quad - (-Qx - A^T(u)\lambda)^T [A(u), A_{12}] x - \lambda^T [A(u), A_{12}] A(u) x \\ &= C(x,\lambda) + uD(x,\lambda)\end{aligned} \tag{28}$$

where

$$\begin{aligned}C(x,\lambda) &:= -x^T(A_2^T Q - QA_2)A_{12} x - 2x^T Q A_{12} A_2 x \\ &\qquad + \lambda^T [A_2, [A_2, A_{12}]] x \\ D(x,\lambda) &:= -\left\{x^T(A_{12}^T Q + QA_{12})A_{12} x - \lambda^T [A_{12}, [A_2, A_{12}]] x\right\}.\end{aligned}$$

In (28), the input u appears explicitly (see Appendix A for the detailed calculation), which means that the problem order of the singular control problem (Def. 3.2 in [11]) is one. Substituting $u^*(t) = \alpha^*$, $\lambda^*(t) = P_{\alpha^*} x^*(t)$, and $x^*(t) = e^{s^*t} x_0$ $(x_0 = v^*)$ into (26) results in

$$\begin{aligned}\left.\frac{\partial H}{\partial u}\right|_{(x^*, \lambda^*, u^*)} &= x^{*T} P_{\alpha^*} A_{12} x^* \\ &= e^{2s^*t} v^{*T} P_{\alpha^*} A_{12} v^* \ \ (\forall t \geq 0) \\ &= 0,\end{aligned}$$

where the equality (16) in Theorem 1 is used. Note that P_{α^*} is the solution of the Lyapunov equation (15), and the adjoint $\lambda^*(t) = P_{\alpha^*} x^*(t)$ satisfies the differential equation (6). Thus, $(x^*(t), \lambda^*(t), u^*(t))$ is a singular arc, i.e., the equality (9) holds for $[0,\infty)$. The quadratic cost in (10) is computed as $\frac{1}{2} x_0^T P_{\alpha^*} x_0$ using the relation (14) with $\alpha = \alpha^*$.

$D(x^*, \lambda^*)$ is calculated as

$$D(x^*, \lambda^*) = D(e^{s^*t} x_0, \lambda^*(t)) = e^{2s^*t} X, \tag{29}$$

and X is non-zero by the assumption, which shows that the arc order (Def. 3.3 in [11]) is one. When the arc order is one, the necessary condition for the optimality of the singular control is given by the generalized Legendre–Clebsch condition (Theorem 3.6 in [11], [9]):

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) < 0,\ \forall t \in [0,\infty). \tag{30}$$

From (28), the left-hand side of (30) is calculated as $D(x,\lambda)$ and thus the necessary condition for the optimality is given by

$$\begin{gathered}D(x^*, \lambda^*) = e^{2s^*t} X < 0,\ \forall t \in [0,\infty) \\ \Updownarrow \\ X < 0.\end{gathered}$$

■

Remark 1: The condition (24) ensures that the arc order for the considered singular arc is one. If $X = 0$, higher time derivatives of (26) are required to obtain the necessary condition.

Remark 2: For a nonlinear affine system with state dimension up to three, determinant-based algebraic conditions to compute singular arcs have been proposed [11]. In this paper, Theorem 2 provides the explicit characterization of the singular control and singular arcs where the singular control is restricted to a constant for a bimodal switched linear system with arbitrary system dimension. Although the singular control is given by a constant for two-dimensional systems [16], note that, for a system with state dimension higher than two, not all the possible singular controls are represented by a constant as in Theorem 2. Our results assume the real eigenvalue condition. If the assumption does not hold, one can use the algebraic conditions [11] for systems with dimension up to three to identify the singular arcs; note that the explicit characterization of the singular arcs was not provided.

Remark 3: We considered only bimodal systems and not multi-mode systems with more than two subsystems. This limitation comes from the adopted technique in the proof of Theorem 1 where we used the formula related to the derivative of matrix exponential functions with single variable (Fact 11.14.3, [17]). When we consider the multi-mode systems, we need a similar formula for multi variables, which we are not sure if there exists. Although our results are only for bi-modal systems, it does not mean that this is too restrictive because there are some practical applications that can be represented bi-modal systems, such as DC-DC converters ([2], [16], [18]) and wheeled mobile robots (Example 1.3, [2]).

Using the above results, singular arcs are identified as follows. For the states on a unitary semisphere in the state-space, we find the states satisfying the conditions mentioned in Lemma 1 and Theorem 1. Scalar multiples of such vectors are candidate singular trajectories. Then we check whether each candidate singular trajectory satisfies the optimality condition in Theorem 2. Note that the optimization problem for each state x_0 in Lemma 1 can be solved using a general nonlinear programming solver such as `fminbnd` in MATLAB. The linear dependency condition for v^* and x_0 in Theorem 1 is easily checked by computing their inner product.

Note that the method mentioned above requires to solve the optimization problem (12) with the states on the unitary semisphere where the state-space gridding (on the unitary semisphere) is necessary for computation. Thus it requires more computational effort when the system dimension becomes higher. The state-space gridding-based method has been used in other optimal control literature (e.g., [6], [7], [11]), and a method for generating the gridding points on the unitary semisphere has been reported (APPENDIX B in [6]). Note here that the computation of our method is performed off-line, and the applicability of the method should be evaluated in terms of a tolerable off-line computation time. Although the method may not be viable for higher dimensional systems because of the requirement of a large amount of computation time, this drawback could be alleviated by adjusting the gridding resolution adaptively. Since we only want to identify the states x_0 satisfying the linear dependency condition, the gridding resolution can be low over the state-space regions where the linear dependency level is relatively low and should be high where the linear dependency level is relatively high. Based on this observation, we could use the following gridding strategy. First we use a uniform coarse gridding, and find the state-space regions where the linear dependency level is relatively high. Then only for such regions we increase the resolution of the gridding, and evaluate the linear dependency level. We further increase the gridding resolution only for the regions where linear dependency level is relatively high, and repeat this procedure until a prescribe level of accuracy is achieve.

IV. Relationship to the Stabilizing Switching Law

We show that the singular control and a well-defined stabilizing switching signal have similar features. We propose a method for approximating the singular control by a discrete-valued switching signal whose switching intervals have a positive lower bound, which shows that for any finite interval $[a,b]$ $(a<b)$ the number of switching times is finite. This switching signal is a well-defined switching signal (see 1.3.3 in [2] for the detailed definition of well-definedness).

Let $\alpha^* \notin \{0,1\}$ be singular control, and define

$$M_i := A_i^T P_{\alpha^*} + P_{\alpha^*} A_i,\ i = 1,2.$$

Consider the following well-defined stabilizing switching law (Chap. 3.4, [2]):

$$S_{stab}: \begin{cases} t_{k+1} &= \inf\left\{t > t_k : x^T(t) M_{\kappa(t_k)} x(t) \right. \\ & \left. > -r_{\kappa(t_k)} x^T(t) Q x(t)\right\}, r_i \in (0,1), \\ \kappa(t_{k+1}) &= \arg \min_{i\in\{1,2\}} \left\{x^T(t_{k+1}) M_i x(t_{k+1})\right\}, \\ & k = 1,2,\dots, \\ \kappa(0) &= \arg \min_{i\in\{1,2\}} \left\{x_0^T M_i x_0\right\}. \end{cases} \tag{31}$$

This switching law is obtained by modifying a switching law

$$\kappa(t) = \arg \min_{i\in\{1,2\}} \left\{x^T(t) M_i x(t)\right\} \tag{32}$$

so that the resulting switching intervals $t_{k+1} - t_k$ $(k = 1,2,\dots)$ have a positive lower bound. Now, notice that the equation (32) can be further rewritten as

$$\begin{aligned} \kappa(t) &= \arg \min_{i\in\{1,2\}} \left\{x^T(t) M_i x(t)\right\} \\ &= \arg \min_{i\in\{1,2\}} \left\{x^T(t)(A_i^T P_{\alpha^*} + P_{\alpha^*} A_i) x(t)\right\} \\ &= \arg \min_{i\in\{1,2\}} \left\{2x^T(t) P_{\alpha^*} A_i x(t)\right\} \qquad (33) \\ &= \begin{cases} 1, & x^T(t) P_{\alpha^*}(A_1 - A_2) x(t) < 0, \\ 2, & x^T(t) P_{\alpha^*}(A_1 - A_2) x(t) > 0. \end{cases} \qquad (34) \end{aligned}$$

Since the region Ω where the singular control occurs is included in

$$\Omega_{sw} := \left\{x \in \mathbb{R}^n | x^T P_{\alpha^*}(A_1 - A_2) x = 0\right\},$$

Algorithm 1 Computation of r_1, r_2 in (31).

1: Set a desired relative error level $\varepsilon > 0$, an initial value $r^* \in (0,1)$, and a sufficiently large evaluation time $t_f > 0$.
2: For an initial state x_0 on the singular trajectory of interest, compute the quadratic cost J^* by (23). Choose a perturbed initial state $\tilde{x}_0$.
3: **loop**
4: Compute a quadratic cost J of the well-defined stabilizing law (31) with $x_0 = \tilde{x}_0$ and $r_1 = r_2 = r^*$ over $[0, t_f]$.
5: **if** $\frac{J-J^*}{J^*} < \varepsilon$ **then**
6: break.
7: **else**
8: $r^* \leftarrow \frac{r^*+1}{2}$.
9: **end if**
10: **end loop**
11: Set $r_1 = r_2 = r^*$.

the singular control is approximated by the switching law S_{stab} in the neighborhood of Ω if Ω_{sw} is attractive. The switching law S_{stab} only gives a larger cost value, J, than the optimal cost, J^*, obtained by the singular control. However, the level of approximation can be adjusted by the parameter $r_i \in (0,1)$ in (31). If we choose a larger value close to 1, the switching intervals $t_{k+1} - t_k$ become small and the state trajectory becomes close to the singular trajectory. We give a simple algorithm, Algorithm 1, to obtain the parameters r_1 and r_2 such that the resulting state response satisfies a prescribed level of approximation. Note that a perturbed initial state $\tilde{x}_0$ is used on line 4 in Algorithm 1, because $x_0^T M_1 x_0 = x_0^T M_2 x_0$ holds in theory for an initial state x_0 on a singular trajectory and the initial switching signal $\kappa(0)$ in (31) cannot be determined. In practice, however, the exact equality does not hold in numerical computation, and replacement of x_0 with $\tilde{x}_0$ may not be necessary.

Remark 4: The way of the approximating the singular control in the stabilizing switching law (31) is done by introducing a hysteresis in (32). Construction of the approximating discrete-valued switching signal itself is not new; for example, a feedback switching law with a parameter adjusting the switching frequency has been reported in [18] and a frequency modulation method for creating a discrete-valued signal over a time interval has been used in [19]. In this section, however, we pointed out the relationship between the well-known switching law and the singular control, which we think has not been explicitly addressed so far.

A state-feedback switching law for linear switched systems has been proposed based on the min-type Lyapunov function in [16] and it is shown that the switching law gives a good approximate optimal control with respect to the infinite horizon quadratic cost. We here discuss the relationship between this switching law and (32). The switching law presented in Theorem 6 in [16] for the bimodal case with the state weight $Q_1 = Q_2 = Q > 0$ is given by

$$i^*(x) \in I(x) = \arg \min_{i \in \{1,2\}, \lambda \in L(x)} x^T (P_\lambda - P_i) A_i x \tag{35}$$

where

$$P_i : A_i^T P_i + P_i A_i + Q = 0, \tag{36}$$

$$P_\lambda : A(\lambda)^T P_\lambda + P_\lambda A(\lambda) + Q = 0 \ \ (A(\lambda) = \lambda_1 A_1 + \lambda_2 A_2) \tag{37}$$

$$L(x) = \arg \min_{\lambda \in \Lambda^+(M)} x^T P_\lambda x, \tag{38}$$

$$\Lambda^+(M) = \{\{\lambda_1, \lambda_2\} : \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0, \\ P_\lambda > 0, \max(\mathrm{eig}(P_\lambda)) \leq M\}, M > 0. \tag{39}$$

The objective function in (35) can be computed as

$$\begin{aligned} x^T (P_\lambda - P_i) A_i x &= x^T P_\lambda A_i x - x^T P_i A_i x \\ &= x^T P_\lambda A_i x - \frac{1}{2} x^T (P_i A_i + A_i^T P_i) x \\ &= x^T P_\lambda A_i x + \frac{1}{2} x^T Q x. \end{aligned}$$

Then, since the term $x^T Q x$ does not affect the minimization, $I(x)$ becomes

$$\begin{aligned} I(x) &= \arg \min_{i \in \{1,2\},\ \lambda \in L(x)} x^T P_\lambda A_i x \\ &= \arg \min_{i \in \{1,2\}} x^T P_{\lambda^*(x)} A_i x \end{aligned} \tag{40}$$

where $\lambda^*(x) \in L(x)$, assuming $L(x)$ is a singleton set. The control law (40) has the same form as (33) except that the solution of the Lyapunov equation $P_{\lambda^*(x)}$ in (40) is state-dependent whereas the one in (33) is fixed. The two control laws give the same switching signal when the state x is in the neighborhood of the singular trajectory of interest. Note that the control law (40) requires to solve the optimization problem (38) online, whereas (33) requires offline optimization of (12) over the states (on the unitary semisphere) to obtain the fixed P_{α^*}. If the state is in the neighborhood of a different singular trajectory, the control law (33) with the fixed P_{α^*} does not necessarily yield an optimal one for this singular trajectory; optimality loss in this case depends on the difference of the two sets Ω_{sw} (below (34)) with two different P_{α^*} corresponding to two different singular trajectories.

V. Illustrative Example

We consider the bimodal switched system [12] (see also [13]):

$$\dot{x}(t) = A_{\kappa(t)} x(t), \kappa(t) \in \{1,2\}$$
$$A_1 = \begin{bmatrix} -10 & 5 \\ 3 & -2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 3 & 1 \\ 7 & -20 \end{bmatrix}.$$

The subsystem A_1 is stable and A_2 is unstable (eigenvalues $A_1 : \{-11.5678, -0.4322\}$, $A_2 : \{3.3004, -20.30042\}$). The convex combination $A(\alpha) = \alpha A_1 + (1-\alpha) A_2$ is stable for $\alpha :\ 0.2975 < \alpha \leq 1$. We choose the weighting matrix $Q = \mathrm{diag}(1,1)$ in (10). In the following, we find the singular arcs using the main results (Lemma 1, Theorem 1 and Theorem 2), and also demonstrate how singular control is approximated by the well-defined stabilizing switching law addressed in Section IV.

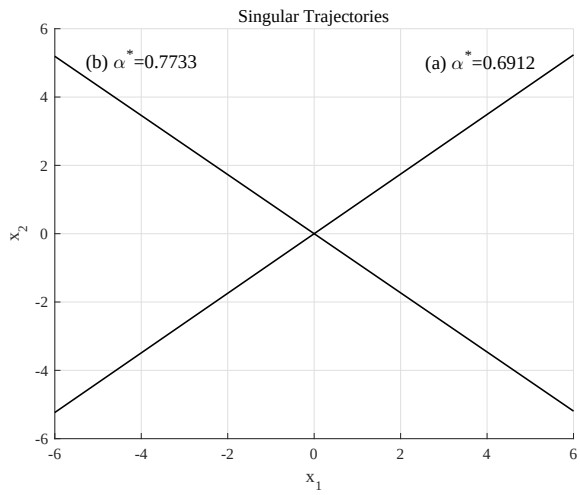

Fig. 1. Singular trajectories.

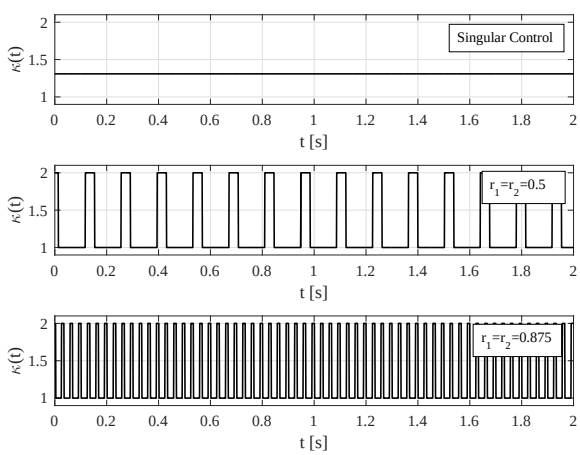

Fig. 2. Singular control and approximate stabilizing switching signals.

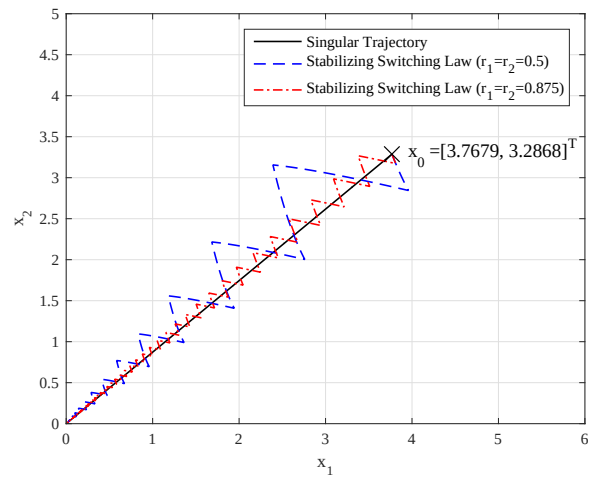

Fig. 3. Singular and approximate state-space trajectories.

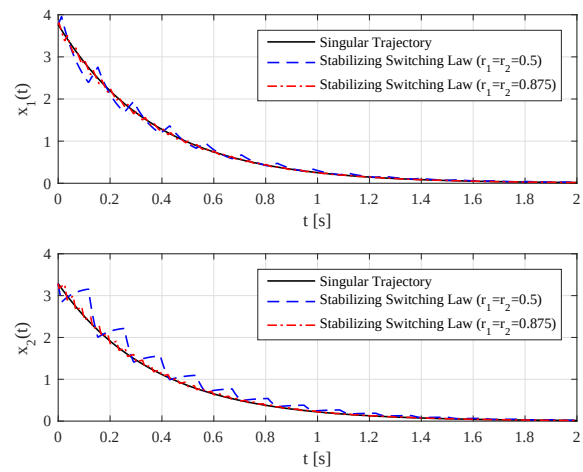

Fig. 4. Singular and approximate state responses.

We first find singular trajectories using Lemma 1 and Theorem 1. Among the points $x = [\cos\theta, \sin\theta]^T$ $(0 \le \theta \le \pi)$, we found vectors that are linearly dependent with an eigenvector v^* of $A(\alpha^*)$ $(\alpha^* = \arg\min_\alpha\ g(\alpha, x))$. The optimization problem (12) was solved using `fminbnd` in MATLAB. Two points satisfying the described condition were found and the corresponding α^* values are 0.6912 and 0.7733. The scalar multiples of the points constitute singular trajectories, and they are plotted in Fig. 1. The optimality condition (24) in Theorem 2 is satisfied with $X = -1003.8$ $(\alpha^* = 0.6912)$ and $X = -426.3$ $(\alpha^* = 0.7733)$.

We compare the singular control with the well-defined stabilizing switching law addressed in Section IV. For the initial condition $x_0 = [3.7679, 3.2868]^T$ on the singular trajectory (a), the singular control is $u(t) = 0.6912$ $(\kappa(t) = 2 - u(t) = 1.3088)$, and the corresponding state trajectories and time responses are plotted in Figs. 2, 3, and 4 (black solid curves). The quadratic cost is computed as $J^* = \frac{1}{2}x_0^T P_{\alpha*} x_0 = 2.3129$. Although the approximate stabilizing switching law (31) only achieves a larger cost value than J^*, we can compute an approximating parameter r_1 and r_2 in (31) to attain a prescribed approximation level. Now we choose a relative error level $\varepsilon = 0.01$ (1%), an initial approximating parameter $r^* = 0.5$, and evaluation time $t_f = 2.0$. Algorithm 1 is then performed and the approximating parameter $r_1 = r_2 = 0.875$ in the well-defined stabilizing switching law (31) is obtained. The responses with $r_1 = r_2 = 0.5$ and 0.875 are shown in Figs. 2, 3, and 4. We see that more frequent switching occurs in the control law with larger r_i, and the corresponding state responses are closer to the singular trajectory than those with smaller r_i. The quadratic costs for the stabilizing switching law with $r_1 = r_2 = 0.5$ and 0.875 are $J = 2.5266$ and 2.3309, respectively (relative error: 9.2% and 0.78%, respectively).

Note that when the initial condition x_0 is not on the singular trajectory, a kind of backward integration is used to compute a non-singular arc segment (Proposition 4.2, [11]). Let us illustrate this point. For an initial condition $x_0 = [1, 6]^T$, the state trajectories are plotted in Fig. 5: the subsystem A_2 is first used, and then the switching signal turns into the singular control. The trajectory resembles the one for a finite-horizon

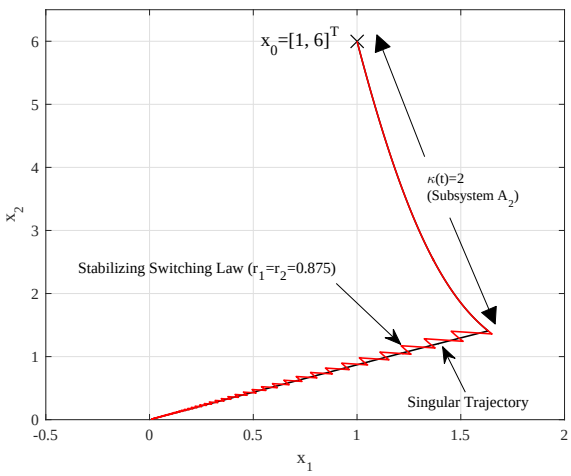

Fig. 5. State-space trajectories for the initial state $x_0 = [1,6]^T$.

optimal control case reported in [12], [13].

VI. Conclusion

We have considered a method for characterizing singular arcs in the optimal control of bimodal switched linear systems. We showed that a well-defined switching signal can be used to approximate a singular control. Future research topics include extension of the method to switched systems with more than two subsystems and classification of possible optimal controls by spectra of subsystems A_i. We think that the type of possible optimal controls is deeply related to the spectra of A_i, and the present work with the real eigenvalue assumption is a step toward this direction.

Appendix
Derivation of (28) in Theorem 2

The representation (28) is obtained as follows. We have

$$\begin{aligned}
\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) &= \frac{d}{dt}\left(-x^T QA_{12}x - \lambda^T[A_2, A_{12}]x\right)\\
&= -\dot{x}^T QA_{12}x - x^T QA_{12}\dot{x} - \dot{\lambda}^T[A_2, A_{12}] - \lambda^T[A_2, A_{12}]\dot{x}\\
&= -x^T A^T(u)QA_{12}x - x^T QA_{12}A(u)x\\
&\quad -(-Qx - A^T(u)\lambda)^T[A_2, A_{12}]x - \lambda^T[A_2, A_{12}]A(u)x\\
&= -x^T(A^T(u)QA_{12} + QA_{12}A(u))x\\
&\quad -(-Qx - A^T(u)\lambda)^T[A(u), A_{12}]x - \lambda^T[A(u), A_{12}]A(u)x,
\end{aligned}$$

and substituting $A(u) = A_2 + uA_{12}$ in the above equation and arranging with respect to u leads to

$$\begin{aligned}
&-x^T(A_2^T QA_{12} + uA_{12}^T QA_{12} + QA_{12}A_2 + uQA_{12}^2)x\\
&-(-Qx - A_2^T\lambda - uA_{12}^T\lambda)^T[A_2, A_{12}]x - \lambda^T[A_2, A_{12}]A_2x\\
&-u\lambda^T[A_2, A_{12}]A_{12}x\\
=&-x^T(A_2^T Q - QA_2)A_{12}x - 2x^T QA_{12}A_2x + \lambda^T[A_2, [A_2, A_{12}]]x\\
&-u\{x^T(A_{12}^T Q + QA_{12})A_{12}x - \lambda^T[A_{12}, [A_2, A_{12}]]x\}\\
=&C(x,\lambda) + uD(x,\lambda).
\end{aligned}$$

References

[1] D. Liberzon, *Switching in Systems and Control*, Boston, U.S.A.: Birkhäuser, 2003.

[2] Z. Sun and S.S. Ge, *Switched Linear Systems: Control and Design*, London, U.K.:Springer-Verlag, 2005.

[3] F. Zhu and P.J. Antsaklis, "Optimal control of hybrid switched systems: A brief survey," *Discrete Event Dyn. Syst.*, vol. 25, no. 3, pp. 345–364, 2015.

[4] X. Xu and P.J. Antsaklis, "Optimal control of switched systems based on parameterization of the switching instants," *IEEE Trans. Autom. Control*, vol. 49, no. 1, pp. 2–16, Jan. 2004.

[5] A. Bemporad, A. Giua and C. Seatzu, "Synthesis of state-feedback optimal controllers for continuous-time switched linear systems," in *Proc. 41st IEEE Conf. Decision Control*, 2002, pp. 3182–3187.

[6] C. Seatzu, D. Corona, A. Giua and A. Bemporad, "Optimal control of continuous-time switched affine systems," *IEEE Trans. Autom. Control*, vol. 51, no. 5, pp. 726–741, May 2006.

[7] D. Corona, A. Giua, and C. Seatzu, "Stabilization of switched systems via optimal control," *Nonlinear Analysis: Hybrid Systems*, vol. 11, pp. 1–10, 2014.

[8] P. Riedinger, J. Daafouz and C. Iung, "Suboptimal switched controls in context of singular arcs," in *Proc. 42nd IEEE Conf. Decision Control*, 2003, pp. 6524–6259.

[9] D.J. Bell and D.H. Jacobson, *Singular Optimal Control Problems*, London, U.K.: Academic Press, 1975.

[10] S.C. Bengea and R.A. DeCarlo, "Optimal control of switching systems," *Automatica*, vol. 41, no. 1, pp. 11–27, 2005.

[11] D. Patino, P. Riedinger and C. Iung, "Practical optimal state feedback control law for continuous-time switched affine systems with cyclic steady state," *Int. J. Control*, vol. 82, no. 7, pp. 1357–1376, 2009.

[12] T. Das and R. Mukherjee, "Optimally switched linear systems," *Automatica*, vol. 44, no. 5, pp. 1437–1441, 2008.

[13] P. Riedinger, "Comments on "Optimally switched linear systems"," *Automatica*, vol. 45, no. 9, pp. 1588–1590, 2009.

[14] N. Hara and K. Konishi, "Singular arcs in optimal control of bimodal switched linear systems and approximate switching signals," in *Proc. 54th IEEE Conf. Decision Control*, 2015, pp. 3876–3881.

[15] H. Lin and P.J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 308–322, Feb. 2009.

[16] P. Riedinger, "A switched LQ regulator design in continuous time," *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 1322–1328, 2014.

[17] D.S. Bernstein, *Matrix Mathematics*, 2nd ed., Princeton, U.S.A.: Princeton University Press, 2009.

[18] C. Albea, G. Garcia and L. Zaccarian, "Hybrid dynamic modeling and control of switched affine systems: application to DC-DC converters," in *Proc. 54th IEEE Conf. Decision Control*, 2015, pp. 2264–2269.

[19] H. Chen and W. Zhang, "On weak topology for optimal control of switched nonlinear systems," *Automatica*, vol. 81, pp. 409–415, 2017.