Amplitude death in a pair of one-dimensional complex Ginzburg-Landau systems coupled by diffusive connections

Amplitude death in a pair of one-dimensional complex Ginzburg-Landau systems coupled by diffusive connections

Hakui Teki, Keiji Konishi,[*] and Naoyuki Hara

Department of Electrical and Information Systems, Osaka Prefecture University, 1-1 Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531 Japan

This paper shows that, in a pair of one-dimensional complex Ginzburg-Landau (CGL) systems, diffusive connections can induce amplitude death. Stability analysis of a spatially uniform steady state in coupled CGL systems reveals that amplitude death never occurs in a pair of identical CGL systems coupled by no-delay connection, but can occur in the case of delay connection. Moreover, amplitude death never occurs in coupled identical CGL systems with zero nominal frequency. Based on these analytical results, we propose a procedure for designing the connection delay time and the coupling strength to induce spatial-robust stabilization, that is, a stabilization of the steady state for any system size and any boundary condition. Numerical simulations are performed to confirm the analytical results.

I. INTRODUCTION

Over the past few decades, a number of studies have examined amplitude death, in which oscillators cease their oscillatory behavior by their mutual diffusive connections [1–4]. Diffusive connections that induce amplitude death can be roughly classified into the no-delay connection [1,2], the delay connection [5,6], and the dynamic connection [7–12]. Among these connections, the no-delay connection [13–18] and the delay connection [19–27] have been actively used in studies on amplitude death. This is because the no-delay connection is the simplest way to couple oscillators diffusively, and the delay connection is considered to be a natural way to connect oscillators with a finite propagation speed for information interactions [28,29].

There has been considerable research on spatiotemporal nonlinear phenomena in coupled reaction-diffusion systems [30–35]. One of the most popular reaction-diffusion systems, the complex Ginzburg-Landau (CGL) system, has been widely used in the field of nonlinear science, because the CGL system describes dynamics around the start of a Hopf bifurcation [36–39]. Spatiotemporal nonlinear phenomena in a pair of one-dimensional CGL systems coupled by the no-delay connection have been studied [40–42]. These studies have been expanded to the one-way connection [43,44], asymmetric connection [45,46], delay connection [47,48], and two-dimensional CGL systems [49,50]. However, to the best of our knowledge, there have been few reports concerning amplitude death in coupled reaction-diffusion systems [51,52].

The purpose of this study is to show that amplitude death can occur in a pair of one-dimensional CGL systems coupled by diffusive connections. We focus on two types of connections: the no-delay connection [40–42] and the delay connection [47]. The stability of a spatially uniform steady state in coupled CGL systems is investigated analytically. The analytical results reveal that amplitude death never occurs in a pair of *identical* CGL systems coupled by the no-delay connection, but can occur in the case of the delay connection. Moreover, amplitude death never occurs, even in the case of

the time delay connection when the nominal frequency of identical CGL systems is zero. Furthermore, we provide a procedure for designing the connection delay time and the coupling strength to stabilize the steady state for any system size and any boundary condition. We confirm these analytical results through numerical simulations.

II. COUPLED COMPLEX GINZBURG-LANDAU SYSTEMS

Let us consider a pair of one-dimensional complex Ginzburg-Landau (CGL) systems:

$$\frac{\partial W_{1,2}}{\partial t} = \{(1 + i\omega_{1,2}) - (1 + i\beta)|W_{1,2}|^2\}W_{1,2}$$

$$+ (1 + i\alpha)\frac{\partial^2}{\partial x^2}W_{1,2} + U_{1,2}, \qquad (1)$$

as illustrated in Fig. 1. Here, $W_{1,2}(t,x) \in \mathbb{C}$ are complex state variables at time $t \geqslant 0$ and location $x \in [0,L]$, where $L > 0$ is the system size. These systems have common parameters, $\alpha, \beta \in \mathbb{R}$, and have different frequencies,

$$\omega_1 := \omega_0 + \Delta\omega, \quad \omega_2 := \omega_0 - \Delta\omega, \qquad (2)$$

where $\omega_0 \in \mathbb{R}$ is the nominal frequency and $2\Delta\omega \in \mathbb{R}$ represents the frequency difference.

Each CGL system (1) without a connection (i.e., $U_{1,2} \equiv 0$) has a spatially uniform steady state

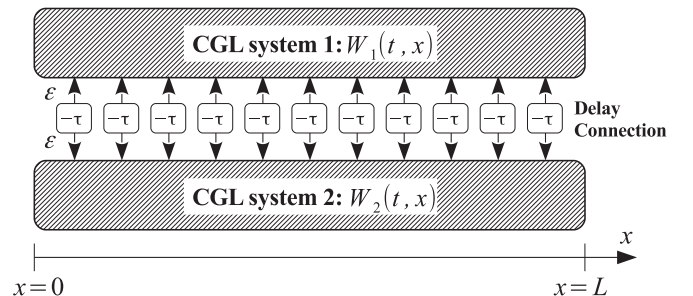$$[W_1(t,x) \ W_2(t,x)]^T = [0 \ 0]^T, \quad \forall x \in [0,L]. \qquad (3)$$

FIG. 1. Sketch of CGL systems (1) coupled by the time delay connection (4).
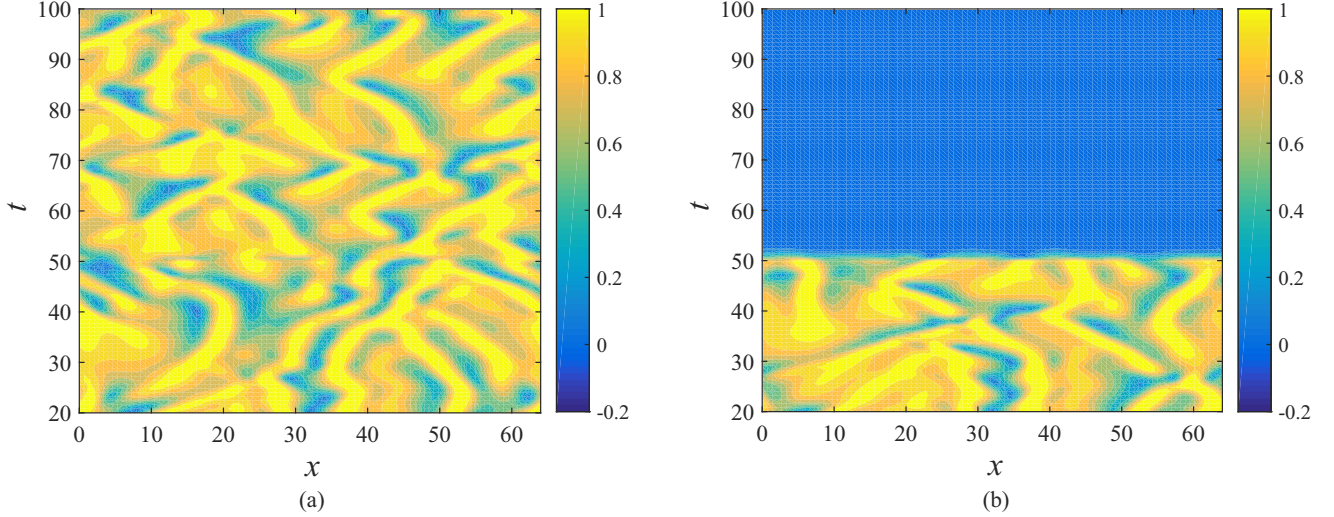
*http://www.eis.osakafu-u.ac.jp/~ecs

FIG. 2. Spatiotemporal behavior of amplitude $|W_1(t,x)|$ in CGL systems (1) coupled by the *no-delay* connection (4) without delay ($\tau = 0$, $\alpha = 3.0$, $\beta = -1.2$, $\omega_0 = 6.0$, $\varepsilon = 5.0$, $L = 64$): (a) identical systems ($\Delta\omega = 0.0$), (b) nonidentical systems ($\Delta\omega = 4.0$). These systems are coupled at $t = 50$.

The connection signals with coupling strength $\varepsilon \geqslant 0$ and connection delay $\tau \geqslant 0$, i.e.,

$$U_{1,2}(t,x) = \varepsilon\{W_{2,1}(t - \tau,x) - W_{1,2}(t,x)\}, \qquad (4)$$

implement the no-delay connection for $\tau = 0$ and the time delay connection[1] for $\tau > 0$. The coupled CGL systems of (1) and (4) also have a steady state (3), but the stability of this state depends on the connection parameters ε and τ.

Let us investigate the spatiotemporal behavior of CGL systems (1) coupled by no-delay connection (4) ($\tau = 0$) and by time delay connection (4) ($\tau > 0$) on numerical simulations. These simulations are performed under periodic boundary conditions, and have a system size of $L = 64$ and a random initial condition. We use the explicit Euler method with a time step of $\Delta t = 1 \times 10^{-4}$ and $N = 512$ space mesh points. Throughout this paper, the parameters $\beta = -1.2$, $\omega_0 = 6.0$, and $\varepsilon = 5.0$ are fixed, and α is set to 3.0 or 7.0. These parameters satisfy the Benjamin-Feir criterion $1 + \alpha\beta < 0$; then, amplitude turbulence occurs in isolated CGL systems (1) without connection [$U_{1,2}(t,x) \equiv 0$].

The spatiotemporal behaviors of amplitude $|W_1(t,x)|$ of the first system with the no-delay connection ($\tau = 0$) in coupled identical systems ($\Delta\omega = 0.0$) and nonidentical systems ($\Delta\omega = 4.0$) for $\alpha = 3.0$ are shown in Figs. 2(a) and 2(b), respectively. These systems run without connection until $t = 50$ and are then coupled at $t = 50$. The amplitude turbulence remains in coupled *identical* systems after coupling, but vanishes in coupled *nonidentical* systems. We have observed that the same behavior occurs for amplitude $|W_2(t,x)|$ of the second system.[2] In what follows, we will show that the

above-mentioned vanishing in coupled *nonidentical* systems is caused by the stabilization induced by both the no-delay connection and parameter mismatch.

Although amplitude death does not occur in coupled *identical* systems with the no-delay connection ($\tau = 0$) [see Fig. 2(a)], the time delay connection ($\tau = 0.25$) can induce stabilization [see Fig. 3(a)]. On the other hand, we observe that the time delay connection fails to induce stabilization when the parameter α is changed from 3.0 to 7.0 [see Fig. 3(b)]. As shown above, the condition under which stabilization occurs depends on several parameters in coupled CGL systems of (1) and (4). The relationship between these parameters and the above-mentioned condition will be analytically investigated in the following sections.

III. SPATIAL-ROBUST STABILITY

The coupled CGL systems can be transformed into

$$\frac{\partial w_{1,2}}{\partial t} = \{(1 \pm i\,\Delta\omega) - (1 + i\beta)|w_{1,2}|^2\}w_{1,2}$$

$$+ (1 + i\alpha)\frac{\partial^2}{\partial x^2}w_{1,2} + u_{1,2}, \qquad (5)$$

$$u_{1,2}(t,x) = \varepsilon\{e^{-i\omega_0\tau}w_{2,1}(t - \tau,x) - w_{1,2}(t,x)\}, \qquad (6)$$

via the transformation

$$w_{1,2}(t,x) := e^{-i\omega_0 t}W_{1,2}(t,x). \qquad (7)$$

The transformed systems of (5) and (6) have a spatially uniform steady state

$$[w_1(t,x) \; w_2(t,x)]^T = [0 \; 0]^T, \quad \forall\, x \in [0,L], \qquad (8)$$

which corresponds to state (3). The stability of state (8) is equivalent to that of state (3). Thus, we will investigate the stability of state (8).

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[1]The delay connection considered herein is a special case of the distributed delay connection in Ref. [47].

[2]Each system shows different behavior before coupling due to different initial conditions. The amplitude turbulence in the both systems vanishes after coupling.
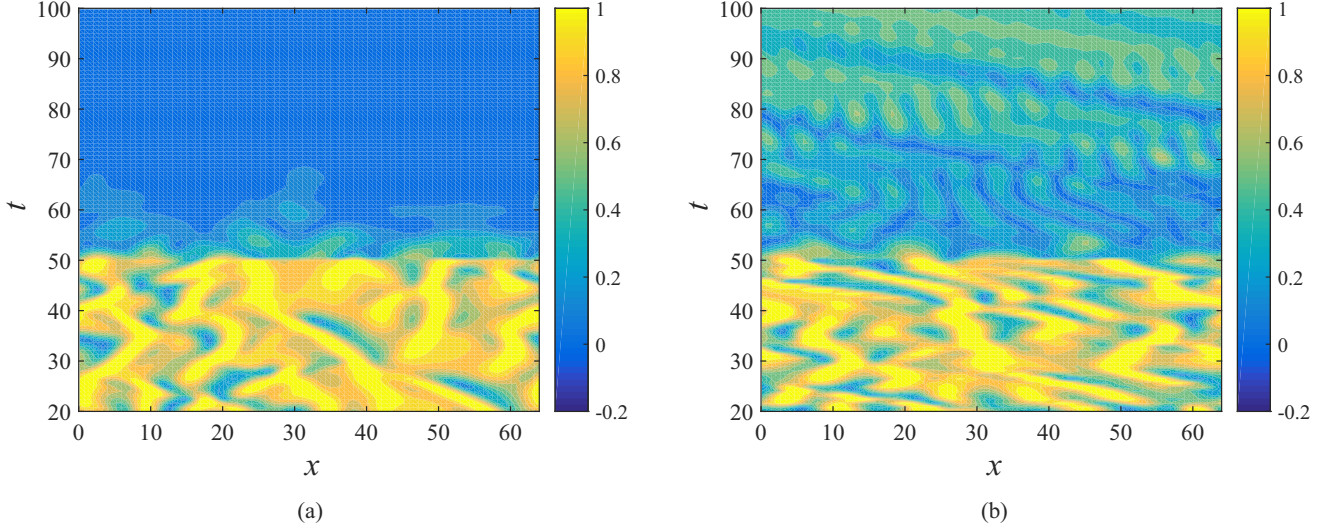
FIG. 3. Spatiotemporal behavior of amplitude $|W_1(t,x)|$ in *identical* CGL systems (1) coupled by the *time delay* connection (4) ($\tau = 0.25$, $\Delta\omega = 0.0$, $\beta = -1.2$, $\omega_0 = 6.0$, $\varepsilon = 5.0$, $L = 64$): (a) $\alpha = 3.0$, (b) $\alpha = 7.0$. These systems are coupled at $t = 50$.

The linearized dynamics around state (8) is described by

$$\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{\mathcal{X}}_1 \\ \boldsymbol{\mathcal{X}}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}(+\Delta\omega) - \varepsilon\boldsymbol{I}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}(-\Delta\omega) - \varepsilon\boldsymbol{I}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\mathcal{X}}_1 \\ \boldsymbol{\mathcal{X}}_2\end{bmatrix}$$

$$+ \varepsilon\begin{bmatrix}\boldsymbol{0} & \boldsymbol{R}(\omega_0\tau) \\ \boldsymbol{R}(\omega_0\tau) & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\mathcal{X}}_1(t-\tau) \\ \boldsymbol{\mathcal{X}}_2(t-\tau)\end{bmatrix}$$

$$+ \frac{\partial^2}{\partial x^2}\begin{bmatrix}\boldsymbol{H}(\alpha) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}(\alpha)\end{bmatrix}\begin{bmatrix}\boldsymbol{\mathcal{X}}_1 \\ \boldsymbol{\mathcal{X}}_2\end{bmatrix}, \quad (9)$$

where

$$\boldsymbol{A}(\Delta\omega) := \begin{bmatrix}1 & -\Delta\omega \\ \Delta\omega & 1\end{bmatrix}, \quad \boldsymbol{H}(\alpha) := \begin{bmatrix}1 & -\alpha \\ \alpha & 1\end{bmatrix},$$

$$\boldsymbol{R}(\omega_0\tau) := \begin{bmatrix}\cos\omega_0\tau & \sin\omega_0\tau \\ -\sin\omega_0\tau & \cos\omega_0\tau\end{bmatrix}, \quad \boldsymbol{\mathcal{X}}_{1,2} := \begin{bmatrix}\mathrm{Re}(w_{1,2}) \\ \mathrm{Im}(w_{1,2})\end{bmatrix}. \quad (10)$$

The perturbation is set to

$$\begin{bmatrix}\boldsymbol{\mathcal{X}}_1^T & \boldsymbol{\mathcal{X}}_2^T\end{bmatrix}^T = (e^{st+ikx} + e^{st-ikx})\boldsymbol{\Gamma}_k, \quad (11)$$

where we have $k \geqslant 0$, $s \in \mathbb{C}$, $\boldsymbol{\Gamma}_k \in \mathbb{R}^4$. Substituting Eq. (11) into linear system (9) leads its characteristic function

$$F(s,\gamma,\Delta\omega,\tau)$$

$$:= \det\left(s\boldsymbol{I}_4 - \begin{bmatrix}\boldsymbol{A}(+\Delta\omega) - \varepsilon\boldsymbol{I}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}(-\Delta\omega) - \varepsilon\boldsymbol{I}_2\end{bmatrix}\right.$$

$$\left. -\varepsilon e^{-s\tau}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{R}(\omega_0\tau) \\ \boldsymbol{R}(\omega_0\tau) & \boldsymbol{0}\end{bmatrix} + \gamma\begin{bmatrix}\boldsymbol{H}(\alpha) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}(\alpha)\end{bmatrix}\right). \quad (12)$$

The parameter γ is defined as $\gamma := k^2$.

Here, in order to simplify the following discussion, this paper provides a definition of spatial-robust stability for spatially uniform steady state (3).

Definition 1. The uniform steady state (3) is said to be spatial-robust stable if it is locally stable for any bound-

ary condition and for any system size, i.e., all roots of $F(s,\gamma,\Delta\omega,\tau) = 0$ are in the open left-half of the complex plane for any $\gamma \geqslant 0$.

The parameter $\gamma \geqslant 0$ depends on the boundary condition and the system size L. Thus, the steady state (3) is locally stable for any boundary condition and for any system size, if $F(s,\gamma,\Delta\omega,\tau)$ is stable for all $\gamma \geqslant 0$. In what follows, the spatial-robust stability without delay ($\tau = 0$) and with delay ($\tau > 0$) will be investigated analytically based on functions (12), $F(s,\gamma,\Delta\omega,0)$ and $F(s,\gamma,\Delta\omega,\tau)$, respectively.

IV. NO-DELAY CONNECTION

This section investigates the local stability of state (3) in the CGL systems (1) coupled by no-delay connection (4) ($\tau = 0$).

A. Limitation and preliminaries

Let us consider a situation in which CGL systems are *identical* ($\Delta\omega = 0$). Then, we have $\boldsymbol{A}(+\Delta\omega) = \boldsymbol{A}(-\Delta\omega) = \boldsymbol{A}(0)$, and function (12) is simplified to $F(s,\gamma,0,0) = F_1(s,\gamma)F_2(s,\gamma)$, where

$$F_1(s,\gamma) := \det[s\boldsymbol{I}_2 - \boldsymbol{A}(0) + \gamma\boldsymbol{H}(\alpha)],$$

$$F_2(s,\gamma) := \det[s\boldsymbol{I}_2 - \boldsymbol{A}(0) + 2\varepsilon\boldsymbol{I}_2 + \gamma\boldsymbol{H}(\alpha)]. \quad (13)$$

The simplified function allows us to derive the following limitation.

Lemma 1. Amplitude death at spatially uniform steady state (3) in identical CGL systems (1) coupled by no-delay connection (4) ($\tau = 0$) never occurs for any boundary condition, system size L, coupling strength ε, and common parameters α and β.

Proof. See Appendix A.

This lemma is supported by the numerical simulation in Fig. 2(a), where amplitude death does not occur in identical CGL systems coupled by no-delay connection.

Next, we consider the situation in which the CGL systems (1) have different frequencies $\Delta\omega > 0$. The characteristic function (12) is described by $F(s,\gamma,\Delta\omega,0) =$
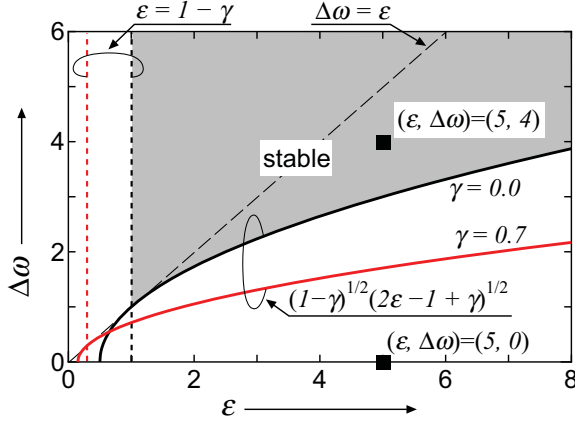
FIG. 4. Stability region of uniform steady state (3) in parameter space $(\varepsilon, \Delta\omega)$ for coupled CGL systems (1) and (4) without time delay $(\tau = 0)$. Steady state (3) is spatial-robust stable in the gray region $(\gamma = 0)$.

$F_+^{\mathrm{s}}(s,\gamma)F_-^{\mathrm{s}}(s,\gamma)$, where

$$F_\pm^{\mathrm{s}}(s,\gamma) := s^2 + 2(\gamma - 1 + \varepsilon)s + (\gamma - 1 + \varepsilon)^2$$
$$+ (\alpha\gamma \pm \sqrt{\Delta\omega^2 - \varepsilon^2})^2. \qquad (14)$$

These functions are polynomials of degree 2 in $s \in \mathbb{C}$. First, we investigate the stability of function $F_+^{\mathrm{s}}(s,\gamma)$ (see Fig. 4).

For $\Delta\omega > \varepsilon$, polynomial $F_+^{\mathrm{s}}(s,\gamma)$ has real coefficients. The coefficient of s^0 is obviously positive real. Thus, the necessary and sufficient condition for $F_+^{\mathrm{s}}(s,\gamma)$ to be stable is that the coefficient of s^1 be positive, i.e.,

$$\varepsilon > 1 - \gamma. \qquad (15)$$

For $\Delta\omega < \varepsilon$, note that $F_+^{\mathrm{s}}(s,\gamma)$ has a complex coefficient. Based on Appendix B, we easily derive the necessary and sufficient condition for the complex polynomial $F_+^{\mathrm{s}}(s,\gamma)$ to be stable. Namely, both condition (15) and condition

$$(\gamma - 1 + \varepsilon)^2 - \varepsilon^2 + \Delta\omega^2 > 0 \qquad (16)$$

hold. These arguments are summarized as follows. The stability of $F_+^{\mathrm{s}}(s,\gamma)$ is described by condition (15) for $\Delta\omega > \varepsilon$ and condition (16) for $\Delta\omega < \varepsilon$. The summarized condition, i.e., the stability of $F_+^{\mathrm{s}}(s,\gamma)$, can be simplified to

$$\varepsilon > 1 - \gamma, \quad \Delta\omega > \sqrt{(1-\gamma)(2\varepsilon - 1 + \gamma)}, \qquad (17)$$

as illustrated in Fig. 4.

On the other hand, it is easy to similarly confirm that $F_-^{\mathrm{s}}(s,\gamma)$ has the same stability of $F_+^{\mathrm{s}}(s,\gamma)$. Therefore, the stability of $F^{\mathrm{s}}(s,\gamma)$ is equivalent to that of $F_+^{\mathrm{s}}(s,\gamma)$.

B. Main result and examples

Let us describe the necessary and sufficient condition for uniform steady state (3) to be spatial-robust stable.

Theorem 1. The uniform steady state (3) in CGL systems (1) coupled by no-delay connection (4) $(\tau = 0)$ is spatial-robust stable for any common parameters α and β if and only if coupling strength ε and frequency difference $\Delta\omega$ satisfy

$$\varepsilon > 1, \quad \Delta\omega > \sqrt{2\varepsilon - 1}. \qquad (18)$$

Proof. See Appendix C.

The condition (17) indicates that the stability region of $F(s,\gamma,\Delta\omega,0)$ in parameter space $(\varepsilon, \Delta\omega)$ shrinks with a decrease in γ, as shown in Fig. 4. This fact indicates that, if $(\varepsilon, \Delta\omega)$ are within the region with $\gamma = 0$ (gray region), then $F(s,\gamma,\Delta\omega,0)$ is stable for all $\gamma \geqslant 0$.

Let us design the coupling strength ε and the frequency difference $\Delta\omega$ which induce the spatial-robust stabilization of the uniform steady state (3). Here, ε and $\Delta\omega$ are set to an instability set $(\varepsilon, \Delta\omega) = (5.0, 0.0)$ outside of the stability region in Fig. 4 and to a stability set $(\varepsilon, \Delta\omega) = (5.0, 4.0)$ inside the stability region in Fig. 4 (see the filled squares in Fig. 4). As shown in Figs. 2(a) and 2(b), corresponding to these two sets, amplitude death does not occur for the instability set, but does occur for the stability set. The numerical simulations agree with the theoretical analysis.

Here, we show that $F_+^{\mathrm{s}}(s,0)$ is a characteristic function of a steady state in coupled Stuart-Landau oscillators. The coupled CGL systems of (1) and (4) without time delay $(\tau = 0)$, without diffusive term $(1 + i\alpha \rightarrow 0)$, and without imaginary coefficient $(\beta \equiv 0)$ are identical to the coupled Stuart-Landau oscillators

$$\frac{dW_{1,2}}{dt} = \{(1 + i\omega_{1,2}) - |W_{1,2}|^2\}W_{1,2} + \varepsilon\{W_{2,1} - W_{1,2}\}. \qquad (19)$$

This allows us to obtain the following corollary.

Corollary 1. The uniform steady state (3) in CGL systems (1) coupled by no-delay connection (4) $(\tau = 0)$ is spatial-robust stable for any common parameters α and β if and only if the equilibrium point $W_{1,2} \equiv 0$ in coupled Stuart-Landau oscillators (19) is stable.

Proof. The stability of $W_{1,2} \equiv 0$ in coupled Stuart-Landau oscillators (19) is governed by $F_+^{\mathrm{s}}(s,0)$, which is equivalent to condition (18) in Theorem 1. ∎

Note that condition (18) in the parameter space $(\varepsilon, \Delta\omega)$ illustrated in Fig. 4 has been described in previous studies [5,6] dealing coupled Stuart-Landau oscillators.

It is well known that the uniform steady state (3) in isolated CGL systems (1) for $\gamma = 0$ mode is the most unstable among all $\gamma \geqslant 0$ modes [39]. In addition, we notice that the dynamics of the isolated CGL systems for $\gamma = 0$ mode can be reduced to that of the isolated Stuart-Landau oscillators. The main results in this section suggest that the above-mentioned relationship between the isolated CGL systems and the isolated Stuart-Landau oscillators is maintained even in the case of no-delay connection (4) $(\tau = 0)$. The next section will show that the relationship is not always maintained in the case of time delay connection (4) $(\tau > 0)$.

V. TIME DELAY CONNECTION

The preceding section revealed that the stabilization of the uniform steady state (3) in identical CGL systems (1) was never induced by no-delay connection (4) $(\tau = 0)$. This section demonstrates that time delay connection (4) $(\tau > 0)$ can induce stabilization even in identical CGL systems (1).

A. Limitation

The characteristic function (12) with $\Delta\omega = 0$ and $\tau > 0$ can be simplified to $F(s,\gamma,0,\tau) := F_+^d(s,\gamma)F_-^d(s,\gamma)$, where

$$F_\pm^d(s,\gamma) := \det[(s-1)\boldsymbol{I}_2 + \gamma\boldsymbol{H}(\alpha)$$
$$+ \varepsilon\{\boldsymbol{I}_2 \pm e^{-s\tau}\boldsymbol{R}(\omega_0\tau)\}]. \quad (20)$$

This simplified function allows us to derive the following lemma.

Lemma 2. The uniform steady state (3) in identical CGL systems (1) coupled by time delay connection (4) ($\tau > 0$) is spatial-robust stable for any common parameter β if and only if functions

$$f_\pm(s,\gamma) := s - i\omega_0 - 1 + (1 + i\alpha)\gamma + \varepsilon(1 \pm e^{-s\tau}), \quad (21)$$

are both stable for all $\gamma \geqslant 0$.

Proof. See Appendix D.

These functions $f_\pm(s,\gamma)$ will play an important role in our analysis of stability. From $f_\pm(s,\gamma)$ in Lemma 2, we obtain a limitation, which is a sufficient condition for the uniform steady state (3) to be unstable.

Corollary 2. Amplitude death at the uniform steady state (3) in identical CGL systems (1) coupled by time delay connection (4) ($\tau > 0$) never occurs for any boundary condition, system size L, and parameters $\varepsilon, \tau, \alpha$, and β, if the nominal frequency ω_0 is zero.

Proof. See Appendix E.

It is well known that delayed feedback control [53] has the odd number property [54,55]. This property indicates that some types of unstable equilibrium points within oscillators cannot be stabilized by delayed feedback control. In addition, for delay-coupled oscillators [5], this property holds: a time delay connection never induces amplitude death at such equilibrium points [56]. Corollary 2 suggests that this property still holds even in coupled CGL systems of (1) and (4).

B. Main result and design procedure

In order to prove our main result, we now provide two lemmas.

Lemma 3. The roots of $f_+(s,\gamma) = 0$ and $f_-(s,\gamma) = 0$ with $\gamma > 1$ never intersect the imaginary axis for any ε, τ, and α.

Proof. See Appendix F.

This lemma indicates that the roots never intersect the imaginary axis for any $\gamma > 1$. The following lemma provides a condition under which the roots never intersect the imaginary axis for $\gamma \in [0,1]$.

Lemma 4. Consider the two equations, $f_\pm(i\lambda_I,\gamma) = 0$. If there does not exist $\lambda_I \in \mathbb{R}$ such that at least one of these equations holds in $\gamma \in [0,1]$, then the roots of $f_\pm(s,\gamma) = 0$ never intersect the imaginary axis for any $\gamma \in [0,1]$.

Proof. See Appendix G.

The main result of this paper, based on Lemmas 2, 3, and 4, is given in the following.

Theorem 2. Design ε and τ such that both

$$f_\pm(s,0) = s - i\omega_0 - 1 + \varepsilon(1 \pm e^{-s\tau}) \quad (22)$$

are stable. If the designed ε and τ satisfy Lemma 4, then the uniform steady state (3) in identical CGL systems (1) coupled

by time delay connection (4) is spatial-robust stable for any common parameter β.

Proof. See Appendix H.

The delay-coupled CGL systems of (1) and (4) with $\beta = 0$ and without the diffusive term is equivalent to the delay-coupled Stuart-Landau oscillators [5,6]

$$\frac{dW_{1,2}}{dt} = \{(1 + i\omega_0) - |W_{1,2}|^2\}W_{1,2}$$
$$+ \varepsilon\{W_{2,1}(t-\tau) - W_{1,2}\}. \quad (23)$$

The stability of equilibrium point $W_{1,2}(t) \equiv 0$ in the delay-coupled oscillators is governed by functions (22). This fact implies that ε and τ in Theorem 2 can be easily designed using the analytical results in previous studies [5,6], which deal with the delay-coupled Stuart-Landau oscillators. In addition, Lemma 4 can be easily confirmed by numerical calculations. Consequently, a systematic procedure for designing ε and τ, which induce the spatial-robust stabilization independently of the common parameter β, is provided as follows:

(Step 0) The nominal frequency ω_0 and the common parameter α are given.

(Step 1) If $\omega_0 = 0$ is satisfied, then stop designing ε and τ (Corollary 2); otherwise, go to the next step.

(Step 2) Design ε and τ such that both of functions (22) are stable (refer to studies [5,6]).

(Step 3) Draw the curves of $\text{Re}[f_+(i\lambda_I,\gamma)] = 0$ and $\text{Im}[f_+(i\lambda_I,\gamma)] = 0$ on the γ-λ_I plane. If an intersection of $\text{Re}[f_+(i\lambda_I,\gamma)] = 0$ and $\text{Im}[f_+(i\lambda_I,\gamma)] = 0$ and that of $\text{Re}[f_-(i\lambda_I,\gamma)] = 0$ and $\text{Im}[f_-(i\lambda_I,\gamma)] = 0$ are not within the range $\gamma \in [0,1]$, then ε and τ designed in Step 2 definitely induce the spatial-robust stabilization of the uniform steady state (3).

Now, let us design ε and τ in accordance with the design procedure, as follows. For Step 0, $\omega_0 = 6$ and $\alpha = 3.0$ are given. For Step 1, $\omega_0 \neq 0$ holds, then go to the next step. For Step 2, the boundary curves of functions $f_\pm(i\lambda_I,0) = 0$ are plotted as shown in Fig. 5. Both ε and τ are chosen from the stability region (gray area): $\varepsilon = 5.0$ and $\tau = 0.25$. For Step 3, the curves of $\text{Re}[f_+(i\lambda_I,\gamma)] = \text{Im}[f_+(i\lambda_I,\gamma)] = 0$ and $\text{Re}[f_-(i\lambda_I,\gamma)] = \text{Im}[f_-(i\lambda_I,\gamma)] = 0$ are plotted on the γ-λ_I plane, as shown in Fig. 6(a). These curves do not
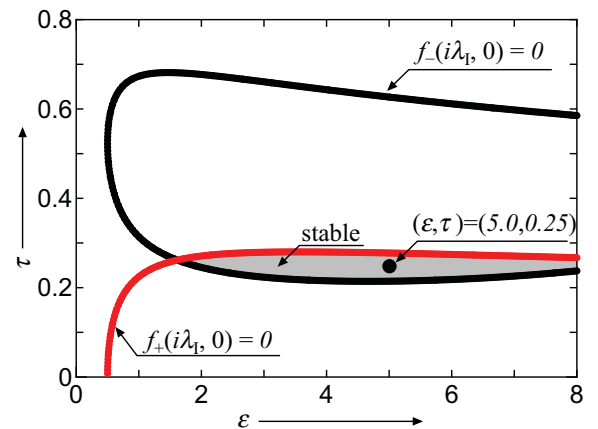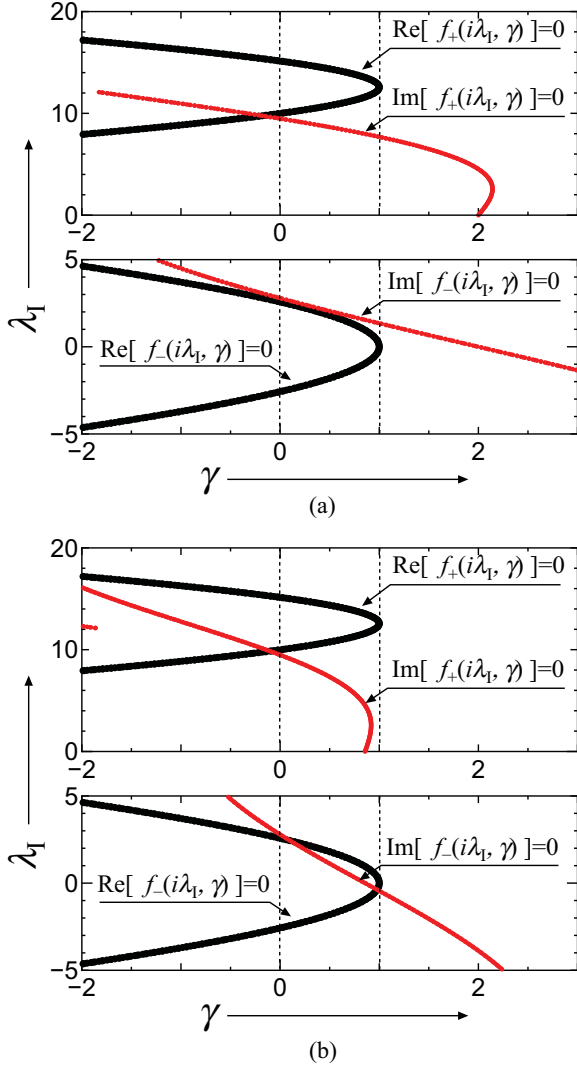

FIG. 5. Stability region and boundary curves in the ε-τ plane ($w_0 = 6.0$).

(a)

(b)

FIG. 6. Curves of $\mathrm{Re}[f_+(i\lambda_I,\gamma)] = \mathrm{Im}[f_+(i\lambda_I,\gamma)] = 0$ and $\mathrm{Re}[f_-(i\lambda_I,\gamma)] = \mathrm{Im}[f_-(i\lambda_I,\gamma)] = 0$ on the γ-λ_I plane ($\tau = 0.25$, $\omega_0 = 6.0$, $\varepsilon = 5.0$): (a) $\alpha = 3.0$, (b) $\alpha = 7.0$.

intersect within the range $\gamma \in [0,1]$. Then, we can guarantee that $\varepsilon = 5.0$ and $\tau = 0.25$ designed in Step 2 induce the spatial-robust stabilization of the uniform steady state (3). We now confirm our procedure on numerical simulations. The spatiotemporal behavior of the coupled identical CGL systems with the designed ε and τ is shown in Fig. 3(a), where amplitude death occurs after coupling $t > 50$.

Here, we change the parameter α from 3.0 to 7.0. For $\alpha = 7.0$, the curves in Step 3, as illustrated in Fig. 6(b), intersect within the range $\gamma \in [0,1]$. This fact states that we cannot guarantee the induction of stabilization. The spatiotemporal behavior for $\alpha = 7.0$ is shown in Fig. 3(b), where amplitude death does not occur after coupling $t > 50$. These numerical results agree with our analytical results.

The above results suggest that the stability depends on the parameter α. In order to clarify the dependence of the stability on the parameter α, the boundary curves of $f_\pm(i\lambda_I,\gamma) = 0$ with the designed $\varepsilon = 5.0$ and $\tau = 0.25$ are plotted in the γ-α plane as shown in Fig. 7. We cannot see the curve of $f_+(i\lambda_I,\gamma) = 0$,
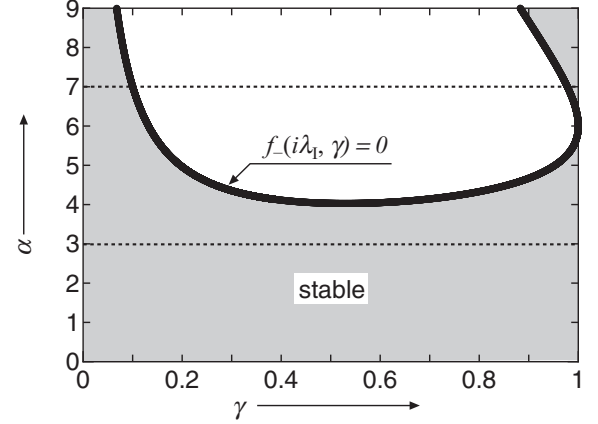

FIG. 7. Boundary curve of $f_-(i\lambda_I,0) = 0$ with $\varepsilon = 5.0$ and $\tau = 0.25$, below which both of $f_\pm(s,\gamma)$ are stable.

since it is located outside of the range of Fig. 7. Here, both of $f_\pm(s,\gamma)$ are stable below the curve of $f_-(i\lambda_I,\gamma) = 0$. The dotted line at $\alpha = 3.0$ in Fig. 7 is located inside of the stability region for any $\gamma \in [0,1]$. On the other hand, the dotted line at $\alpha = 7.0$ intersects with the curve of $f_-(i\lambda_I,\gamma) = 0$. These results agree with our numerical results of Figs. 3(a) and 3(b) and with our analytical results of Figs. 6(a) and 6(b). Furthermore, we see from Fig. 7 that the spatial-robust stability cannot be guaranteed even when both of $f_\pm(s,\gamma)$ are stable at $\gamma = 0$. This fact is different from the results of the no-delay connection.

VI. DISCUSSIONS

This section will discuss previous studies that are closely related to the subject of this paper: (1) the stabilization of a uniform steady state in a single CGL system with inhomogeneous frequency, (2) the stabilization of a uniform steady state in single CGL systems with delayed feedback control, and (3) comparison of amplitude death in coupled CGL systems with amplitude death in coupled oscillators.

To the best of our knowledge, there have been no reports on amplitude death in coupled CGL systems. On the other hand, Sakaguchi reported that stabilization occurs in a one-dimensional inhomogeneous CGL system in which the natural frequency depends on location x [51,52]. Although the inhomogeneous CGL system differs greatly from our coupled CGL systems, we obtained similar results. Stabilization occurs for large frequency differences and strong diffusion (see Fig. 1 in Ref. [51] and Fig. 4 in this paper).

A pair of CGL systems (1) coupled by time delay connection (4) can be considered to be an extension of single CGL systems controlled by time delay feedback. Let us review previous studies on single CGL systems with time delay feedback control and clarify relations through the analytical results obtained herein. It has been reported that global delay invasive feedback can stabilize a uniform oscillation in CGL systems, whereas noninvasive feedback cannot [57–59]. A traveling wave in one-dimensional CGL systems can be stabilized by local delay noninvasive feedback [60], but not in two-dimensional CGL systems [61]. This limitation was overcome by introducing the combination of

local delay- and spatial-noninvasive feedbacks [62,63] and an asymmetric multiple-delay noninvasive feedback [64,65]. The combination of local and global delay noninvasive feedbacks was proposed, and several phenomena [66,67], such as uniform oscillations, traveling waves, and standing waves [68,69], were observed. Although most of these studies did not deal with the stabilization of a uniform steady state in single CGL systems, Stich and Beta analytically investigated such stabilization in detail, where the stabilization was referred to as *amplitude death* [67]. Note that, in previous studies [66,67], this term was used to refer to the stabilization of the uniform steady state in single CGL systems with delay feedback. On the other hand, in this paper, the term *amplitude death* is used to refer to the stabilization of the uniform steady state in a pair of CGL systems coupled by diffusive connections.

In past decades, amplitude death in Stuart-Landau oscillators coupled by no-delay connection ($\tau = 0$) [2] and delay connection ($\tau > 0$) [3–5] has been the subject of intense interest.[3] This paper considers several stability conditions similar to those considered in previous studies. In recent years, significant efforts have been made to design the connection parameters for inducing amplitude death in coupled oscillators [24,70,71] from the viewpoint of control theory because of its potential applications in engineering systems [72–74]. Based on the same concept, this paper also designed connection parameters for inducing amplitude death in coupled CGL systems.

VII. CONCLUSION

In this paper, we showed that a spatially uniform steady state in coupled CGL systems can be spatial-robust stable by both diffusive no-delay and delay connections. In addition, the stability analysis of this study revealed the following results. Amplitude death never occurs in coupled identical CGL systems with no-delay connection, but can occur in the case of delay connection. Amplitude death never occurs in coupled identical CGL systems with zero nominal frequency. Finally, a systematic procedure for designing the connection delay time and the coupling strength for inducing spatial-robust stabilization of the steady state was provided.

APPENDIX A: PROOF OF LEMMA 1

The wave number $k = 0$ exists for any boundary condition and for any system size L. Here, we consider the stability of $F_1(s,\gamma)$ at $\gamma = k^2 = 0$. We see that $F_1(s,0) = \det[s\boldsymbol{I}_2 - \boldsymbol{A}(0)]$, i.e., the characteristic function of unstable matrix $\boldsymbol{A}(0)$, is unstable. As $F(s,0) = F_1(s,0)F_2(s,0)$, we see

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[3]It may be worth mentioning that the similar quenching phenomenon, oscillation death, can occur in coupled oscillators [4]. This study does not deal with oscillation death in coupled CGL systems. There is room for further investigation.

that $F(s,0)$ is unstable for any boundary condition and for any L, ε, α, and β. Therefore, amplitude death does not occur at state (3) for any boundary condition and for any L, ε, α, and β. ∎

APPENDIX B: STABILITY OF TWO-DEGREE COMPLEX POLYNOMIALS

A two-degree complex polynomial,

$$s^2 + (a_1 + ib_1)s + a_0 + ib_0, \tag{B1}$$

is stable if and only if

$$a_1 > 0, \quad a_0 a_1 + \left(b_1 - \frac{b_0}{a_1}\right)b_0 > 0 \tag{B2}$$

holds [75].

APPENDIX C: PROOF OF THEOREM 1

Condition (18) is equivalent to the stability condition of $F_+^{\mathrm{s}}(s,0)$, i.e., inequities (17) ($\gamma = 0$). Thus, this proof will indicate that uniform steady state (3) in CGL systems (1) coupled by no-delay connection (4) with $\tau = 0$ is spatial-robust stable for any common parameters α and β if and only if $F_+^{\mathrm{s}}(s,0)$ is stable. For this indication, we need only show that the stability of $F_+^{\mathrm{s}}(s,0)$ is equivalent to that of $F_+^{\mathrm{s}}(s,\gamma)$, $\forall \gamma \geqslant 0$.

First, if $F_+^{\mathrm{s}}(s,\gamma)$ is stable for all $\gamma \geqslant 0$, then $F_+^{\mathrm{s}}(s,0)$ is stable. Second, we show below that, if $F_+^{\mathrm{s}}(s,0)$ is stable, then $F_+^{\mathrm{s}}(s,\gamma)$ is also stable for all $\gamma \geqslant 0$. Note that the necessary and sufficient condition for $F_+^{\mathrm{s}}(s,\gamma)$ to be stable is that both conditions (15) and (16) be satisfied. We can easily see that if condition (15) with $\gamma = 0$ holds, then the condition with all $\gamma > 0$ holds. It is straightforward to show that condition (16) holds for all $\gamma \geqslant 1$. Next, note that the condition (16) can be rewritten as $\Delta\omega^2 > -(\gamma - 1)(\gamma + 2\varepsilon - 1)$. The right-hand side of this inequality, a polynomial of degree 2 in γ, is convex upward and positive in $\gamma \in [0,1)$. Therefore, if this inequality holds at $\gamma = 0$, then it also holds for all $\gamma \in [0,1)$. As a consequence, if $F_+^{\mathrm{s}}(s,0)$ is stable, then $F_+^{\mathrm{s}}(s,\gamma)$ is also stable for all $\gamma \geqslant 0$. ∎

APPENDIX D: PROOF OF LEMMA 2

The stability of the uniform steady state (3) is obviously described by the function $F_+^{\mathrm{d}}(s,\gamma)F_-^{\mathrm{d}}(s,\gamma)$. This proof will show that the stability of $F_+^{\mathrm{d}}(s,\gamma)$ is equivalent to that of $f_\pm(s,\gamma)$. First, we consider the stability of $F_+^{\mathrm{d}}(s,\gamma)$. This function can be described by $F_+^{\mathrm{d}}(s,\gamma) = F_+^{\mathrm{d},(+)}(s,\gamma)F_+^{\mathrm{d},(-)}(s,\gamma)$, where

$$F_+^{\mathrm{d},(\pm)}(s,\gamma) := s - 1 + (1 \pm i\alpha)\gamma + \varepsilon\{1 + e^{-(s\pm i\omega_0)\tau}\}. \tag{D1}$$

Since the stability of $F_+^{\mathrm{d},(+)}(s,\gamma)$ is equivalent to that of $F_+^{\mathrm{d},(-)}(s,\gamma)$, it is sufficient to deal only with $F_+^{\mathrm{d},(+)}(s,\gamma)$. The function $F_+^{\mathrm{d},(+)}(s,\gamma)$ can be transformed into $f_+(s,\gamma)$ via the transformation $s + i\omega_0 \to s$, which does not influence the stability criterion. Moreover, $f_-(s,\gamma)$ can be similarly obtained from $F_-^{\mathrm{d}}(s,\gamma)$. ∎

APPENDIX E: PROOF OF COROLLARY 2

There exists $\gamma = 0$ corresponding to a spatially uniform steady state, regardless of the boundary condition and the system size L. This proof will show that if $\omega_0 = 0$ holds, then $f_-(s,\gamma)$ with $\gamma = 0$ defined in Lemma 2,

$$f_-(s,0) = s - 1 + \varepsilon(1 - e^{-s\tau}), \qquad \text{(E1)}$$

is unstable for any ε and τ. This function satisfies

$$\lim_{s \to +\infty} f_-(s,0) = +\infty, \quad f_-(0,0) = -1. \qquad \text{(E2)}$$

Thus, $f_-(s,0) = 0$ has at least one real positive root independently of ε, τ. We can say that, if $\omega_0 = 0$ holds, then the uniform steady state (3) is unstable due to the instability of $f_-(s,0)$. Consequently, amplitude death does not occur at the uniform steady state (3). ∎

APPENDIX F: PROOF OF LEMMA 3

There exits at least one root $\lambda_I \in \mathbb{R}$ satisfying $f_+(i\lambda_I,\gamma)=0$ or $f_-(i\lambda_I,\gamma) = 0$ when roots of $f_+(s,\gamma) = 0$ or $f_-(s,\gamma) = 0$ intersect the imaginary axis. Thus, this proof will show that, for any $\gamma > 1$, there exists no $\lambda_I \in \mathbb{R}$ satisfying $f_+(i\lambda_I,\gamma) = 0$ or $f_-(i\lambda_I,\gamma) = 0$, regardless of ε, τ, or α. Here, the real parts

of $f_\pm(i\lambda_I,\gamma) = 0$ are given by

$$\text{Re}[f_\pm(i\lambda_I,\gamma)] := -1 + \gamma + \varepsilon(1 \pm \cos \lambda_I \tau) = 0. \qquad \text{(F1)}$$

Due to the structure of (F1), if $\gamma > 1$, then there exists no $\lambda_I \in \mathbb{R}$ for any ε, τ, and α, such that (F1) holds. As $\text{Re}[f_\pm(i\lambda_I,\gamma)] \neq 0$, we have $f_\pm(i\lambda_I,\gamma) \neq 0$ for any λ_I, γ, ε, τ, and α. ∎

APPENDIX G: PROOF OF LEMMA 4

Equations $f_\pm(i\lambda_I,\gamma) = 0$ hold when roots of $f_\pm(s,\gamma) = 0$ intersect the imaginary axis. Therefore, if there exists no $\lambda_I \in \mathbb{R}$ such that $f_+(i\lambda_I,\gamma) = 0$ or $f_-(i\lambda_I,\gamma) = 0$ hold in $\gamma \in [0,1]$, then roots of $f_+(s,\gamma) = 0$ and $f_-(s,\gamma) = 0$ never intersect the imaginary axis for any $\gamma \in [0,1]$. ∎

APPENDIX H: PROOF OF THEOREM 2

In order to prove this theorem, based on Lemma 2, we must show that both $f_\pm(s,\gamma)$ are stable for all $\gamma \geqslant 0$. As both $f_\pm(s,0)$ with the designed ε and τ are stable, all roots of $f_\pm(s,0) = 0$ are within the open left-half complex plane at $\gamma = 0$. If the designed ε and τ satisfy Lemma 4, then these roots do not intersect the imaginary axis for any $\gamma \in [0,1]$. Furthermore, Lemma 3 always holds, independently of ε and τ. As a result, we can say that if the designed ε and τ satisfy Lemma 4, these roots remain in the open left-half complex plane for any $\gamma > 0$. As a consequence, it has been proved that both $f_\pm(s,\gamma)$ are stable for all $\gamma \geqslant 0$. ∎

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