Covering a bounded set of functions by an increasing chain of slaloms

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Masaru Kada*

Abstract

A slalom is a sequence of finite sets of length ω. Slaloms are ordered by coordinatewise inclusion with finitely many exceptions. Improving earlier results of Mildenberger, Shelah and Tsaban, we prove consistency results concerning existence and non-existence of an increasing sequence of a certain type of slaloms which covers a bounded set of functions in ω^ω.

1 Introduction

We use standard terminology and refer the readers to [2] for undefined set-theoretic notions.

Bartoszyński [1] introduced the combinatorial concept of *slalom* to study combinatorial aspects of measure and category on the real line.

We call a sequence of finite subsets of ω of length ω a *slalom*. For a function $g \in \omega^\omega$, let $\mathcal{S}^g$ be the set of slaloms φ such that $|\varphi(n)| \leq g(n)$ for all $n < \omega$. $\mathcal{S}$ denotes $\mathcal{S}^g$ for $g(n) = 2^n$. For two slaloms φ and ψ, we write $\varphi \sqsubseteq \psi$ if $\varphi(n) \subseteq \psi(n)$ for all but finitely many $n < \omega$. For a function $f \in \omega^\omega$ and a slalom φ, $f \sqsubseteq \varphi$ if $\langle \{f(n)\} : n < \omega \rangle \sqsubseteq \varphi$.

Mildenberger, Shelah and Tsaban [9] defined cardinals θ_h for $h \in \omega^\omega$ and θ_* to give a partial characterization of the cardinal $\mathfrak{od}$, the critical cardinality of a certain selection principle for open covers.

The definition of θ_h in [9] is described using a combinatorial property which is called *o-diagonalization*. Here we redefine θ_h to fit in the present

*Graduate School of Science, Osaka Prefecture University. Sakai, Osaka 599–8531 JAPAN

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context. It is easy to see that the following definition is equivalent to the original one. For a function $h \in (\omega \smallsetminus \{0,1\})^\omega$, let $h-1$ denote the function $h' \in \omega^\omega$ which is defined by $h'(n) = h(n) - 1$ for all n.

Definition 1.1. For a function $h \in (\omega \smallsetminus \{0,1\})^\omega$, θ_h is the smallest size of a subset Φ of $\mathcal{S}^{h-1}$ which satisfies the following, *if such a set* Φ *exists*:

1. Φ is well-ordered by $\sqsubseteq$;

2. For every $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f \sqsubseteq \varphi$.

If there is no such Φ, we define $\theta_h = \mathfrak{c}^+$.

It is easy to see that $h_1 \leq^* h_2$ implies $\theta_{h_1} \geq \theta_{h_2}$.

Definition 1.2 ([9]). $\theta_* = \min\{\theta_h : h \in \omega^\omega\}$.

In Section 2, we will show that $\theta_* = \mathfrak{c}^+$ is consistent with ZFC.

We say a proper forcing notion $\mathbb{P}$ has the *Laver property* if, for any $h \in \omega^\omega$, $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ for a function in ω^ω such that $p \Vdash_{\mathbb{P}} \dot{f} \in \prod_{n<\omega} h(n)$, there exist $q \in \mathbb{P}$ and $\varphi \in \mathcal{S}$ such that q is stronger than p and $q \Vdash_{\mathbb{P}} \dot{f} \sqsubseteq \varphi$.

Mildenberger, Shelah and Tsaban proved that $\theta_* = \aleph_1$ holds in all forcing models by a proper forcing notion with the Laver property over a model for CH, the continuum hypothesis [9]. In section 2, we refine their result and state a sufficient condition for $\theta_* \leq \mathfrak{c}$. As a consequence, we will show that Martin's axiom implies $\theta_* = \mathfrak{c}$.

In Section 3, we give an application of the lemma presented in Section 2 to another problem in topology. We answer a question on approximations to the Stone–Čech compactification of ω by Higson compactifications of ω, which was posed by Kada, Tomoyasu and Yoshinobu [6].

2 Facts on the cardinal θ_*

First we observe that $\theta_* = \mathfrak{c}^+$ is consistent with ZFC. We use the following theorem, which is a corollary of Kunen's classical result [7]. For the readers' convenience, we present a complete proof in Section 4.

Theorem 2.1. *Suppose that $\kappa \geq \aleph_2$. The following holds in the forcing model obtained by adding κ Cohen reals over a model for CH: Let $\mathcal{X}$ be a Polish space and $A \subseteq \mathcal{X} \times \mathcal{X}$ a Borel set. Then there is no sequence $\langle r_\alpha : \alpha < \omega_2 \rangle$ in $\mathcal{X}$ which satisfies*

$$\alpha \leq \beta < \omega_2 \text{ if and only if } \langle r_\alpha, r_\beta \rangle \in A.$$

Fix $h \in \omega^\omega$. We may regard $\mathcal{S}^{h-1}$ as a product space of countably many finite discrete spaces, and then the relation $\sqsubseteq$ on $\mathcal{S}^{h-1}$ is a Borel subset of $\mathcal{S}^{h-1} \times \mathcal{S}^{h-1}$.

Theorem 2.2. *$\theta_* = \mathfrak{c}^+$ holds in the forcing model obtained by adding $\aleph_2$ Cohen reals over a model for CH.*

Proof. Fix $h \in \omega^\omega$. By Theorem 2.1, in the forcing model obtained by adding $\aleph_2$ Cohen reals over a model for CH, there is no $\sqsubseteq$-increasing chain of length ω_2 in $\mathcal{S}^{h-1}$. This means that θ_h must be $\aleph_1$ whenever $\theta_h \leq \mathfrak{c}$.

On the other hand, $\mathsf{cov}(\mathcal{M}) = \aleph_2$ holds in the same model. Also, by [9] we have $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{od} \leq \theta_h$. This means that θ_h cannot be $\aleph_1$ in this model, and hence $\theta_h = \mathfrak{c}^+$. $\square$

Next we state a sufficient condition for $\theta_* \leq \mathfrak{c}$. We use the following characterization of $\mathsf{add}(\mathcal{N})$.

Theorem 2.3 ([2, Theorem 2.3.9]). *$\mathsf{add}(\mathcal{N})$ is the smallest size of a subset F of ω^ω such that, for every $\varphi \in \mathcal{S}$ there is an $f \in F$ such that $f \not\sqsubseteq \varphi$.*

Definition 2.4 ([5, Section 5]). For a function $h \in \omega^\omega$, $\mathfrak{l}_h$ is the smallest size of a subset Φ of $\mathcal{S}$ such that for all $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f \sqsubseteq \varphi$. Let $\mathfrak{l} = \sup\{\mathfrak{l}_h : h \in \omega^\omega\}$.

Note that $h_1 \leq^* h_2$ implies $\mathfrak{l}_{h_1} \leq \mathfrak{l}_{h_2}$.

If CH holds in a ground model V, $h \in \omega^\omega \cap V$, and a proper forcing notion $\mathbb{P}$ has the Laver property, then $\mathfrak{l}_h = \aleph_1$ holds in the model $V^{\mathbb{P}}$. Consequently, if CH holds in V, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ is a countable support iteration of proper forcings, $\mathbb{P} = \lim_{\alpha<\omega_2} \mathbb{P}_\alpha$ and

$$\Vdash_{\mathbb{P}_\alpha} \text{“}|\dot{\mathbb{Q}}_\alpha| \leq \aleph_1 \text{ and } \dot{\mathbb{Q}}_\alpha \text{ has the Laver property”}$$

holds for every $\alpha < \omega_2$, then $\mathfrak{l} = \aleph_1$ holds in $V^{\mathbb{P}}$, since every function h in $V^{\mathbb{P}}$ appears in $V^{\mathbb{P}_\alpha}$ for some $\alpha < \omega_2$, where CH holds.[1]

Now we define a subset $\mathcal{S}^+$ of $\mathcal{S}$ as follows:

$$\mathcal{S}^+ = \left\{ \varphi \in \mathcal{S} : \lim_{n\to\infty} \frac{|\varphi(n)|}{2^n} = 0 \right\}.$$

Let $\mathfrak{l}'_h$ be the smallest size of a subset Φ of $\mathcal{S}^+$ such that for all $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f \sqsubseteq \varphi$. Clearly we have $\mathfrak{l}_h \leq \mathfrak{l}'_h$, and it is easy to see that for every $h \in \omega^\omega$ there is an $h^* \in \omega^\omega$ such that $\mathfrak{l}'_h \leq \mathfrak{l}_{h^*}$. Hence we have $\mathfrak{l} = \sup\{\mathfrak{l}'_h : h \in \omega^\omega\}$.

[1] In the paper [6], the authors state “If CH holds in a ground model V, and a proper forcing notion $\mathbb{P}$ has the Laver property, then $\mathfrak{l} = \aleph_1$ holds in the model $V^{\mathbb{P}}$”. But it is inaccurate, since we do not see the values of $\mathfrak{l}_h$ for functions $h \in V^{\mathbb{P}}$ which are not bounded by any function from V.

Lemma 2.5. *For a subset Φ of $\mathcal{S}^+$ of size less than $\mathsf{add}(\mathcal{N})$, there is a $\psi \in \mathcal{S}^+$ such that $\varphi \sqsubseteq \psi$ for all $\varphi \in \Phi$.*

Proof. For each $\varphi \in \mathcal{S}^+$, define an increasing function $\eta_\varphi \in \omega^\omega$ by letting

$$\eta_\varphi(m) = \min\left\{ l < \omega : \forall k \geq l \ \left(|\varphi(k)| < \frac{2^k}{m \cdot 2^m} \right) \right\}$$

for all $m < \omega$. η_φ is well-defined by the definition of $\mathcal{S}^+$.

Suppose $\kappa < \mathsf{add}(\mathcal{N})$ and fix a set $\Phi \subseteq \mathcal{S}^+$ of size κ arbitrarily. Since $\kappa < \mathsf{add}(\mathcal{N}) \leq \mathfrak{b}$, there is a function $\eta \in \omega^\omega$ such that $\lim_{n\to\infty} \eta(n)/2^n = \infty$ and for all $\varphi \in \Phi$ we have $\eta_\varphi \leq^* \eta$. For each $m < \omega$, let $I_m = \{\eta(m), \eta(m)+1, \ldots, \eta(m+1)-1\}$ and enumerate $\prod_{n\in I_m} [\omega]^{\leq \lfloor 2^n/(m\cdot 2^m)\rfloor}$ as $\{s_{m,i} : i < \omega\}$, where $\lfloor r \rfloor$ denotes the largest integer which does not exceed the real number r.

For $\varphi \in \Phi$, define $\tilde{\varphi} \in \omega^\omega$ as follows. If there is an $i < \omega$ such that $\varphi \restriction I_m = s_{m,i}$, then let $\tilde{\varphi}(m) = i$; otherwise $\tilde{\varphi}(m)$ is arbitrary.

Since $|\Phi| = \kappa < \mathsf{add}(\mathcal{N})$ and by Theorem 2.3, there is a $\hat{\psi} \in \mathcal{S}$ such that, for all $\varphi \in \Phi$ we have $\tilde{\varphi} \sqsubseteq \hat{\psi}$. Define ψ by letting for each n, if $n \in I_m$ then $\psi(n) = \bigcup\{s_{m,i}(n) : i \in \hat{\psi}(m)\}$, and if $n < \eta(0)$ then $\psi(n) = \emptyset$. It is straightforward to check that $\psi \in \mathcal{S}^+$ and $\varphi \sqsubseteq \psi$ for all $\varphi \in \Phi$. $\square$

Lemma 2.6. *Suppose that $h \in \omega^\omega$ satisfies $h(n) > n^2$ for all $n < \omega$. If $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_h = \kappa$, then there is an $\sqsubseteq$-increasing sequence $\langle \sigma_\alpha : \alpha < \kappa \rangle$ in $\mathcal{S}^+$ such that, for all $f \in \prod_{n<\omega} h(n)$ there is an $\alpha < \kappa$ such that $f \sqsubseteq \varphi_\alpha$.*

Proof. Fix a sequence $\langle \varphi_\alpha : \alpha < \kappa \rangle$ in $\mathcal{S}^+$ so that for all $f \in \prod_{n<\omega} h(n)$ there is an $\alpha < \kappa$ such that $f \sqsubseteq \varphi_\alpha$. Using the previous lemma, inductively construct an $\sqsubseteq$-increasing sequence $\langle \sigma_\alpha : \alpha < \kappa \rangle$ of elements of $\mathcal{S}^+$ so that $\varphi_\alpha \sqsubseteq \sigma_\alpha$ holds for each $\alpha < \omega_2$. Then $\langle \sigma_\alpha : \alpha < \kappa \rangle$ is as required. $\square$

Define $H_1 \in \omega^\omega$ by letting $H_1(n) = 2^n + 1$ for all n.

Theorem 2.7. *If $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_1}$, then $\theta_* = \mathfrak{od} = \mathsf{add}(\mathcal{N})$.*

Proof. Let $\kappa = \mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_1}$. Since $\mathcal{S}^+ \subseteq \mathcal{S} \subseteq \mathcal{S}^{H_1 - 1}$, the previous lemma shows that $\theta_* \leq \theta_{H_1} \leq \kappa$. On the other hand, by [9], we have $\kappa = \mathsf{add}(\mathcal{N}) \leq \mathsf{cov}(\mathcal{M}) \leq \mathfrak{od} \leq \theta_*$. $\square$

Corollary 2.8 ([9]). *If a ground model V satisfies CH, and a proper forcing notion $\mathbb{P}$ has the Laver property, then $\theta_* = \aleph_1$ holds in the model $V^{\mathbb{P}}$.*

Proof. Follows from Theorem 2.7 and the fact that $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_1} = \mathfrak{l}_{H_1^*} = \aleph_1$ holds in the model $V^{\mathbb{P}}$. $\square$

Corollary 2.9. *Martin's axiom implies $\theta_* = \mathfrak{c}$.*

Proof. Follows from Theorem 2.7 and the fact that $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_1} = \mathfrak{l} = \mathfrak{c}$ holds under Martin's axiom. $\square$

3 Application

In this section, we give an answer to a question which was posed by Kada, Tomoyasu and Yoshinobu [6]. We refer the reader to [6] for undefined topological notions.

For compactifications αX and γX of a completely regular Hausdorff space X, we write $\alpha X \leq \gamma X$ if there is a continuous surjection from γX to αX which fixes the points from X, and $\alpha X \simeq \gamma X$ if $\alpha X \leq \gamma X \leq \alpha X$. The Stone–Čech compactification βX of X is the maximal compactification of X in the sense of the order relation $\leq$ among compactifications of X.

For a proper metric space (X, d), $\overline{X}^d$ denotes the Higson compactification of X with respect to the metric d.

$\mathfrak{ht}$ is the smallest size of a set D of proper metrics on ω such that

1. $\{\overline{\omega}^d : d \in D\}$ is well-ordered by $\leq$;
2. There is no $d \in D$ such that $\overline{\omega}^d \simeq \beta\omega$;
3. $\beta\omega \simeq \sup\{\overline{\omega}^d : d \in D\}$, where sup is in the sense of the order relation $\leq$ among compactifications of ω;

if such a set D exists. We define $\mathfrak{ht} = \mathfrak{c}^+$ if there is no such D.

Kada, Tomoyasu and Yoshinobu [6, Theorem 6.16] proved the consistency of $\mathfrak{ht} = \mathfrak{c}^+$ using a similar argument to the proof of Theorem 2.2. But the consistency of $\mathfrak{ht} \leq \mathfrak{c}$ was not addressed. Here we state a sufficient condition for $\mathfrak{ht} \leq \mathfrak{c}$, and show that it is consistent with ZFC.

Define $H_2 \in \omega^\omega$ by letting $H_2(n) = 2^{2^{(n^4)}}$ for all n. The following lemma is obtained as a corollary of the proof of [6, Theorem 6.11].

Lemma 3.1. *Let κ be a cardinal. If there is an $\sqsubseteq$-increasing sequence $\langle \varphi_\alpha : \alpha < \kappa \rangle$ of slaloms in $\mathcal{S}$ such that for all $f \in \prod_{n<\omega} H_2(n)$ there is an $\alpha < \kappa$ such that $f \sqsubseteq \varphi_\alpha$, then $\mathfrak{ht} \leq \kappa$.*

Now we have the following theorem.

Theorem 3.2. *If $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_2}$, then $\mathfrak{ht} = \mathsf{add}(\mathcal{N})$.*

Proof. $\mathsf{add}(\mathcal{N}) \leq \mathfrak{ht}$ is proved in [6, Section 6]. To see $\mathfrak{ht} \leq \mathsf{add}(\mathcal{N})$, apply Lemma 2.6 for $h = H_2$ to get a sequence of slaloms which is required in Lemma 3.1. $\square$

Corollary 3.3. *If a ground model V satisfies CH, and a proper forcing notion $\mathbb{P}$ has the Laver property, then $\mathfrak{ht} = \aleph_1$ holds in the model $V^{\mathbb{P}}$.*

Proof. Follows from Theorem 3.2 and the fact that $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_2} = \mathfrak{l}_{H_2^*} = \aleph_1$ holds in the model $V^{\mathbb{P}}$. $\square$

Corollary 3.4. *Martin's axiom implies* $\mathfrak{ht} = \mathfrak{c}$.

Proof. Follows from Theorem 3.2 and the fact that $\mathsf{add}(\mathcal{N}) = \mathfrak{l}'_{H_2} = \mathfrak{l} = \mathfrak{c}$ holds under Martin's axiom. $\square$

4 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. The idea of the proof is the same as the one in Kunen's original proof [7], which is known as the "isomorphism of names" argument. The same argument is also found in [4].

For an infinite set I, let $\mathbb{C}(I) = \mathrm{Fn}(I, 2, \aleph_0)$, the canonical Cohen forcing notion for the index set I. As described in [8, Chapter 7], for any $\mathbb{C}(I)$-name $\dot{r}$ for a subset of ω, we can find a countable subset J of I and a *nice* $\mathbb{C}(J)$-*name* $\dot{s}$ for a subset of ω such that $\Vdash_{\mathbb{C}(I)} \dot{s} = \dot{r}$. For a countable set I, there are only $\mathfrak{c}$ nice $\mathbb{C}(I)$-names for subsets of ω.

Proof of Theorem 2.1. Suppose that $\kappa \geq \aleph_2$. Let $\mathcal{X}$ be a Polish space, $\dot{A}$ a $\mathbb{C}(\kappa)$-name for a Borel subset of $\mathcal{X} \times \mathcal{X}$, and $\langle \dot{r}_\alpha : \alpha < \omega_2 \rangle$ a sequence of $\mathbb{C}(\kappa)$-names for elements of $\mathcal{X}$.

We will prove the following statement:

$$\Vdash_{\mathbb{C}(\kappa)} \exists \alpha < \omega_2\, \exists \beta < \omega_2\, (\alpha < \beta \wedge (\langle \dot{r}_\alpha, \dot{r}_\beta \rangle \notin \dot{A} \vee \langle \dot{r}_\beta, \dot{r}_\alpha \rangle \in \dot{A})).$$

There is nothing to do if it holds that

$$\Vdash_{\mathbb{C}(\kappa)} \exists \alpha < \omega_2\, \exists \beta < \omega_2\, (\alpha < \beta \wedge \langle \dot{r}_\alpha, \dot{r}_\beta \rangle \notin \dot{A}).$$

So we assume that it fails, and fix any $p \in \mathbb{C}(\kappa)$ which satisfies

$$p \Vdash_{\mathbb{C}(\kappa)} \forall \alpha < \omega_2\, \forall \beta < \omega_2\, (\alpha < \beta \rightarrow \langle \dot{r}_\alpha, \dot{r}_\beta \rangle \in \dot{A}). \tag{$*$}$$

We will find $\alpha, \beta < \omega_2$ such that $\alpha < \beta$ and $p \Vdash_{\mathbb{C}(\kappa)} \langle \dot{r}_\beta, \dot{r}_\alpha \rangle \in \dot{A}$, which concludes the proof.

Let $J_p = \mathrm{dom}(p)$. Find a set $J_A \in [\kappa]^{\aleph_0}$ and a nice $\mathbb{C}(J_A)$-name $\dot{C}_A$ for a subset of ω such that

$$\Vdash_{\mathbb{C}(\kappa)} \text{“}\dot{C}_A \text{ is a Borel code of } \dot{A}.\text{”}$$

For each $\alpha < \omega_2$, find a set $J_\alpha \in [\kappa]^{\aleph_0}$ and a nice $\mathbb{C}(J_\alpha)$-name $\dot{C}_\alpha$ for a subset of ω such that

$$\Vdash_{\mathbb{C}(\kappa)} \text{“}\dot{C}_\alpha \text{ is a Borel code of } \{\dot{r}_\alpha\}.\text{”}$$

Using the Δ-system lemma [8, II Theorem 1.6], take $S \in [\kappa]^{\aleph_0}$ and $K \in [\omega_2]^{\aleph_2}$ so that $J_p \cup J_A \cup (J_\alpha \cap J_\beta) \subseteq S$ for any $\alpha, \beta \in K$ with $\alpha \neq \beta$. Without loss of generality we may assume that $|J_\alpha \smallsetminus S| = \aleph_0$ for all $\alpha \in K$. For each $\alpha \in K$, enumerate $J_\alpha \smallsetminus S$ as $\langle \delta_n^\alpha : n < \omega \rangle$.

For $\alpha, \beta \in K$, and let $\sigma_{\alpha,\beta}$ be the involution (automorphism of order 2) of $\mathbb{C}(\kappa)$ obtained by the permutation of coordinates which interchanges δ_n^α with δ_n^β for each n. $\sigma_{\alpha,\beta}$ naturally induces an involution of the class of all $\mathbb{C}(\kappa)$-names: We simply denote it by $\sigma_{\alpha,\beta}$. Since $J_p \cup J_A \subseteq S$, for all $\alpha, \beta \in K$ we have $\sigma_{\alpha,\beta}(p) = p$, $\sigma_{\alpha,\beta}(\dot{C}_A) = \dot{C}_A$ and $\Vdash_{\mathbb{C}(\kappa)} \sigma_{\alpha,\beta}(\dot{A}) = \dot{A}$.

Since $|K| = \aleph_2$ and there are only $\mathfrak{c} = \aleph_1$ nice names for subsets of ω over a countable index set, we can find $\alpha, \beta \in K$ with $\alpha < \beta$ such that $\sigma_{\alpha,\beta}(\dot{C}_\alpha) = \dot{C}_\beta$. Then $\sigma_{\alpha,\beta}(\dot{C}_\beta) = \dot{C}_\alpha$ and

$$\Vdash_{\mathbb{C}(\kappa)} \text{“}\sigma_{\alpha,\beta}(\dot{r}_\alpha) = \dot{r}_\beta \text{ and } \sigma_{\alpha,\beta}(\dot{r}_\beta) = \dot{r}_\alpha.\text{”}$$

By $(*)$, we have $p \Vdash_{\mathbb{C}(\kappa)} \langle \dot{r}_\alpha, \dot{r}_\beta \rangle \in \dot{A}$. Since $\sigma_{\alpha,\beta}$ is an automorphism of $\mathbb{C}(\kappa)$, we have

$$\sigma_{\alpha,\beta}(p) \Vdash_{\mathbb{C}(\kappa)} \langle \sigma_{\alpha,\beta}(\dot{r}_\alpha), \sigma_{\alpha,\beta}(\dot{r}_\beta) \rangle \in \sigma_{\alpha,\beta}(\dot{A})$$

and hence $p \Vdash_{\mathbb{C}(\kappa)} \langle \dot{r}_\beta, \dot{r}_\alpha \rangle \in \dot{A}$. □

Remark 1. Fuchino pointed out that Theorem 2.1 is generalized in the following two ways [3]: (1) The set A is not necessarily Borel, but is “definable” by some formula. (2) We can prove a similar result for a forcing extension by a side-by-side product of the same forcing notions, each generically adds a real in a natural way. The argument in the above proof also works in those generalized settings.

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