Fractal Dimensions of Coupled Chaotic Systems

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The correlation dimension, which is one of the fractal dimensions, in coupled chaotic systems cannot be calculated with Takens' phase space in some situations. In this paper we investigate the correlation and the Lyapunov dimensions of coupled Rössler systems and coupled Rössler-Lorenz systems in various situations, and clear that the situations depend on the degree of unsymmetry and the coupling strength in coupled chaotic systems.

1. Introduction

Chaotic phenomena had been studied with academic interest since the beginning of 1980. Most studies were achieved to understand roots to chaos or characterize chaotic attractors (e.g., fractal dimensions, Lyapunow exponents, entropy, and so on) in the only one nonlinear system[1-8].

Chaotic motions in nature, however, are influenced by other chaotic systems. Recently, Kaneko proposed a chaotic system which consists of sub-chaotic systems coupled each other for the purpose of understanding turbulence in fluid[9]. The chaotic system is called *Caupled Map Lattice* (CML), and now it is used to investigate the hyper chaos. Very recently Pecora and Carroll reported on the synchronization of a coupled chaotic system. The system consisted of the two subsystems. The behavior of the second subsystem was dependent on the first, but the first was not dependent on the second[10-13]. Endo and Chua suggested that the synchronization of the chaotic phase-locked loops was able to be used for secure communications[14,15].

However, the measure of chaotic attractors in coupled chaotic systems has not been established. Lorenz has shown that if we calculate the correlation dimension with the Taken's phase space, we may get erroneous results[16]. He proposed a set of seven weak coupled identical systems, and estimated the correlation dimension from one time series data. Moreover he calculated the Lyapunov dimension (i.e., the Kaplan-Yorke dimension) using the equations of the whole system. The correlation dimension had to be close to the Lyapunov dimension, but they were significant difference. The correlation dimension estimated from combined several time series data was close to the

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Lyapunov dimension. He argued that this difference was due to the weak coupling among the subsystems. On the contrary, Stewart pointed out that this difference was due to the symmetry in the system[17]. Landa and Rosenblum argued that Stewart's opinion is not true since the set proposed by Lorenz is not completely symmetric system[18]. They calculated the difference on weak coupled two identical systems and coupled two different systems, and showed that the use of combined several time series data is effective in coupled different systems, but not effective in weak coupled identical systems.

We think that these erroneous results are due to not only the weak coupling but also the degree of the unsymmetry in the whole system. In this paper we discuss the mechanism of these erroneous results with the coupling strength and the degree of the unsymmetry.

2. Coupled Chaotic Systems

The coupled Rössler systems are shown in Fig.1. The equations can be described as follows:

$$\begin{aligned}
\dot{x}_1 &= -d_1 \cdot z_1 - y_1 + k \cdot x_2 \\
\dot{y}_1 &= x_1 + a \cdot y_1 \\
\dot{z}_1 &= b + z_1 \cdot (x_1 - c) \\
\dot{x}_2 &= -d_2 \cdot z_2 - y_2 + k \cdot x_1 \\
\dot{y}_2 &= x_2 + a \cdot y_2 \\
\dot{z}_2 &= b + z_2 \cdot (x_2 - c)
\end{aligned} \tag{1}$$

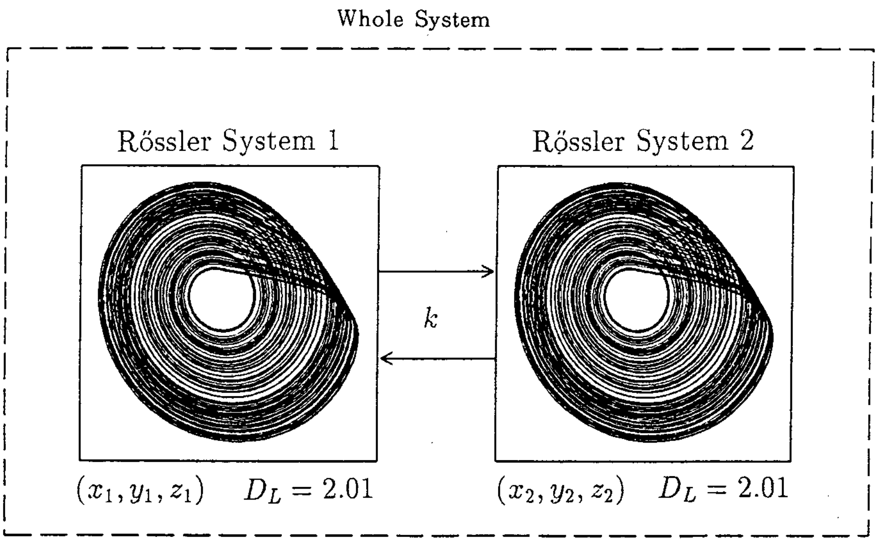

Fig. 1 Coupled Rössler systems.

where a, b, c are fixed as $a=0.15$, $b=0.2$, $c=10.0$. k is the coupling coefficient and d_1, d_2 are parameters for each subsystem. If $d_1=d_2$, the first subsystem (Rössler system 1) is equal to the second (Rössler system 2) and the whole system is symmetry. When d_1 is not equal to d_2, the whole system is unsymmetry. We think that one of the causes of the erroneous results is due to the degree of the unsymmetry in the whole system. Unsymmetry parameter α, which is equivalent to the defference between the first and second subsystems, is introduced as follows.

$$\alpha=\left|\frac{d_1-d_2}{d_1+d_2}\right| \tag{2}$$

We think that the erroneous results must be influenced by the synchronization effect in coupled chaotic systems. To quantify the synchronization effect, we introduce the value *dis* which is obtained by averaging $|x_1(t)-x_2(t)|$ for a long time.

$$dis=\lim_{N_d\to\infty}\frac{1}{N_d}\sum_{i=1}^{N_d}|x_1(i\Delta t)-x_2(i\Delta t)| \tag{3}$$

where $x_j(i\Delta t)$ is the variable of subsystem j at time $i\times\Delta t$ (Δt:time step). If *dis* is close to 0, the coupled chaotic systems synchronize (we use $N_d=100$). In order to make clear the causes of the erroneous results, we investigate the relations among the degree of the unsymmetry, coupling strength of the whole system and the erroneous results.

Furthermore, we examine coupled Rössler-Lorenz systems shown in Fig.2,

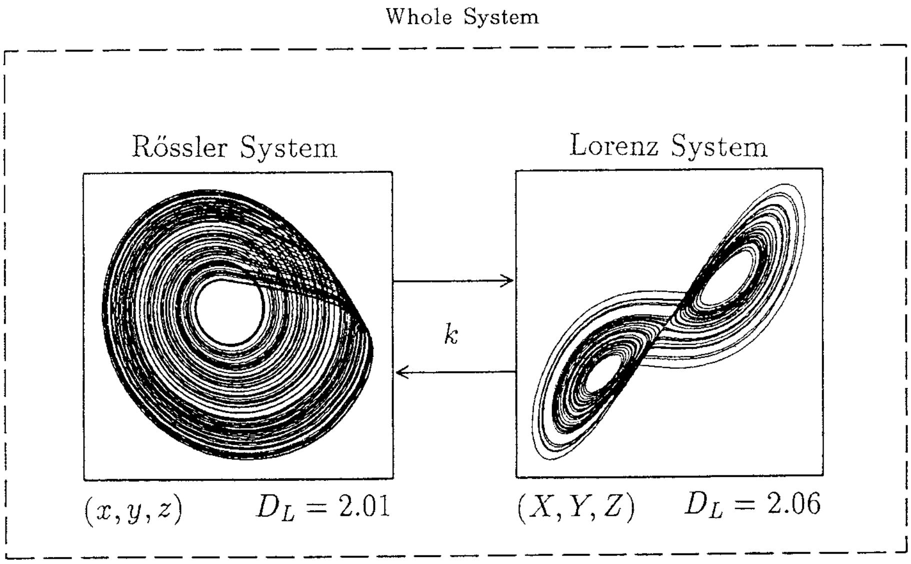

Fig. 2 Coupled Rössler-Lorenz systems.

$$\begin{aligned}
&\dot{x}=-d\cdot z-y+k\cdot X\\
&\dot{y}=x+a\cdot y\\
&\dot{z}=b+z\cdot(x-c)\\
&\dot{X}=\{-e\cdot(X-Y)\}/n+k\cdot x\\
&\dot{Y}=\{f\cdot X-Y-X\cdot Y\}/n\\
&\dot{Z}=\{X\cdot Y-g\cdot Z\}/n
\end{aligned} \tag{4}$$

where (a, b, c, d, e, f, g) are fixed as (0.15, 0.2, 10.0, 1.0, 10.0, 28.0, 8/3), and k is the coupling coefficient. In order to make the Rössler and Lorenz systems have almost the same oscillations, the parameter n is introduced (n=10.0).

In this paper the fractal dimensions (the correlation and the Lyapunov dimensions) in the coupled Rössler systems and the coupled Rössler-Lorenz systems are examined to clear the mechanism of the erroneous results.

3. Fractal Dimensions

This section deals with the two procedures for estimation of fractal dimensions. One of them is the Lyapunov dimension and another is the correlation dimension with Takens' phase space.

The Lyapunov exponents (λ_1, λ_2, $\cdots$, λ_n) are obtained using the Shimada and Nagashima method[19]. The Lyapunov dimension D_L can be calculated with several exponents as follows[3,4].

$$D_L=j+\frac{\sum_{i=1}^{j}\lambda_i}{|\lambda_{j+1}|} \tag{5}$$

where following conditions are satisfied.

$$\sum_{i=1}^{j}\lambda_i\geq 0,\ \sum_{i=1}^{j+1}\lambda i<0 \tag{6}$$

The Lyapunov dimension is the only measure to accurately quantify high dimensional chaos.

The correlation dimension with the Takens' phase space proposed by Grassberger and Procaccia is very powerful tool to quantify chaotic motion in the unknown system[20,21]. We consider a time series $x(t)$ sampled at times $(t+i\Delta t)$, i integer, and build a vectorial temporal sequence $\boldsymbol{X}(i\Delta t)=\boldsymbol{X}(i)\in\mathbf{R}^P$. The components of vector $\boldsymbol{X}(i)$ are ($x(i)$, $x(i-\tau)$, $\cdots$, $x(i-(p-1)\tau)$), in which a discrete time i is meant for a time $i\Delta t$. i and τ are the discrete time and the discrete time delay, respectively. We define a local correlation $C_i(\varepsilon)$ at point $\boldsymbol{X}(i)$ by the relation

$$C_i(\varepsilon)=\frac{1}{N}\sum_{j=1}^{N}H(\varepsilon-|\boldsymbol{X}(i)-\boldsymbol{X}(j)|) \tag{7}$$

in which N is the size of the temporal sequence. Spatially averaging local correlation moments, m central vectors are chosen at random.

$$C(\varepsilon)=\frac{1}{m}\sum_{i=1}^{m}C_i(\varepsilon) \tag{8}$$

The correlation dimension D_{cor} can be estimated by calculating the slope.

$$D_{cor}=\lim_{\varepsilon\to 0}\frac{\log C(\varepsilon)}{\log\varepsilon} \tag{9}$$

In practice, size N of the temporal sequence and number m of central vectors are finite preventing the reaching of the limit $\varepsilon\to 0$ in Eq.(9). This scaling relation is only observed in a finite domain(ε_{min}, ε_{max}) as shown in Fig.3. In theory, correlation dimension D_{cor} must be close to the Lyapunov dimension D_L in all situations. In the numerical experiment in some situations (coupled chaotic systems), however, this equivalent relation is not satisfied[16-18].

$$D_{cor}\ll D_L \tag{10}$$

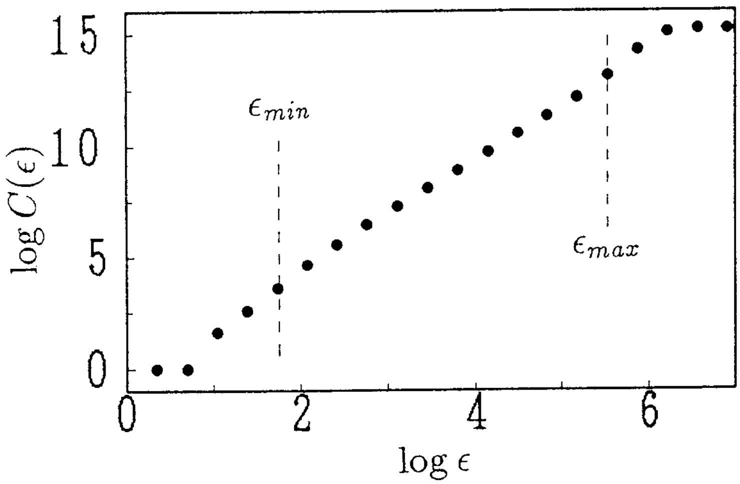

Fig. 3 An example of $\log\varepsilon-\log C(\varepsilon)$ plot (coupled Rössler systems $k=0.2$, $\alpha=0.1$).

4. Numerical Results

The fourth-order Runge-Kutta procedure with a time step 0.05 units is used in all numerical integration. For the correlation dimension with the Taken's phase space, a step of time series data Δt, a time lag τ, the size of the time series data N, central vectors m and the embedding dimension p are chosen $\Delta t=20\times 0.05$, $\tau=1$, $N=20000$, $m=200$, $p=10$, respectively. A finite domain (ε_{min}, ε_{max}) in which the scaling relation can be observed is determined with $\log\varepsilon - \log C(\varepsilon)$ plots (Fig.3).

We investigate the relations between coupling strength and the erroneous results for each of values of the unsymmetry parameter α.

Table 1. The measures for coupled Rossler systems ($\alpha=0$)

k	dis	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	D_L	$D_{cor}(x_1)$	$D_{cor}(x_1+y_2)$
0.05	3.09	+, +, 0, −	4.01	2.36	2.55
0.10	5.61	+, +, 0, −	4.01	2.80	3.18
0.15	0.07	+, +, 0, −	4.00	2.34	2.63
0.20	0.00	+, +, 0, −	3.86	2.04	2.10
0.25	0.00	+, 0, −, −	3.66	2.00	2.03

(1) $\alpha=0.0(d_1=d_2=1.0)$: Values of the synchronization effect dis, four largest Lyapunow exponents $\lambda_1 \sim \lambda_4$, the Lyapunov dimension D_L, the correlation dimension from one time series data and combined two time series data $D_{cor}(x_1)$, $D_{cor}(x_1+y_2)$ for coupled two identical Rossler systems are shown in Table 1. For weak coupling coefficient ($k=0.05\sim0.15$), two identical systems aren't synchronized, as values of the synchronization effect aren't close to 0. Since two identical systems behave independently, the whole system has two stretching derections and the Lyapunov dimension ($D_L \simeq 4$) which is equal to the sum of two dimensions of sub-Rössler systems ($D_L \simeq 2$). On the contrary, for $k=0.20$, 0.25, two identical systems are synchronized. The correlation dimension from one time series $D_{cor}(x_1)$ is much smaller than the Lyapunov dimension D_L of the whole system. This result agrees with results of Lorenz, Landa and Rosenblum. The correlation dimensions from combined two time series data $D_{cor}(x_1+y_2)$, however, are slightly greater than $D_{cor}(x_1)$ for $k=0.05\sim0.15$. We find out that in the unsynchronization regime ($k=0.05\sim0.15$) the use of combined time series data is effective in coupled two identical systems, but this result doesn't agree with Landa and Rosenblum's result. Our result can be explained as follows: If we calculate the correlation dimension with the embedding dimension $p=4\sim6$, the same result of Landa and Rosenblum are estimated. However, we can get the different result with sufficiently large embedding dimension $p=10$.

(2) $\alpha=0.1(d_1=1.1,\ d_2=0.9)$, $\alpha=0.5(d_1=1.5,\ d_2=0.5)$: Systems 1, 2 aren't synchronized for all coupling coefficient $k=0.05\sim0.25$ (Table 2, 3). It is cleared that $D_{cor}(x_1)$ approaches to D_L as unsymmetry parameter α increases (Table 1, 2, 3). Of course, $D_{cor}(x_1+y_2)$ approaches to D_L as α increases, but it isn't equal to D_L for even if

$\alpha=0.5$.

We investigate the fractal dimensions of coupled Rössler-Lorenz systems, and deal with the systems as coupled entirely unsymmetric chaotic systems, since structures of them are quite different. Table 4 indicates the Lyapunow exponents, Lyapunow dimension and the correlation dimensions for each coupling coefficient k. These systems aren't synchronized, as structures of them are quite different. For strong coupling $k=0.2\sim0.4$, $D_{cor}(x)$ is almost equal to D_L. $D_{cor}(x)$ is much smaller than D_L for weak coupling $k=0.01\sim0.1$, but $D_{cor}(x+Y)$ is almost equal to D_L. We can say that using combined time series data $(x+Y)$ is very effective for weak coupled entirely unsymmetric chaotic systems. However, for strong coupled entirely unsymmetric chaotic systems, the combining is not need.

Table 2. The measures for coupled Rössler systems ($\alpha=0.1$)

k	dis	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	D_L	$D_{cor}(x_1)$	$D_{cor}(x_1+y_2)$
0.05	2.37	+, +, 0, −	4.01	2.39	2.64
0.10	5.01	+, +, 0, −	4.01	2.86	3.30
0.15	5.10	+, +, 0, −	4.00	2.45	2.83
0.20	0.23	+, +, 0, −	3.94	2.43	2.63
0.25	0.26	+, 0, −, −	3.66	2.04	2.15

Table 3. The measures for coupled Rössler systems ($\alpha=0.5$)

k	dis	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	D_L	$D_{cor}(x_1)$	$D_{cor}(x_1+y_2)$
0.05	4.10	+, +, 0, −	4.01	2.51	2.82
0.10	7.28	+, +, 0, −	4.01	2.91	3.24
0.15	4.77	+, +, 0, −	4.00	2.72	3.14
0.20	0.98	+, +, 0, −	4.00	2.70	3.07
0.25	1.42	+, 0, −, −	3.76	2.34	2.62

Table 4. The measures for coupled Rössler-Lorenz systems

k	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	D_L	$D_{cor}(x_1)$	$D_{cor}(x+Y)$
0.01	+, +, 0, −	4.12	2.03	3.91
0.05	+, +, 0, −	4.12	3.18	4.19
0.10	+, +, 0, −	4.01	3.17	3.22
0.20	+, +, 0, −	4.01	3.64	3.19
0.30	+, +, 0, −	4.00	3.81	3.05
0.40	+, +, 0, −	4.00	3.91	3.20

5. Conclusions

In this paper the fractal dimensions of coupled chaotic systems which have the same structures (coupled Rössler systems) and coupled different chaotic systems (coupled Rössler-Lorenz systems) are investigated, and following characteristics are cleared:

(1) Contrary to Landa and Rosenblum's result, for coupled identical chaotic systems (coupled Rössler systems $\alpha=0$), using combined time series data is slightly effective with large embedding dimension p at which the synchronization doesn't occur.

(2) In the case of coupled chaotic systems which have same structures (coupled Rössler systems $\alpha\neq 0$), the correlation dimension from one time series data approaches to the Lyapunov dimension as the degree of unsymmetry increases. The effect of combined time series data is a little when the degree of unsymmetry is even large.

(3) In the case of coupled different chaotic systems (coupled Rössler-Lorenz systems), the use of combined time series data is very effective for weak coupling, but it isn't need for strong coupling case.

Reflections on some of above characteristics will make clear the reason why the dimension of whole system can't be estimated from one time series data. We think that there are two cases. In the case of coupled chaotic systems which have the same structures, if the degree of unsymmetry of the whole system is small, the dimension of whole system can't be estimated from one time series data with no reference to the strength of coupling. In the case of coupled different chaotic systems, if the strength of coupling is weak, the dimension of whole system can't be estimated from one time series data.

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References

1) J. M. T. Thomspon and H. B. Stewart, Nonlinear Dynamics and Chaos, Springer-Verlag, 1989.
2) J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields, Springer-Verlag, 1983.
3) P. Berge, Y. Pomeau and C. Vidal, Order within Chaos Towards a Deterministic Approach to Turbulence, Wiley and Sons, New York, 1984.

4) T. S. Parker and L. O. Chua, Practical Numerical Algorithm for Chaotic Systems, Springer-Verlag, 1989.
5) M. J. Feigenbaum, J. Stat. Phys., 21, pp.669-706, 1979.
6) J. Stavans, F. Heslot and A. Libchaber, Phys. Rev. Lett., 55, pp.596-599, 1985.
7) M. H. Jensen, L. P. Kadanoff, A. Libchaber, I. Procaccia and J. Starams, Phys. Rev. Lett., 55, pp.2798-2801, 1985.
8) T. C. Halsy, M. H. Jensen, L. P. Kandannof, I. Procaccia and B. I. Shiraima, Phys. Rev., A33, pp. 1141-1151, 1986.
9) K. Kaneko, Chaos, vol.2, No.3, 1992.
10) L. M. Pecora and T. L. Carroll, Phys. Rev. Lett., 64, pp.821-824, 1990.
11) J. M. Kowalski, G. L. Albert and G. W. Gross, Phys. Rev., A42, pp.6260-6263, 1990.
12) L. M. Pecora and T. L. Carroll, Phys. Rev., A44, pp.2374-2383, 1991.
13) T. L. Carroll and L. M. Pecora, IEEE Trans. Circuits Syst., vol.CAS-38, pp.453-456, 1991.
14) T. Endo and L. O. Chua, Int. J. Bifurcation and Chaos, vol.1, pp.701-710, 1991.
15) T. Endo and L. O. Chua, Tech. Rep. M91/59, Electronics Research Laboratory University of California Berkeley, May 1, 1991.
16) E. N. Lorenz, Nature 353, pp.241-244, 1991.
17) L. Stewart, Nature 354, pp.113, 1991.
18) P. S. Landa and M. G. Rosenblum, Chaos, Soliton and Fractals, vol.2, pp.251-258, 1992.
19) L. Shimada and T. Nagashima, Prog. Theor. Phys., 61, pp.1605-1616, 1979.
20) P. Grassberger and L. Procaccia, Phys. Rev. Lett., 50, pp.346-349, 1983.
21) P. Grassberger and L. Procaccia, Physica 9D, pp.189-208, 1983.