A Method of Yield Estimation for Analog Integrated Circuits

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This paper presents a new method of yield estimation, which is a very useful criterion for a design of electrical circuits, particularry of semiconductor integrated circuits. It is assumed that the output responses of the circuit are statistically varying with a probability density function.

The region determined by the upper and lower limits of specifications is divided into equal cells. The yield of the circuit is given by the sum total of integration value of the probability density function upon each cell.

Finally the efficiency of the presented algorithm will be demonstrated on a switched capacitor filter.

1. Introduction

For a design of semiconductor integrated circuits, the statistical variations of the circuit parameter values must be considered. The manufacturing yield – which is the proportion of manufactured circuits fulfilling the desired performance specifications – is a very useful design criterion.

During the last years, several proposals for manufacturing yield have been published[1]. Monte Carlo methods may be used to simulate parameter variation in order to estimate the yield[2], but can be rather expensive in terms of computing time for a large number of circuit parameters.

There are also other approaches applicable for practical examples[3,4]. In 3), the method of yield estimation by using the Fourier transform of the probability distribution function of circuit responses is derived. In 4), the tolerance region of possible outcomes is discretized into a set of orthotopic cells, then the yield is defined by the ratio of the number of outcomes which satisfy the specifications to the total number of outcomes. These methods are difficult to deal with a large number of circuit parameters. Furthermore, the accuracy is not satisfactory.

This paper proposes a new approach to yield estimation. It is assumed that the N circuit parameters can be varied according to the joint probability density function (*pdf*) of normal distribution. Suppose that we obtain the Taylor expansion for the circuit responses about the points of the mean values of circuit parameters. Neglecting the second and higher order terms of Taylor expansion, the circuit responses are considered to be M normal random variables. By linear mapping we transform the normal

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random variables into the standardised normal random variables.

After dividing the space determined by the upper and lower specification limits into a set of equal cells, we obtain the yield by the sum of integration of *pdf* over each cell. The factor determines the computing time of this method is the division numbers of specification's space. This is the function of the number of circuit responses M and the width between the lower and upper limits of specifications. Since the computing time increases according to the increasing of M, as one of the counter-measures, we propose a method improving the efficiency of the yield estimation by reducing the integration region.

There is no effect of the numbers of circuit parameters.

This method is useful for yield estimation of amplifiers or the like when the frequency domain specifications are given at frequencies of the order of 5.

2. Definition of Manufacturing Yield

Consider a circuit in which there are N variable components $\boldsymbol{P} = [p_1, p_2, \cdots, p_N]$. A set of circuit responses $\boldsymbol{u}$ is expressed as

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{p}, \boldsymbol{q}) \tag{1}$$

where $\boldsymbol{q} = [q_1, q_2, \cdots, q_M]$ is a vector of physical quantities as temperature or frequency etc..

Let

$$u_i = u(p, q) \mid_{q=q_i} \tag{2}$$

then

$$\boldsymbol{u} = [u_1, u_2, \cdots, u_M]^T \tag{3}$$

The circuit responses $\boldsymbol{u}$ are statistically varying with a *pdf* of $f_u(\boldsymbol{u})$. Thus the manufacturing yield Y of a circuit can be formulated by

$$Y = \int_{\Omega_u} f_u(\boldsymbol{u}) d\boldsymbol{u} \tag{4}$$

where Ω_u is the region of acceptable performance specifications. The region Ω_u can be descriebd as

$$\Omega_u = \left\{ \boldsymbol{u} \,\middle|\, u_{1L} \leqslant u_1 \leqslant u_{1M}, \cdots, u_{ML} \leqslant u_M \leqslant u_{MH} \right\} \tag{5}$$

where u_{iH} and u_{iL} ($i = 1, 2, \cdots, M$) are the upper and lower specification limits on the ith of the circuit responses of interest.

In this paper, we assume that $\boldsymbol{q}$ indicates the angular frequency ω and we call the point $\omega = \omega_i$ the "frequency point".

3. Yield Estimation

We now assume multinormal distributions for the *pdf* $f_p(\boldsymbol{p})$ of $\boldsymbol{p}$. With regard to practical requirements as well as to an especially clear illustration of the new approach this assumption is very useful. Under this assumption we obtain

$$f_p(p) = \frac{|C_p^{-1}|^{1/2}}{(2\pi)^{N/2}} \exp\left[-\frac{1}{2}(p-\mu_p)^T C_p^{-1}(p-\mu_p)\right] \tag{6}$$

where μ_p is the mean value vector of p and C_p is the covariance matrix.

The circuit response u is considered to be a function of only p when q_i is fixed. Using a Taylor series expansion we get

$$u(p) = u(\mu_p) + F_p(p-\mu_p) \tag{7}$$

thereby, the second and higher order terms are neglected. The sensitivity matrix F_p is shown as

$$F_p = \begin{bmatrix} \frac{\partial u_1}{\partial p_1} & \cdots & \frac{\partial u_1}{\partial p_N} \\ \vdots & & \vdots \\ \frac{\partial u_M}{\partial p_1} & \cdots & \frac{\partial u_M}{\partial p_N} \end{bmatrix} \tag{8}$$

By the linear transformation of Eq. (7), u obeys a M-dimensional normal distribution, too.

The mean value vector and the covariance matrix of u can be expressed as μ_u and C_u respectively. Then the *pdf* $f_u(u)$ of u is as follows.

$$f_u(u) = \frac{|C_u^{-1}|^{1/2}}{(2\pi)^{M/2}} \exp\left(-\frac{1}{2}\Delta u^T C_u^{-1}\Delta u\right) \tag{9}$$

where

$$\Delta u = u - u(\mu_p) \tag{10}$$

The $M \times M$ matrix C_u is obtained from the next equation[5].

$$C_u = F_p C_p F_p^T \tag{11}$$

From the following relation

$$C_u = D^T D \tag{12}$$

we obtain the upper triangular matrix D. Define a new variable z by

$$z = (D^T)^{-1}\left\{u - u(\mu_p)\right\} \tag{13}$$

The resulting vector z has the following *pdf* $f_z(z)$ of the standardised normal distribution[6].

$$f_z(z) = \frac{1}{(2\pi)^{M/2}} \exp\left(-\frac{1}{2}\, |z|^2\right) \tag{14}$$

Simultaneously, the region Ω_u is converted to Ω_z using the mapping of Eq. (12). Hence, Eq. (4) can be written in the form

$$Y = \int_{\Omega_z} f_z(z)\, dz \tag{15}$$

The region of acceptable performance specifications Ω_u is a M-dimensional rectangular prism, thus an arbitrary inner point of Ω_u is expressed as follows

$$\begin{bmatrix} u_{1L} \\ \vdots \\ u_{ML} \end{bmatrix} + \alpha_1 \begin{bmatrix} u_{1H} - u_{1L} \\ \\ 0 \\ \end{bmatrix} + \cdots + \alpha_M \begin{bmatrix} \\ 0 \\ u_{MH} - u_{ML} \end{bmatrix}$$
$$(0 \leqslant \alpha_1, \cdots, \alpha_M \leqslant 1)$$

The inner point of Ω_z corresponding to that of Ω_u is expressed as

$$\xi_0 + \alpha_1 \xi_1 + \cdots + \alpha_M \xi_M$$

where

$$\xi_0 = (D^T)^{-1} \left\{ \begin{bmatrix} u_{1L} \\ \vdots \\ u_{ML} \end{bmatrix} - u(\mu_p) \right\} \tag{16}$$

$$\xi_i = (D^T)^{-1} \left\{ \begin{bmatrix} 0 \\ u_{iH} - u_{iL} \\ 0 \end{bmatrix} \right\} \tag{17}$$

The region Ω_z is a M-dimensional parallel polyhedron whose edges are $\xi_1, \xi_2, \cdots, \xi_M$. Figure 1 (a) shows the relation between Ω_u and $u(\mu_p)$ for the case of $M = 2$. Figure 1 (b) shows the region of Ω_z for $M = 2$.

By dividing ξ_i ($i = 1, 2, \cdots, M$) into m_i equal length portions, the region Ω_z is partitioned into $m_1 \times m_2 \times \cdots \times m_M$ cells, such as Ω_z (1, 1, ⋯, 1), Ω_z (1, 1, ⋯, 2), ⋯, Ω_z $(m_1, m_2, \cdots, m_M)$.

The yield Y can now be expressed as

$$Y = \sum_{d_1=1}^{m_1} \cdots \sum_{d_M=1}^{m_M} \int_{\Omega_z(d_1, d_2, \cdots, d_M)} f_z(z)\, dz \tag{18}$$

If m_i is sufficiently large, the value of $f_z(z)$ over each cell is considered to be constant. Therefore, the integration of Eq. (18) can be obtained by applying Newton-Cotes rule.

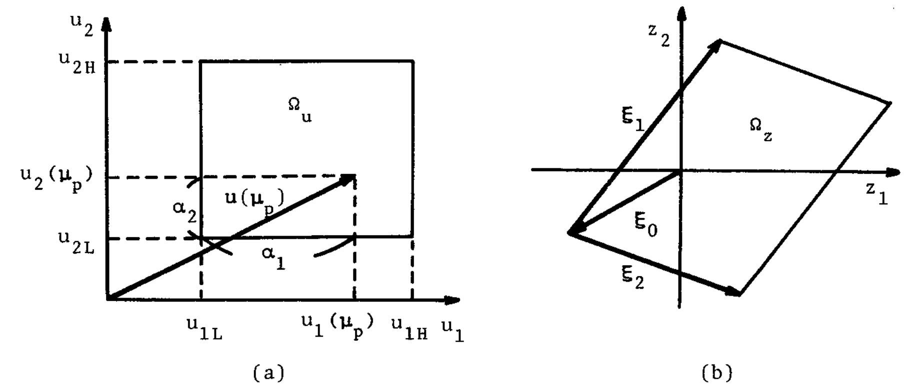

Fig. 1 (a) The region of Ω_u and $u(\mu_p)$ for the case of $M = 2$.
(b) The region of Ω_z and ξ_i ($i = 0, 1, 2$) for the case of $M = 2$.

The volume V of Ω_z is given by

$$V = \left\| \boldsymbol{\xi}_1\ \boldsymbol{\xi}_2 \cdots \boldsymbol{\xi}_M \right\| = \left\| \begin{matrix} \xi_{11} & \cdots & \xi_{1M} \\ \vdots & & \vdots \\ \xi_{M1} & \cdots & \xi_{MM} \end{matrix} \right\| \tag{19}$$

where $\|\cdot\|$ denotes the absolute of determinant. From Eq. (19), the volume ΔV of $\Omega_z\ (d_1, d_2, \cdots, d_M)$ is given as

$$\Delta V = \frac{1}{\prod_{i=1}^{M} m_i} \left\| \begin{matrix} \xi_{11} & \cdots & \xi_{1M} \\ \vdots & & \vdots \\ \xi_{M1} & \cdots & \xi_{MM} \end{matrix} \right\| \tag{20}$$

The mean value of $f_z(z)$ over $\Omega_z(d_1, d_2, \cdots, d_M)$ is obtained by Newton-Cotes rule as

$$f_z(z)_{\Delta V} = \frac{1}{2^M} \left\{ \sum_{k_1=d_1}^{d_1+1} \cdots \sum_{k_M=d_M}^{d_M+1} f_z\left(\boldsymbol{\xi}_0 + \frac{k_1}{m_1} \boldsymbol{\xi}_1 + \cdots + \frac{k_M}{m_M} \boldsymbol{\xi}_M \right) \right\} \tag{21}$$

Substituting Eq. (14) into Eq. (21), we obtain

$$f_z(z)_{\Delta V} = \frac{1}{2^M} \left[\sum_{k_1=d_1}^{d_1+1} \cdots \sum_{k_M=d_M}^{d_M+1} \frac{1}{(2\pi)^{M/2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{M} \left(\xi_{j0} + \frac{k_1}{m_1} \xi_{j1} + \cdots + \frac{k_M}{m_M} \xi_{jM} \right)^2 \right\} \right]$$

$$= \frac{1}{2^{3M/2}\pi^{M/2}} \left[\sum_{k_1=d_1}^{d_1+1} \cdots \sum_{k_M=d_M}^{d_M+1} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{M} \left(\xi_{j0} + \sum_{i=1}^{M} \frac{k_i}{m_i} \xi_{ji} \right)^2 \right\} \right] \tag{22}$$

After all, the yield Y defined by Eq. (18) can be expressed as

$$Y = \frac{\Delta V}{2^{3M/2}\pi^{M/2}} \sum_{d_1=1}^{m_1} \cdots \sum_{d_M=1}^{m_M} \left[\sum_{k_1=d_1}^{d_1+1} \cdots \sum_{k_M=d_M}^{d_M+1} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{M} \left(\xi_{j0} + \sum_{i=1}^{M} \frac{k_i}{m_i} \xi_{ji} \right)^2 \right\} \right] \tag{23}$$

From Eq. (23), it is evident that the manufacturing yield Y can be obtained as the sum of a finite series.

4. Discussion for the Number of Divisions m_i

At first we consider the following simple integration

$$\Phi = \int_{z_L}^{z_H} f_z(z)\, dz \tag{24}$$

where $f_z(z)$ is the *pdf* of one dimensional standardised normal distribution. Dividing the interval $[z_L, z_H]$ into m equal-length portions, Eq. (24) can be calculated by applying the Newton-Cotes rule. The truncation error R_n of Newton-Cotes rule is given by[7)]

$$R_n = \frac{f_z(\gamma)^{(n+1)}}{(n+1)!} \cdot h^{n+2} \int_0^n \prod_{k=0}^{n} (x-k)\, dx \tag{25}$$

$$n; odd$$

where γ is an arbitrary point during $[z_L, z_H]$ and

$$h = (z_H - z_L)/m \tag{26}$$

Substituting the following equation into Eq. (25)

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{27}$$

we get the truncation error R_1 for $n = 1$ as

$$|R_1| = \left| \frac{(\gamma^2-1)e^{-\gamma^2/2}}{12\sqrt{2\pi}} \right| \cdot h^3 \tag{28}$$

Since $|R_1|$ has a maximum value at $\gamma = 0$, it is sufficient that the error ϵ is defined such as

$$\epsilon = \left|R_1\right| \Big|_{\gamma=0} = \frac{1}{12\sqrt{2\pi}} \left(\frac{z_H - z_L}{m}\right)^3 \tag{29}$$

In order that the error ϵ is less than an arbitrary positive number ϵ_0, the division number m is required to satisfy the following relation

$$m \geqslant (12\sqrt{2\pi}\ \epsilon_0)^{-1/3} (z_H - z_L) \tag{30}$$

For the general case when $M > 1$, minor modifications of the above derivation are required. Consider the rectangular prism Ω_z' which contains Ω_z. Ω_z' has M edges of $\boldsymbol{\xi}_1', \boldsymbol{\xi}_2', \cdots, \boldsymbol{\xi}_M'$. Since $f_z(z) > 0$, the next relation holds.

$$\int_{\Omega_z'} f_z(z)dz > \int_{\Omega_z} f_z(z)dz \tag{31}$$

Let the left hand side of Eq. (31) be replaced by Φ',

$$\begin{aligned}\Phi' &= \int_{z_{1L}'}^{z_{1H}'} \cdots \int_{z_{ML}'}^{z_{MH}'} \prod_{i=1}^{M} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} dz_1 \cdots dz_M \\ &= \prod_{i=1}^{M} \int_{z_{iL}'}^{z_{iH}'} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} dz_i \\ &= \prod_{i=1}^{M} \Phi_i' \end{aligned} \tag{32}$$

If the error of Φ_i' is taken as ϵ_i, then the error ϵ of Φ' is

$$\epsilon = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_M \tag{33}$$

The next relation should be held to keep that ϵ is less than ϵ_0

$$\begin{aligned}\epsilon &\simeq \sum_{i=1}^{M} \frac{1}{12\sqrt{2\pi}} \left(\frac{z_{iH}' - z_{iL}'}{k_i'}\right)^3 \\ &= \sum_{i=1}^{M} \frac{1}{12\sqrt{2\pi}} \left(\frac{|\boldsymbol{\xi}_i'|}{\sum_{i=1}^{M} m_i}\right)^3 \\ &\leqslant \epsilon_0 \end{aligned} \tag{34}$$

where k_i' is the number of divisions of $\boldsymbol{\xi}_i'$.

If

$$m_i \geqslant (12\sqrt{2\pi}\ \epsilon_0)^{-1/3} M \left| \boldsymbol{\xi}_i \right| \tag{35}$$

then

$$\sum_{i=1}^{M} \frac{|\boldsymbol{\xi}_i|}{m_i} \leqslant \sum_{i=1}^{M} (12\sqrt{2\pi}\ \epsilon_0)^{1/3} M^{-1}$$

$$= (12\sqrt{2\pi}\ \epsilon_0)^{1/3} \tag{36}$$

The next relation is also true

$$\left| \boldsymbol{\xi}_i' \right| / \sum_{i=1}^{M} m_i \leqslant \left| \boldsymbol{\xi}_i \right| / m_i \tag{37}$$

From Eqs. (36) and (37), we have taken

$$\sum_{i=1}^{M} \left(|\boldsymbol{\xi}_i'| \ / \sum_{i=1}^{M} m_i \right)^3 < 12\sqrt{2\pi}\ \epsilon_0 \tag{38}$$

Hence, we can make the following statement that m_i satisfying Eq. (35) should be chosen for the number of divisions.

Figure 2 illustrates the relation among Ω_z, Ω_z', $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_i'$ for $M = 2$.

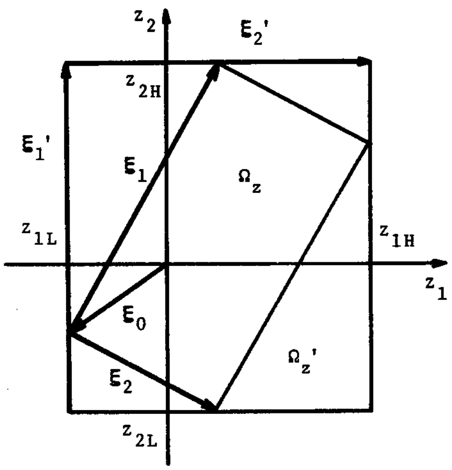

Fig. 2 Relation among Ω_z, Ω_z', ξ_i and ξ_i' (for $M = 2$).

5. Improvement of the Efficiency of Yield Estimation by Reduction of Integration Region

As shown in Chapter 3, the yield Y can be obtained as the sum of the volumes of $m_1 \times m_2 \times \cdots \times m_M$ cells. Therefore, the amount of calculations increases terribly according to the increasing of M. To avoid the difficulty we propose a method to improve the efficiency of yield estimation by reduction of integration region. The circuit response u_i is a normal random variable. Hence, the probability that u_i satisfies the next condition should be equal to 1.

$$\mu_{u_i} - 4\sigma_{u_i} \leqslant u_i \leqslant \mu_{u_i} + 4\sigma_{u_i} \tag{39}$$

In Eq. (39) μ_{u_i} is the mean value of u_i and σ_{u_i} is the standard deviation of u_i. Therefore, the given specifications can be reduced such as

$$\max\left\{u_{iL}, \mu_{u_i} - 4\sigma_{u_i}\right\} \leqslant u_i \leqslant \min\left\{u_{iH}, \mu_{u_i} + 4\sigma_{u_i}\right\} \tag{40}$$

From the consideration above, Ω_z is compressed to the interior of M dimensional sphere with radius 4, and $|\boldsymbol{\xi}_i| \leqslant 8$ holds.

Figure 3 (a) shows the relation between the region Ω_u and the region recognized the existence of $f_u(\boldsymbol{u})$ (inside of the ellipse) for $M = 2$. Note that the region of specifications outside of the ellipse can be ignored for the yield calculations. Figure 3 (b) shows the relation between the region Ω_z and the existence region of $f_z(z)$ for $M = 2$. The value of $f_z(z)$ is considered to be zero outside of the circle with its center at the origin and with a radius of 4. Hence, for the yield calculation of Eq. (35), it is sufficient to consider only the shadowed portions. Furthermore, if we begin calculating from the neighbourhood of the origin and stop calculating at the point crossing the circle, the efficiency of the yield estimation is improved. In this paper we use the method of calculating the yield in numerical order shown in Fig. 3 (b). The judgement that the cell position of interest is in or out of the circle is done such that the distance between the cell and the origin is smaller than 4 or not.

Since the term of Eq. (23)

$$-\frac{1}{2}\sum_{j=1}^{M}\left(\xi_{j0} + \sum_{i=1}^{M}\frac{k_i}{m_i}\,\xi_{ji}\right)^2$$

represents a square of the distance between the cell position and the origin, we may form the judgement mentioned above after this calculation.

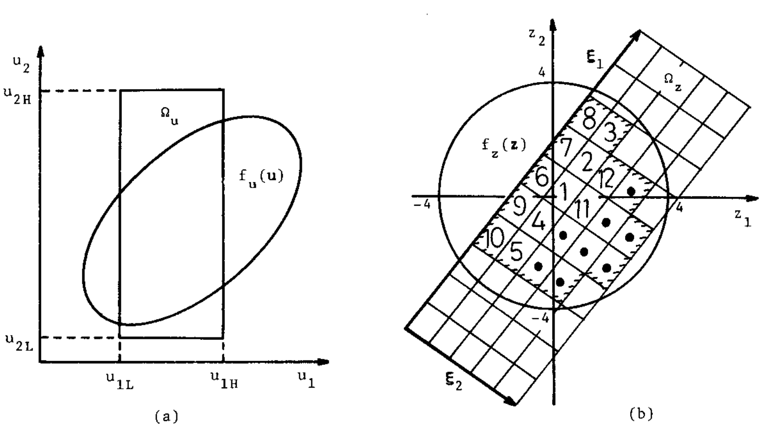

Fig. 3 (a) Relation between the region recognized the existence of $f_u(u)$ and Ω_u (for $M = 2$).
(b) Relation between the region recognized the existence of $f_z(z)$ and Ω_z (for $M = 2$).

6. Example

This yield estimation method was applied to the switched capacitor (SC) second order low pass filter shown in Fig. 4. The transfer function $T(z)$ in z domain can be described with Eq. (41).

$$T(z) = V_o/V_i = \frac{\dfrac{C_1 C_3}{C_4 C_5}}{z^2 - \left(2 - \dfrac{C_2}{C_4}\right)z + \dfrac{C_1 C_3}{C_4 C_5} - \dfrac{C_2}{C_4} + 1} \tag{41}$$

With $z \simeq 1 + T_c s$ (T_c: clock period) we obtain the frequency domain transfer function $T(s)$;

$$T(s) = \frac{\dfrac{C_1 C_3}{C_4 C_5}}{T_c^2 s^2 + \dfrac{C_2}{C_4} T_c s + \dfrac{C_1 C_3}{C_4 C_5}} \tag{42}$$

From Eq. (42), the cutoff angular frequency ω_c and Q are given by

$$\omega_c = \sqrt{\frac{C_1 C_3}{C_4 C_5}} \Big/ T_c\,, \quad Q = \sqrt{\frac{C_1 C_5}{C_4 C_2}} \tag{43}$$

Figure 5 shows the frequency characteristics of this filter. As shown in Fig. 5 we set the upper and lower limits of specifications at the two frequency points.

We assume that the operational amplifiers and switches are ideal.

Although the circuit of Fig. 4 has five circuit parameters $C_1 \sim C_5$, in Eq. (41) only

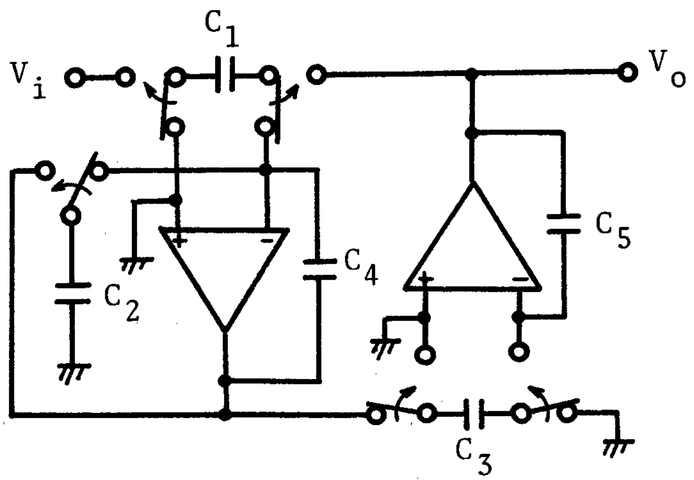

Fig. 4 *SC* second order low pass filter.

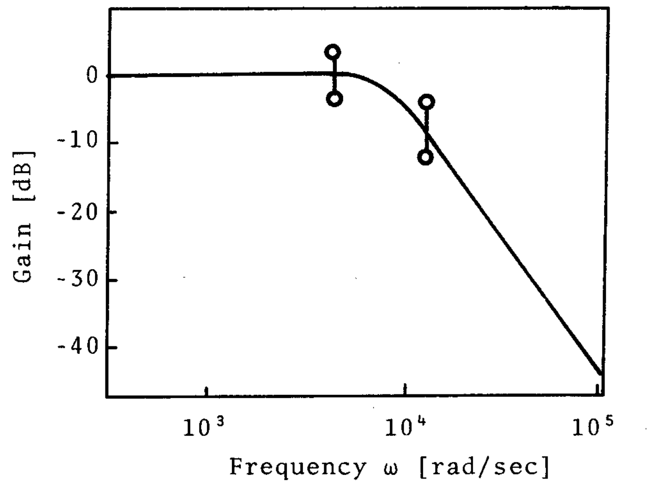

Fig. 5 Frequency Characteristics.

two combinations such as C_2/C_4, C_1C_3/C_4C_5 are appeared. Therefore, we can apply the parameter transformation method proposed in 8).

Consider a ratio C_n to C_m. We now assume a normal distribution $N(\mu_m, \mu_n, \sigma_m^2, \sigma_n^2, \rho_{mn})$ for the joint *pdf* of C_m and C_n, where μ_m and μ_n are the mean values, σ_m and σ_n are the standard deviations and ρ_{mn} is the coefficient of correlation.

Under the conditions that $\sigma_m = S\mu_m$, $\sigma_n = S\mu_n$ $(S < 1)$ and $\rho_{mn} \simeq 1$, the ratio C_n/C_m is statistically varying with a *pdf* of the logarithmic normal distribution. Hence, $\ln(C_n/C_m)$ becomes to obey the normal distribution. (see Appendix)

In this paper, for a set of parameters $p_1 = \ln(C_1/C_4)$ and $p_2 = \ln(C_1C_3/C_4C_5)$ are chosen.

Table 1 shows the results of the division numbers, the yield and CPU time of the computer for the three specifications ① ~ ③ when the parameter values are as follows; $T_c = 10^{-5}$ [sec], $S = 0.1$, $\rho_{mn} = 0.8$, $C_2/C_4 = 0.12$, $C_1C_3/C_4C_5 = 0.008$ and $\epsilon_0 = 0.0001$.

The algorithms for yield estimation have been implemented in Fortran on a ACOS 850 computer. In Table 1, "Reduction" is the case that the effective method mentioned

Table 1 Calculation results

Frequency points [rad/sec]		$\omega_1 = 5{,}000$	$\omega_2 = 15{,}000$	Division numbers Yield (%) CPU time (sec)	
$u\ (\mu_p)$		$u_1 = 0.983$	$u_2 = 0.346$	Non-reduction	Reduction
Specifications	①	$u_{1L} = 0.950$ $u_{1H} = 1.100$	$u_{2L} = 0.330$ $u_{2H} = 0.360$	160 × 133 43.8 3.14	160 × 133 43.8 2.57
	②	$u_{1L} = 0.700$ $u_{1H} = 1.500$	$u_{2L} = 0.220$ $u_{2H} = 0.420$	855 × 728 99.9 92.0	855 × 728 99.9 13.3
	③	$u_{1L} = 0.980$ $u_{1H} = 0.990$	$u_{2L} = 0.340$ $u_{2H} = 0.350$	10 × 16 2.9 0.07	10 × 16 2.9 0.07

in Chapter 5 is applied. In comparison with the case of "Nonreduction" the CPU time is reduced but the accuracy of yield estimation is kept unchanged. Particularly the effect is remarkable for the case that the specifications are fairly broad as ②.

7. Conclusions

In this paper we proposed a new method of yield estimation which is a very useful criterion for a design of integrated circuits.

The calculation time of this method is mainly determined by the number of specifications and it can be considered that there is no effect of the number of circuit parameters. Hence, this method is useful for the yield estimation of a large scale integrated circuits, such as amplifiers, for which the frequency point numbers are not so much in general.

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Appendix

We now assume the normal distribution $N(\mu_{x_m}, \mu_{x_n}, \sigma_{x_m}{}^2, \sigma_{x_n}{}^2, \rho_{mn})$ for the circuit parameters x_m and x_n. For semiconductor integrated circuits it seems to be quite all right to consider that the standard deviation σ_x is proportional to the mean value μ_x and the correlation between circuit parameters is strong. These can be expressed as follows.

$$\sigma_x = S\mu_x \ (S < 1) \tag{A·1}$$

$$\rho_{mn} \simeq 1 \tag{A·2}$$

The joint *pdf* is given as

$$f_x(x_m, x_n) = k \cdot \exp\left[-\frac{1}{2(1-\rho_{mn}{}^2)} \left\{ \left(\frac{x_m - \mu_{x_m}}{\sigma_{x_m}}\right)^2 - 2\rho_{mn}\left(\frac{x_m - \mu_{x_m}}{\sigma_{x_m}}\right)\left(\frac{x_n - \mu_{x_n}}{\sigma_{x_n}}\right) + \left(\frac{x_n - \mu_{x_n}}{\sigma_{x_n}}\right)^2 \right\} \right] \tag{A·3}$$

where

$$k = 1/(2\pi\sigma_{x_m}\sigma_{x_n}\sqrt{1-\rho_{mn}^2}) \tag{A·4}$$

To obtain the *pdf* for $x(= x_n/x_m)$, we express $f_x(x_m, x_n)$ in the polar coordinates. Using Eq. (A·1) and the following equations

$$x_m = r\cos\theta, \quad x_n = r\sin\theta \tag{A·5}$$

we obtain from Eq. (A·3)

$$f(r, \theta) = k\exp(-ar^2 + br - c) \tag{A·6}$$

where

$$a = \frac{1}{2(1-\rho_{mn}^2)S^2}\left(\frac{\cos^2\theta}{\mu_{x_m}^2} - 2\rho_{mn}\cdot\frac{\cos\theta}{\mu_{x_m}}\cdot\frac{\sin\theta}{\mu_{x_n}} + \frac{\sin^2\theta}{\mu_{x_n}^2}\right) \tag{A·7}$$

$$b = \frac{1}{(1+\rho_{mn})S^2}\left(\frac{\cos\theta}{\mu_{x_m}} + \frac{\sin\theta}{\mu_{x_n}}\right) \tag{A·8}$$

$$c = \frac{1}{(1+\rho_{mn})S^2} \tag{A·9}$$

Considering $f(r, \theta)$ as a function of only θ, we replace $f(r, \theta)$ by $g(\theta)$. The probability $g(\theta)\,d\theta$ of θ being in the interval $[\theta, \theta + d\theta]$ is equal to the volume of cubic as shadowed in Fig. A·1 (a). The section dS produced by cutting this cubic with a plane of $r = constant$ becomes a trapezoid. From Fig. A·1 (b), when $d\theta \to 0$, dS is given as

$$dS = f(r, \theta)\, r d\theta \tag{A·10}$$

Then

$$\begin{aligned} g(\theta)\,d\theta &= \int_0^\infty dS dr \\ &= \int_{r=0}^\infty f(r, \theta)\, r dr d\theta \end{aligned} \tag{A·11}$$

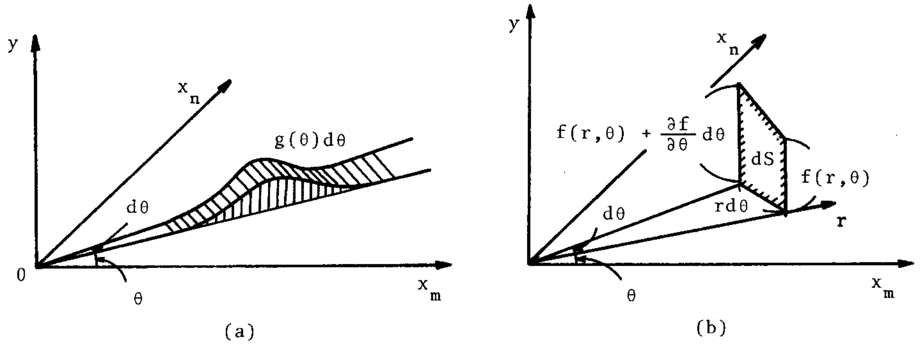

Fig. A·1 (a) Probability $g(\theta)d\theta$ that θ is included in the interval $[\theta, \theta + d\theta]$.
(b) Differential area dS.

From Eqs. (A·6) and (A·11)

$$g(\theta) = \int_0^\infty f(r, \theta)\, rdr$$

$$= k \cdot \exp\left[\frac{b^2}{4a} - c\right] \int_0^\infty e^{-a(r-b/2a)^2}\, rdr \tag{A·12}$$

By applying the condition $\sqrt{b^2/2a} \ll 1$ (correspond to $S < 1$) in the calculation process of Eq. (A·12), the *pdf* $g(\theta)$ becomes

$$g(\theta) = \frac{\sqrt{\pi}\, ke^{-c}}{a} \sqrt{\frac{b^2}{4a}}\; e^{b^2/4a} \tag{A·13}$$

On the other hand, the probability of x being in the interval $[x, x + dx]$ is

$$h(x)\, dx = g(\theta)\, d\theta \tag{A·14}$$

With the relation of $x = \tan\theta$ we obtain

$$d\theta/dx = 1/1 + x^2 \tag{A·15}$$

Thus

$$h(x) = \frac{1}{1 + x^2} \cdot g(\theta) \tag{A·16}$$

Substituting Eq. (A·12) into Eq. (A·16) $h(x)$ is obtained as

$$h(x) = \frac{e^{-c}}{\nu} \sqrt{\frac{1 - \rho_{mn}}{\pi}}\; \frac{\sqrt{\frac{b^2}{4a}}\; e^{b^2/4a}}{1 - 2\rho_{mn}(x/\nu) + (x/\nu)^2} \tag{A·17}$$

where

$$\nu = \mu_{x_n}/\mu_{x_m} \tag{A·18}$$

$$\frac{b^2}{4a} = \frac{1 - \rho_{mn}}{2(1 + \rho_{mn})S^2}\; \frac{1 + 2(x/\nu) + (x/\nu)^2}{1 - 2\rho_{mn}(x/\nu) + (x/\nu)^2} \tag{A·19}$$

From Eqs. (A·17) and (A·19) we can make the following statement. Let $h(x)$ be $h_{\nu_0}(x)$ for $\nu = \nu_0$. We can obtain the next relation

$$h_{\nu_0}(x) = \frac{1}{\nu_0}\, h_1(x/\nu_0) \tag{A·20}$$

where $h(x)$ for $\nu_0 = 1$ is denoted by $h_1(x)$. Equation (A·20) means that $h_\nu(x)$ is in the same form as $h_1(x)$ when the magnitude increases by a factor $1/\nu$ and x increases by a factor ν.

Let the mean value and the standard deviation of $h_\nu(x)$ be μ_{h_ν} and σ_{h_ν}, respectively, then

$$\mu_{h_\nu} = \nu \mu_{h_1} \tag{A·21}$$

$$\sigma_{h_\nu}{}^2 = \nu^2 \sigma_{h_1}{}^2 \tag{A·22}$$

The *pdf* $\lambda(x)$ of logarithmic normal distribution is as

$$\lambda(x) = \frac{1}{\sqrt{2\pi}\,\sigma_l x} \exp\left\{-\frac{1}{2\sigma_l^2}(\ln x - \mu_l)^2\right\} \tag{A·23}$$

where μ_l is the mean value of $\ln x$ and σ_l is the standard deviation.

Using the next relation

$$\begin{aligned}\mu_l &= \ln \nu \\ &= \ln \mu_{x_n} - \ln \nu_{x_m}\end{aligned} \tag{A·24}$$

we get

$$\lambda(x) = \frac{1}{\sqrt{2\pi}\;\sigma_l \nu(x/\nu)} \exp\left\{-\frac{1}{2\sigma_l^2}(\ln x/\nu)^2\right\} \tag{A·25}$$

In the same manner as Eq. (A·20), we denote $\lambda(x)$ for $\nu = \nu_0$ as $\lambda_{\nu_0}(x)$, especially for $\nu_0 = 1$ as $\lambda_1(x)$. Thus the next relation holds.

$$\lambda_{\nu_0} = \frac{1}{\nu_0} \lambda_1(x/\nu_0) \tag{A·26}$$

When $S < 1$, $h_\nu(x)$ has a maximum value at the point of $x = 2$.

Using a Taylor series expansion about the point $x = \nu$, we get

$$h_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\nu^{k+1}} h_1^{(k)}(x) \Big|_{x=1} (x-\nu)^k \tag{A·27}$$

$$\lambda_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\nu^{k+1}} \lambda_1^{(k)}(x) \Big|_{x=1} (x-\nu)^k \tag{A·28}$$

Let

$$\sigma_l = \sqrt{2(1-\rho_{mn})} \cdot S \tag{A·29}$$

hence, the first two terms (for $n = 0, 1$) of Eq. (A·27) agree precisely with those of Eq. (A·28).

The mean value μ_x and the standard deviation σ_x are given as follows

$$\begin{aligned}\mu_x &= \nu \exp(\sigma_l^2/2) \\ &\simeq \nu\left\{1 + (1-\rho_{mn})S^2\right\}\end{aligned} \tag{A·30}$$

$$\begin{aligned}\sigma_x &= \mu_x(\exp \sigma_l^2 - 1)^{1/2} \\ &\simeq \mu_x \sqrt{2(1-\rho_{mn})} \cdot S\end{aligned} \tag{A·31}$$

Let the probability distribution functions of $h_\nu(x)$ and $\lambda_\nu(x)$ be $H_\nu(x)$ and $\Lambda_\nu(x)$, respectively. Th error $E(x)$ appeared when $H_\nu(x)$ is approximated with $\Lambda_\nu(x)$ is expressed as

$$E(x) = \left| H_\nu(x) - \Lambda_\nu(x) \right| \tag{A·32}$$

Limiting the domain of x such as $\mu_x - 3\sigma_x \leqslant x \leqslant \mu_x + 3\sigma_x$, $E(x)$ can be formulated by

$$\begin{aligned} E(x) &= \int_{\mu_x-3\sigma_x}^{\mu_x+3\sigma_x} \left| h_\nu(\zeta) - \lambda_\nu(\zeta) \right| d\zeta \\ &\simeq \int_{\mu_x-3\sigma_x}^{\mu_x+3\sigma_x} \frac{\left| h_1^{(2)}(1) - \lambda_1^{(2)}(1) \right| (\zeta - 1)^2}{2\nu^3}\, d\zeta \\ &= \frac{\left| h_1^{(2)}(1) - \lambda_1^{(2)}(1) \right| \left\{ (\mu_x + 3\sigma_x - 1)^3 - (\mu_x - 3\sigma_x - 1)^3 \right\}}{6\nu^3} \\ &\leqslant \frac{18\sqrt{2}}{\sqrt{\pi}} (1 - \rho_{mn}) S^2 \end{aligned}$$

From Eq. (A·33), it is obvious that $E(x)$ can be ignored for the case that $\rho_{mn} \simeq 1$, $S < 1$. Therefore, under these conditions, the *pdf* of x is considered to be approximated with the *pdf* $\lambda(x)$ of logarithmic normal distribution, $\lambda(x)$ is given by

$$\lambda(x) = \frac{1}{\sqrt{2\pi}\ \sigma_l x} \exp\left\{ -\frac{(\ln x/\nu)^2}{2\sigma_l^2} \right\} \tag{A·34}$$

Accordingly, it is apparent that

$$\ln(x_n/x_m) = \ln x_n - \ln x_m \tag{A·35}$$

is varying with a *pdf* of normal distribution. Now we define the parameter p_{mn} as

$$p_{mn} = \ln x_n - \ln x_m \tag{A·36}$$

We introduce the parameters $\boldsymbol{p} = [p_1, p_2, \cdots, p_i, \cdots, p_N]^T$ by writing p_{mn} to replace p_i, then

$$\begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix} = A \begin{bmatrix} \ln x_1 \\ \vdots \\ \ln x_K \end{bmatrix} \tag{A·37}$$

where $x_1, x_2, \cdots, x_K$ are the circuit parameters. In general N is less than K. Every row of the $N \times K$ matrix A has exactly two nonzero elements, a 1 and a -1, with the rest being zeros.

Since $\boldsymbol{p}$ has the *pdf* of normal distribution, $[\ln x_1, \ln x_2, \cdots, \ln x_k]^T$ also has the *pdf* of normal distribution. Between the covariance matrix C_p of $\boldsymbol{p}$ and the covariance matrix C_l of $[\ln x_1, \ln x_2, \cdots, \ln x_K]^T$ the next relation holds.

$$C_p = A C_l A^T \tag{A·38}$$

Using $C_{x_{ij}}$ which is an element of covariance matrix of the circuit parameters, the element $C_{l_{ij}}$ of C_l is given as

$$C_{l_{ij}} = \ln\left(\frac{C_{x_{ij}}}{\mu_{x_i}\mu_{x_j}} + 1\right) \tag{A·39}$$

Substituting the next relations

$$S = \frac{\sigma_{x_i}}{\mu_{x_i}} = \frac{\sigma_{x_j}}{\mu_{x_j}}, \quad \rho_{ij} \simeq \frac{C_{x_{ij}}}{\sigma_{x_i}\sigma_{x_j}} \tag{A·40}$$

into Eq. (A·39), we obtain

$$C_{l_{ij}} \simeq \ln(\rho_{ij} S^2 + 1) \tag{A·41}$$

Especially $C_{p_{ij}}$ is the variance of p_{mn} when $i = j$, thus

$$C_{p_{ij}} = 2(1 - \rho_{mn}) S^2 \tag{A·42}$$

From the above

$$C_{p_{ij}} = \begin{cases} [A C_l A^T]_{ij} & (i \neq j) \\ 2(1 - \rho_{m_i m_j}) S^2 & (i = j) \end{cases} \tag{A·43}$$

The mean value vector $\boldsymbol{\mu}_p$ of $\boldsymbol{p}$ is given as

$$\boldsymbol{\mu}_p = A \begin{bmatrix} \ln \mu_{x_1} \\ \vdots \\ \ln \mu_{x_k} \end{bmatrix} \tag{A·44}$$

A summary of the results is shown below. When the transfer function of the circuit is considered to be a function of the ratio of circuit parameters, we may choose the logarithm of the ratio of circuit parameters. In this way the number of parameters are reduced for yield estimation or optimal design.

The $\boldsymbol{p}$ has the *pdf* $f_p(\boldsymbol{p})$ of N dimensional normal distribution such as

$$f_p(\boldsymbol{p}) = \frac{|C_p^{-1}|^{1/2}}{(2\pi)^{N/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{p} - \boldsymbol{\mu}_p)^T C_p^{-1}(\boldsymbol{p} - \boldsymbol{\mu}_p)\right\} \tag{A·45}$$