On the Approximate Characteristic between the System Governed by the Unsymmetrical Curvilinear Restoring Force and the Piecewise Linear system with the Unsymmetrical Restoring Force

On the Approximate Characteristic between the System Governed by the Unsymmetrical Curvilinear Restoring Force and the Piecewise Linear system with the Unsymmetrical Restoring Force

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This is a study of periodic solutions of the systems with unsymmetrical restoring force i.e. (i) the piecewise linear restoring force, (ii) the curvilinear restoring force. This report shows that the phenomena represented by the piecewise linear system differ slightly from the curvilinear system since it represents the performance by means of two straight lines disregarding the curvilinear part between them.

But as a result of extensive numerical analysis it has been established that for the qualitative behavior of the periodic solutions, the piecewise linear representation is, generally, sufficient and slight quantitative difference between the two modes of representation is justified by the gain in simplicity inherent in the idealized piecewise linear representation.

1. Introduction

The theory of oscillation has gradually become generalized and assumed definite form. However, new problems have raised new questions and on these the subject is still in a state of evolution. Considerable interest of mathematicians in the problems of nonlinear oscillations has resulted in important advances in the theory of nonlinear differential equations but, as is to be expected, some of these advances have exceeded the immediate needs of the theory of oscillations and belong rather to the theory of differential equations per se.

On the other hand, physicists and engineers continue to supply experimental material, the analysis of which requires special mathematical tools, some of which are not yet available. In view of this it is sometimes difficult to draw a line between what is known definitely and what is known only provisionally and subject to later revisions.

It is well known that the oscillation of order 1/2 is apt to occur when the nonlinearity is unsymmetrical[1]. Our several papers have described [2),3),4)] periodicity conditions, stability conditions and branching phenomena, in the case of the piecewise linear system with the unsymmetrical restoring force. The present paper deals with the qualitative behavior of the periodic solutions of certain second order nonlinear equations with unsymmetrical restoring forces; (i) the piecewise linear (ii) the curvilinear. By comparison of the results of these cases it is shown that the phenomena represented by the piecewise linear system differ slightly from the curvilinear system since it represents the performance by means of two straight lines disregarding the curvilinear part between them. But, by numerical analysis, for the qualitative behavior of the periodic solutions, the piecewise linear representation is, generally, sufficient and slight quantitative difference between the two modes of representation is justified by the gain in simplicity inherent in the idealized piecewise linear representation.

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2. Periodicity Conditions

This section presents the conditions of periodicity of the differential equations (1) and (3).

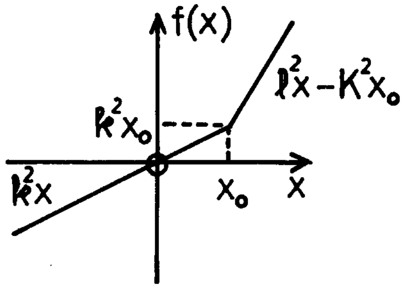

Fig. 1 Piecewise linear restoring force

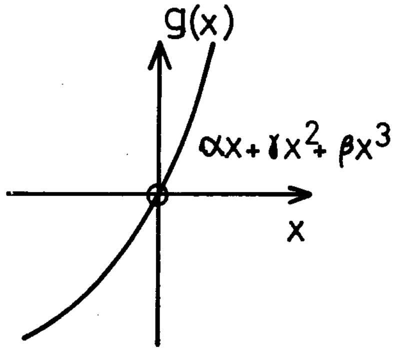

Fig. 2 Curvilinear restoring force

$$\ddot{x} + f(x) = \sum_{i=1}^{m} E_i \cos i\omega t \tag{1}$$

, where

$$\left. \begin{aligned} & f(x) = \begin{cases} l^2 x - K^2 x_0 & (x \geqslant x_0) \\ k^2 x & (x \leqslant x_0) \end{cases} \\ & , l^2 = k^2 + K^2 \end{aligned} \right\} \tag{2}$$

$$\ddot{x} + g(x) = \sum_{i=1}^{m} E_i \cos i\omega t \tag{3}$$

, where

$$g(x) = \alpha x + \gamma x^2 + \beta x^3 \tag{4}$$

In these second order differential equations, the conditions that the solutions will be periodic of period $T\,(= p\dfrac{2\pi}{\omega}$, p: positive integer) under the initial conditions (5)

$$x(O) = M, \dot{x}(O) = O \tag{5}$$

can be written in the next form

$$\left. \begin{aligned} x(T) &= x(O) = M \\ \dot{x}(T) &= \dot{x}(O) = O \end{aligned} \right\} \tag{6}$$

In the piecewise linear system, conditions (6) as to the periodic solutions type ${}_1A$ (shown in Fig. 3) are equivalent to next conditions (7) and (8)[2),3)]

$$(M - \sum_{i=1}^{m} \frac{E_i}{l^2 - (i\omega)^2} - \frac{K^2}{l^2} x_0) \cos l t_1 + \sum_{i=1}^{m} \frac{E_i}{l^2 - (i\omega)^2} \cos i\omega t_1 = \frac{k^2}{l^2} x_0 \tag{7}$$

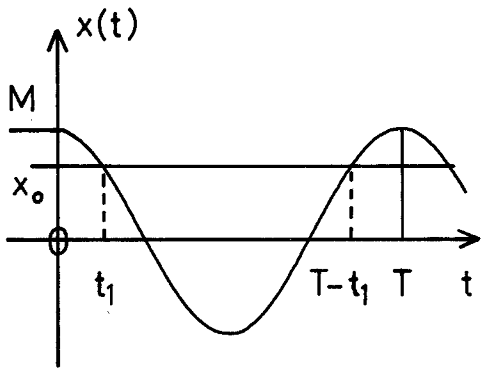

Fig. 3 Periodic solution type ${}_1A$

$$\tan k\,(T/2-t_1)$$
$$=\frac{-l\left(M-\sum_{i=1}^{m}\frac{E_i}{l^2-(i\omega)^2}-\frac{K^2}{l^2}x_0\right)\sin lt_1+\sum_{i=1}^{m}\frac{i\omega K^2E_i}{(l^2-(i\omega)^2)(k^2-(i\omega)^2)}\sin i\omega t_1}{k\left(x_0-\sum_{i=1}^{m}\frac{E_i}{k^2-(i\omega)^2}\cos i\omega t_1\right)} \tag{8}$$

, where t_1 represents the time when the solution reaches the corner point x_0 for the first time.

In the curvilinear system, the periodicity conditions are conditions (6) themselves.

3. Stability

Let $x^0(t)$ be the periodic solution of the equation (1) or the equation (3) with the period $T=2p\pi/\omega$ under the initial conditions (5), then the discussion[5] of the stability of the periodic solution of the equation (1) or the equation (3) depends on the variational equation associated with the periodic solution $x^0(t)$. For the variation $y(t)$, that is given by following equation.

$$\ddot{y}+a(t)y=0 \tag{9}$$

where

$$a(t)\equiv\left.\frac{\partial f}{\partial x}\right|_{x=x^0(t)} \qquad \text{for the equation (1)}$$

or (10)

$$a(t)\equiv\left.\frac{\partial g}{\partial x}\right|_{x=x^0(t)} \qquad \text{for the equation (3)}$$

is an even and T-periodic function. Therefore the equation (9) means a Hill's equation.

Let ρ_1 and ρ_2 be the characteristic multipliers of the equation (9), then

$$\rho_1 \rho_2 = 1$$

$$\rho_1 + \rho_2 = \varphi(T) + \dot{\Psi}(T) \tag{11}$$

where $\varphi(t)$ and $\Psi(t)$ are independent solutions of the equation (9) under the initial conditions $\varphi(O) = \dot{\Psi}(O) = 1, \dot{\varphi}(O) = \Psi(O) = O$

Furthermore the periodic solutions are stable when $|\rho_1 + \rho_2| < 2$ and unstable for $|\rho_1 + \rho_2| > 2$. The condition $|\rho_1 + \rho_2| = 2$ means the border line of stable and unstable regions and has a connection with branching phenomena.

4. Bifurcation

The essential aim of this section is the identification of the branching behavior near bifurcation point of equations (1) and (3).

For simplicity the application of the general solutions derived above is restricted to the case of subharmonic oscillations of order 1/2. As to the case of the piesewise linear restoring force, the bifurcation condition[2),3)] becomes

$$\frac{k}{\omega} < \frac{2n+1}{2} < \frac{l}{\omega} \qquad (n = 0, 1, 2, \dots) \tag{12}$$

and the branching behavior of the equation (1) in M-E plane may be written as

$$M - M_0 \fallingdotseq C_1(E - E_0)^2 \tag{13}$$

where (M_0, E_0) is the bifurcation point and C_1 is a constant. On the other hand, as to the equation (3), the branching condition is too complicated to be represented exactly. But, with regard to the branching behavior in M–E plane, we have the result

$$M - M_0 \fallingdotseq C_2(E - E_0)^2 \qquad (C_2 : \text{constant}) \tag{14}$$

assuming the existence of the solutions of period T of the equation (3) with its multiplier of the first variational equation equal to -1.

One of the most different point of these two systems is whether we clarify the branching condition or not.

5. Numerical Analysis and Discussion

When dealing with nonlinear differential equations it is usual to resort either to a digital (or analog) computer or to approximate methods. It is often inevitable that one should use a computer to produce the final answers, although it may be difficult to obtain physical insights by using a computer.

The oscillations may be calculated in the following steps:

(1) A computer is made use of to understand the behavior of solutions of nonlinear differential equations (1) and (3).

(2) In order to comfirm the same nonlinearity of restoring forces $f(x)$ and $g(x)$, it will be convenient to introduce the notation

$$I = \int_{-a}^{a} \{f(x) - g(x)\}^2 \, dx \tag{5}$$

and use the method of minimizing the cost function I.

(3) For study of the periodic solutions of the equation (1), the periodicity conditions (7) and (8) are applied. The numerical calculations have been made by the Rosenbrock's method[6]. But, in the case of the equation (3) the numerical calculation is made by two methods; (i) Regula falsi method (two-side-attack method) (ii) Harmonic Balance method.

In case $m = 2$ let's rewrite the equation (3)

$$\ddot{x} + g(x) = E_1 \cos\omega t + E_2 \cos 2\omega t \tag{3$'$}$$

and assume a periodic solution of the form

$$x(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t \tag{16}$$

Substitution of (16) in (3)$'$ leads to the relations by means of harmonic balance.

$$\alpha a_0 + \gamma\{a_0^2 + \frac{1}{2}(a_1^2 + a_2^2 + a_3^2)\} + \beta[a_0^3 + \frac{3}{2}a_0a_1^2 + 3\{\frac{1}{4}a_1^2a_2 + \frac{1}{2}a_0a_2^3 + \frac{1}{2}a_0a_3^2 + \frac{1}{2}a_1a_2a_3\}] = 0 \tag{17}$$

$$(\alpha - \omega^2)a_1 + \gamma(2a_0a_1 + a_1a_2 + a_2a_3) + \beta[3a_0^2a_1 + \frac{3}{4}a_1^3 + 3\{a_0a_1a_2 + \frac{1}{4}a_1^2a_3 + a_0a_2a_3 + \frac{1}{2}a_1a_2^2 + \frac{1}{2}a_1a_3^2\} + \frac{3}{4}a_2^2a_3] = E_1 \tag{18}$$

$$(\alpha - 4\omega^2)a_2 + \gamma(\frac{1}{2}a_1^2 + 2a_0a_2 + a_1a_3) + \beta[\frac{3}{2}a_0a_1^2 + 3\{a_0^2a_2 + a_0a_1a_3 + \frac{1}{2}a_1^2a_2 + \frac{1}{2}a_1a_2a_3\} + \frac{3}{4}a_2^3 + \frac{3}{2}a_2a_3^2] = E_2 \tag{19}$$

$$(\alpha - 9\omega^2)a_3 + \gamma(2a_0a_3 + a_1a_2) + \beta[\frac{1}{4}a_1^3 + 3\{a_0^2a_3 + a_0a_1a_2 + \frac{1}{2}a_1^2a_3 + \frac{1}{4}a_1a_2^2\} + \frac{3}{2}a_2^2a_3 + \frac{3}{4}a_3^3] = 0 \tag{20}$$

Fig. 4 shows the two restoring forces in case $l^2 = 3$, $k^2 = 1$, $x_0 = 1$ for the piecewise linear system, corresponding to $\alpha = 1.344$, $\beta = 0.03$, $\gamma = 0.209$ for $a = 4$ in the curvilinear system. In Fig. 5 the results of ω-M plane in case $E_1 = -7.5$ and $m = 1$ are plotted as to the bifurcation of subharmonic solutions from harmonic solutions. In both cases the branching behavior is similar: at the boundary of the stable and unstable harmonic solutions the subharmonic solutions of order 1/2 will appear and exist only one side of the unstable harmonic solutions. In Fig. 6 ($E_1 = -7.5$) are shown the Fourier

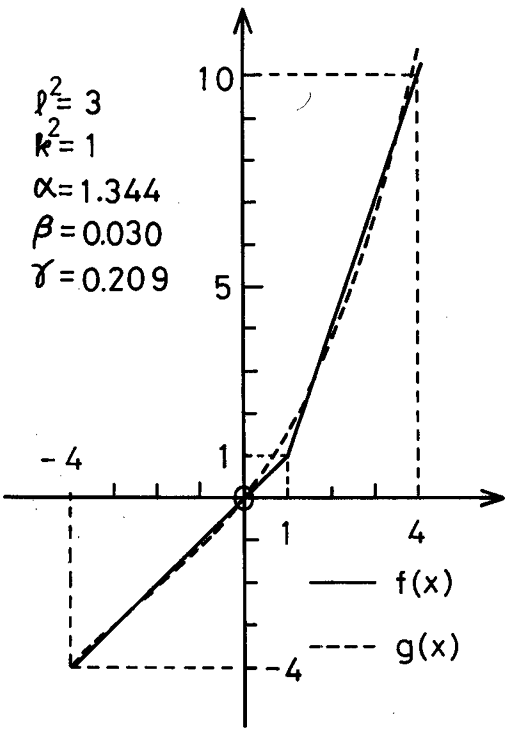

Fig. 4 Restoring force characteristics of $f(x)$ and $g(x)$ in case $x_0 = 1$

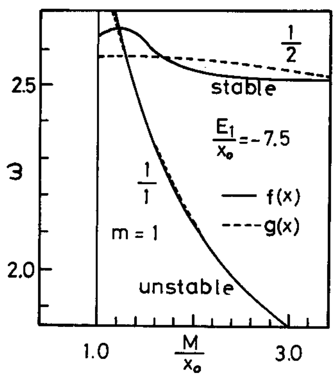

Fig. 5 Branching phenomena of solutions of order 1/2 from harmonic solutions in M-ω plane in case $E_1 = -7.5$, $m = 1$, and $x_0 = 1$

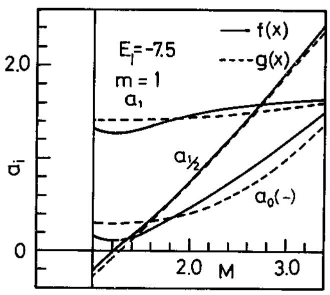

Fig. 6 Initial value responses of subharmonic solutions of order 1/2 in case $E_1 = -7.5$ and $m = 1$

components of subharmonic oscillations of order 1/2. From this figure it is evident that the Fourier component $a_{1/2}$ (whose frequency is a fraction 1/2 of the driving frequency ω) in the piecewise linear system agrees very well with the result in the curvilinear system. Other components differ slightly in the neighborhood of the branching point. As to harmonic solutions in Fig. 5, the Fourier components shown in Fig. 7 ($E_1 = -7.5$)

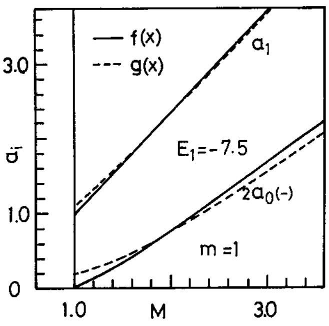

Fig. 7 Initial value responses of harmonic solutions in case E_1 = -7.5 and m = 1

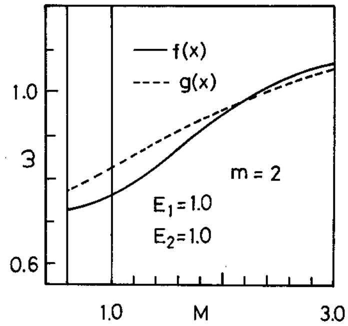

Fig. 8 Regions in which harmonic solutions occur in case m = 2 in ω-M plane

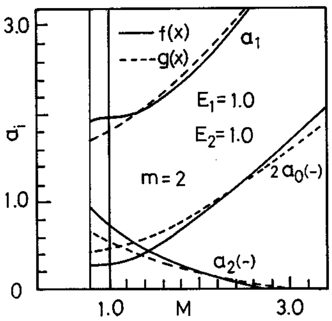

Fig. 9 Initial value responses of Fourier components (a_0, a_1, a_2) of harmonic solutions in case m = 2

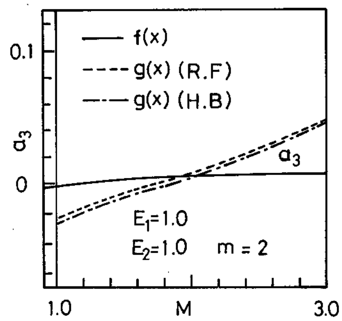

Fig. 10 Initial value responses of Fourier component a_3 of harmonic solutions in case m = 2
R.F and H.B mean regula falsi method and harmonic balance method respectively.

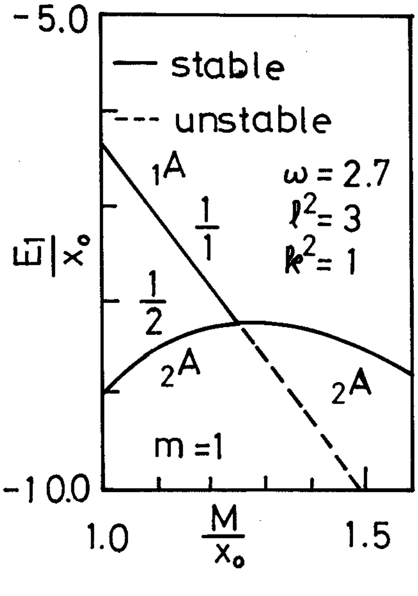

Fig. 11
Branching phenomena of subharmonic solutions of order 1/2 from harmonic solutions in M-E plane in case the piecewise linear system

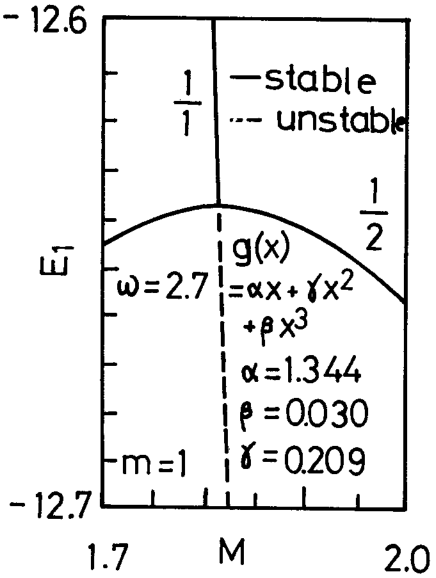

Fig. 12
Branching phenomena of subharmonic solutions of order 1/2 from harmonic solutions in M-E plane in case the curvilinear system

are of the same shape in both cases. In Figs. (8), (9) and (10) the results of harmonic solutions in the case of $m = 2$ are given. In the case of the curvilinear restoring force for the numerical calculations two methods (i) regula falsi, (ii) harmonic balance are used. The components a_0, a_1, a_2 in the case of regula falsi method agree very well with those of harmonic balance method. The third components a_3 differ slightly in two cases.

Finally, branching behavior of subharmonic solutions of order 1/2 from harmonic solutions in M-E plane in case $m = 1$ is shown in Figs. (11) and (12). Comparison of Figs. (11) and (12) tells us that the behavior differs from each other and is no longer of exactly the same value, but having same tendency.

6. Conclusions

Forgoing analysis is restricted to the behavior of the periodic solutions in the case where m external forces are applied to unsymmetrical systems in both the piecewise linear system and the curvilinear system. From this point of view the piecewise linear idealization appears to be sufficiently general to be able to account for a considerable number of phenomena encountered in the theory of oscillations. The results are summarized as follows:

(i) The phenomena represented by the piecewise linear system differ slightly from the curvilinear system since it represents the performance by means of two straight lines disregarding the curvilinear part between them.

(ii) For the qualitative behavior of the periodic solutions, the piecewise linear representation is, generally, sufficient and slight quantitative difference between the two modes of representation is jusified by the gain in simplicity inherent in the idealized piecewise linear representation.

The future work includes investigations of chaotic motions from period doubling motions in dissipative systems.

Finally, it is noted that numerical calculations were performed by using ACOS-700 at the computer center, University of Osaka Prefecture.

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