A Study on First-Order-Second-Moment Method in Structural Reliability

A Study on First-Order-Second-Moment Method in Structural Reliability

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This paper proposes a new method for searching the best linearization point in the Advanced First-Order-Second-Moment method. It is compared with the methods by Rackwits-Fiessler and Chen-Lind, and demonstrated to be efficient. The proposed method is applied to reliability analysis of a column subjected to a combined load effect to examine the effect of the linear and nonlinear failure criteria on the resulting reliability.

1. Introduction

Many studies[1] have been made on the methods of reliability analysis of a structural system where the statistical variations of member strengths and loads are taken into account to evaluate the reliability. Reliability analysis requires the numerical evaluation of the probability of failure events. When non-normal basic random variables are introduced or when a limit state function is non-linear, the evaluation of the convolution integral of the failure probability is a troublesome work. It may be impractical to exactly evaluate the failure probability in case of the statistical data insufficiently provided. Linearizing the limit state function and approximating the non-normal random variables by the normal ones lead to a very simple and efficient method of evaluating the reliability. The validity of such an algorithm depends mainly on the choice of the linearization point and on the approximation of the random variables.

This paper is concerned with a so-called Advanced First-Order-Second-Moment (AFOSM) method[2], in which the best linearization point is selected on the limit state surface, *i.e.,* a surface dividing the transformed basic random variable space into a failure region and a safe region. A new method is developed to search the best linearization point by applying an optimization technique. The proposed method is compared with the methods by Rackwitz-Fiessler[3] and Chen-Lind[4] through numerical examples. Finally, the reliability analysis of a column subjected to a combined load effect of a bending moment and an axial force is carried out with the proposed method and the effect of the linear and nonlinear failure criteria is discussed on the reliability.

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2. Concept of First-Order-Second-Moment (FOSM) Method

At first, a failure criterion or a limit state function is defined in order to evaluate the reliability of a structural system. The limit state function of a structure under operating, accidental and environmental loading conditions can be generally described in the form:

$$Z = g(X_1, X_2, \dots, X_n) \tag{1}$$

where X_i's are basic random variables of member strengths and loads applied to the structure. The structure is in the failure state when $Z \leqq 0$ while it is in the safe state when $Z > 0$. $g(X_1, X_2, \dots, X_n) = 0$ yields a surface to divide the failure and safe regions and it is called the limit state surface. The probability of failure P_f is given by

$$P_f = \underset{D}{\iint\cdots\int} f_{X_1 \cdot X_2 \cdots X_n}(x_1, \dots, x_n)\, dx_1 \dots dx_n \tag{2}$$

where $f_{X_1 \cdot X_2 \cdots X_n}(x_1, \dots, x_n)$ is the joint probability density function of x_i $(i = 1, 2, \dots, n)$ and D is the failure domain in the n-dimensional space in which $Z \leqq 0$ is satisfied. The multi-dimensional integration over the irregular domain D is impossible to carry out analytically and quite costly to perform numerically. Therefore, a simplified method has been proposed for reliability analysis, using the first order approximation and the statistical moments up to the second order.

2.1 Simple method

A linear approximation of the limit state surface $Z = g(X_1, X_2, \dots, X_n) = 0$ is obtained by expanding it into the Taylor series around the mean value point $\bar{\boldsymbol{X}} = (\mu_1, \mu_2, \dots, \mu_n)$ and by neglecting the second order and higher terms. Then, the reliability index β is defined by using the mean value μ_Z and standard deviation σ_Z of the linearized limit state function:

$$\beta = \frac{\mu_Z}{\sigma_Z} \tag{3}$$

where

$$\mu_Z = g(\mu_1, \mu_2, \dots, \mu_n)$$

$$\sigma_Z^2 = \mathrm{E}[(Z - \mu_Z)^2] \cong \sum_i^n \sum_j^n A_i A_j \mathrm{COV}[X_i, X_j]$$

$$A_i = [\partial g / \partial X_i]_{\bar{\boldsymbol{X}}}$$

$$\mathrm{COV}[X_i, X_j] = \mathrm{E}[(X_i - \mu_i)(X_j - \mu_j)]$$

2.2 Advanced method

The FOSM method mentioned above is easy to calculate, but it has some drawbacks;

1) The linearization error will increase in the region far from the linearizing point especially when the limit state function is highly nonlinear.
2) The FOSM method does not give invariant reliability index for the failure criteria which are equivalent but expressed in the different forms.

These drawbacks are eliminated[2)] by choosing the linearizing point on the limit state surface $g(X_1, X_2, \dots, X_n)=0$. This method is refered to as Advanced First-Order-Second-Moment (AFOSM) method, which is briefly explained in the following.

All the basic random variables $X_i(i=1, 2, \dots, n)$ are treated as the uncorrelated ones. They are transformed into their standardized form:

$$U_i = (X_i - \mu_i)/\sigma_i \quad (i = 1,2,\dots,n) \tag{4}$$

Where μ_i and σ_i are the mean value and standard deviation of X_i, respectively. The limit state surface can be correspondingly rewritten in terms of the standardized variables as follows:

$$Z = h(\boldsymbol{u}) = h(u_1, u_2,\dots, u_n) \tag{5}$$

where $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$

Consider a point $\boldsymbol{u}^0 = (u_1{}^0, u_2{}^0, \dots, u_n{}^0)$ on the limit state surface in the standardized space (U-space) and linearized the transformed limit state function $h(\boldsymbol{u})$ by expanding it into the Taylor series around $\boldsymbol{u}^0$. The mean value μ_Z^0 and standard deviation σ_Z^0 are calculated with the linearized failure surface, and an index β_0 is defined by:

$$\begin{aligned}\beta_0 &= \frac{\mu_Z{}^0}{\sigma_Z{}^0} \\ &= \frac{-\sum_{i=1}^{n} u_i^0[\partial h(\boldsymbol{u})/\partial u_i]_{\boldsymbol{u}^0}}{\left\{\sum_{i=1}^{n}[\partial h(\boldsymbol{u})/\partial u_i]^2_{\boldsymbol{u}^0}\right\}^{1/2}}\end{aligned} \tag{6}$$

The value of β_0 obviously depends on the point $\boldsymbol{u}^0$ around which the limit state function is expanded. The minimum value β of β_0 is called the reliability index, and the corresponding point $\boldsymbol{u}$ is called either a β point or a design point. β is also shown to be equivalent to the shortest distance from the origin to the limit state surface $h(\boldsymbol{u})=0$ in the U-space.

Reliability indices Eqs. (3) and (6) are expressed in terms of the first and second statistical moments of the basic random variables and the gradients of the limit state function. No account is taken into the shapes of the probability distributions although the basic random variables, *i.e.*, strengths and loads, generally follow the various non-normal probability distributions. To overcome the problem, a normalizing mapping is introduced in the following section to transform a non-normal random variable into a standardized normal one.

3. Methods for Searching β-point

The problem here is how efficiently to find the β - point. In the following, three conventional methods are summarized and a new method is proposed by using an optimization technique.

3.1 Conventional methods

A transformation of a non-normal random variable X_i into a standard normal one U_i is realized by a mapping:

$$U_i = \Phi^{-1}[F_i(X_i)] \tag{7}$$

where $F_i(X_i)$: probability distribution function of X_i

$\Phi(U_i)$: standard normal probability distribution function

Let U_i be the standardized variable of X_i, *i.e.*, $U_i = (X_i - \mu x_i) / \sigma x_i$ where μx_i and σx_i are the approximating mean and standard deviation of x_i, respectively. Then, μx_i and σx_i are given by

$$\begin{aligned} \mu x_i &= X_i - \sigma x_i \Phi^{-1}[F_i(X_i)] \\ \sigma x_i &= \phi\{\Phi^{-1}[F_i(X_i)]\}/f_i(X_i) \end{aligned} \tag{8}$$

Where ϕ (.) : standard normal probability density function

f_i (.) : probability density function of X_i

The algorithm by Rackwitz and Fiessler[3] (abridged as R–F method) iteratively finds the β-point with the following procedure.

The searching is performed in the U-space. Let $\boldsymbol{\delta}^k$ be a unit normal vector at a point $\boldsymbol{u}^k$ of the limit state surface $h(\boldsymbol{u}) = 0$, as shown in Fig. 1. Then, a one-dimensional search in the direction $\boldsymbol{\delta}^k$ is performed from the origin to find the distance b^k to the limit state surface, *i.e.*,

$$h(b^k \boldsymbol{\delta}^k) = 0 \tag{9}$$

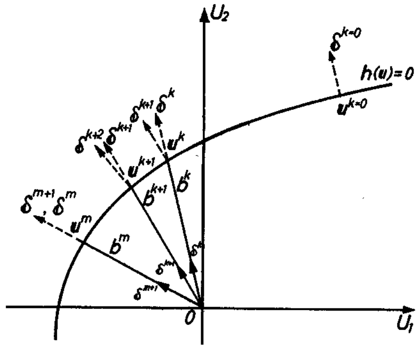


Fig1 Rackwitz–Fiessler method

at the point $\boldsymbol{u}^{k+1} = b^k \delta^k$, calculate the unit normal vector δ^{k+1} of the limit state surface and repeat the one-dimensional search. These procedures are iterated m times when a convergence condition is satisfied. The distance b^m from the origin to the point $\boldsymbol{u}^{m+1}$ corresponds to the reliability index β. The failure probability P_f is then estimated by

$$P_f \cong \Phi(-\beta) \tag{10}$$

When the probability distribution of a basic random variable is very skew, the R–F algorithm tends to give a considerable discrepancy and it is said that a serious numerical error may occur unless double precision is used in the computer calculations[4].

Then a new algorithm for the calculation of reliability index is proposed by Chen and Lind[4),5)]. The method abridged as C–L method is similar to the R–F algorithm, but it uses the following transformation of a non-linear random variable:

$$F_i(X_i) = \alpha_i \Phi\left\{\frac{X_i - \mu x_i}{\sigma x_i}\right\} \tag{11}$$

where α_i: constant

The three parameter μx_i, σx_i and α_i are determined such that the values of $F_i(X_i)$, $f_i(X_i)$, and its derivative $f_i(X_i)$ are identical at the approximation point $\boldsymbol{x}^*$ to the corresponding values of the approximating function $\alpha_i \Phi \{ (X_i - \mu x_i)/\sigma x_i \}$. The reliability index and the probability of failure are approximated as follows:

$$\beta^* = \frac{\sum_{i=1}^{n} \alpha_i(\mu x_i - x_i^*)}{\left[\sum_{i=1}^{n} (a_i \cdot \sigma x_i)^2\right]^{1/2}} \tag{12}$$

where
$$a_i = \left.\frac{\partial g(\boldsymbol{x})}{\partial x_i}\right|_{\boldsymbol{x}^*}$$

$$P_f \cong \Phi(-\beta^*) \cdot \prod_{i=1}^{n} a_i \tag{13}$$

The reliability index β, *i. e.*, the minimum value of β_0, Eq. (6), can also be obtained by solving the following optimization problem[6]:

$$\left.\begin{array}{l} \text{minimize}: \quad \gamma_0 = (\boldsymbol{u}^T \cdot \boldsymbol{u})^{1/2} \\ \text{subject to the constraint}: \\ \qquad h(\boldsymbol{u}) = 0 \end{array}\right\} \tag{14}$$

To solve this problem, a Lagrange's undetermined multiplier is introduced to construct a Lagrangean function L,

$$L = (\boldsymbol{u}^T \cdot \boldsymbol{u})^{1/2} + \lambda h(\boldsymbol{u}) \tag{15}$$

Optimality conditions are given by
$$\frac{\partial L}{\partial u_i} = \frac{\partial L}{\partial \lambda} = 0 \quad (i = 1,2,\dots,n)\ . \tag{16}$$
By solving the set of algebraic equations Eq. (16), the optimum values $\boldsymbol{u}^*$ and λ^* are obtained and β is given by

$$\beta = \min(\boldsymbol{u}^T \cdot \boldsymbol{u})^{1/2} = (\boldsymbol{u}^{*T} \cdot \boldsymbol{u}^*)^{1/2} \tag{17}$$

The probability of failure P_f is calculated approximately by
$$P_f \cong \Phi(-\beta) \tag{18}$$

It should be noted here that Eq. (16) is not, in general, solved easily either analytically or numerically.

3.2 Proposed method

An extended Lagrangean function is introduced to solve the optimization problem Eq. (14) :

$$L_r(\boldsymbol{u}; \mu) = (\boldsymbol{u}^T \cdot \boldsymbol{u})^{1/2} + \mu h(\boldsymbol{u}) + 0.5 r \{h(\boldsymbol{u})\}^2 \tag{19}$$

where μ, r are constants (μ, $r > 0$).
Eq. (19) can be solved easily by making use of an unconstrained optimization technique. An algorithmic procedure[7] is as follows:
Step 1 : Specify the initial values of r and μ, (for example, $r^0 = 5, \mu^0 = 0$) and set $k = 0$, $r^k = r^0$, and $\mu^k = \mu^0$.

Step 2 : Input the initial value of $\boldsymbol{u}$, *i.e.*, $\boldsymbol{u}^k=\boldsymbol{u}^0=(u_1^0, u_2^0, \ldots, u_n^0)$ (for example, $u_i^0 = 0, i = 1, 2, \ldots, n$).

Step 3 : Solve the unconstrained optimization problem of minimizing $L_r\ (\boldsymbol{u},\mu)$ by a conjugate gradient method. Then, obtain the solution $\boldsymbol{u}^{k+1}$.

Step 4 : If the convergence condition $|\ h(\boldsymbol{u})\ | < \varepsilon$ is satisfied for a sufficiently small value of ε (>0), stop the calculation. Otherwise, go to step 5.

Step 5 : Set $r^{k+1}=\omega\cdot r^k,\ \ \mu^{k+1}=\mu^k+r^k\cdot h(\boldsymbol{u})$
where ω is a constant ($\omega > 1$)(for example, $\omega = 5$).

Step 6 : Set $k=k+1$, then go to Step 3.

By substituting the optimum solution $\boldsymbol{u}^*$ thus obtained into Eq. (17), the probability of failure is estimated as

$$P_f \cong \Phi(-\beta) \tag{20}$$

4. Comparison of Searching Methods

The proposed method is compared with the methods by Rackwitz-Fiessler and Chen-Lind through numerical examples.

The strengths of members and applied loads are assumed to be mutually independent random variables whose types of distributions, mean values and coefficients of variation are specified for the calculations. The combination of the coefficients of variation of the member strengths and the loads are denoted by CV(0.05, 0.3), *etc.* The types of distribution are designated by IPD(3, 4), IPD(1, 4), *etc.*, where the first number in the parentheses indicates that of the strengths, and the second one does that of the loads. Further, the numbers correspond to the following distributions : 1-normal, 3-Weibul, 4-Gumbel.

4.1 One-member-one-load structure

Consider a simple structure with one-member-one-load as shown in Fig. 2. The limit state function is given by

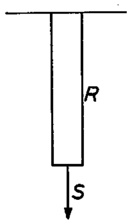
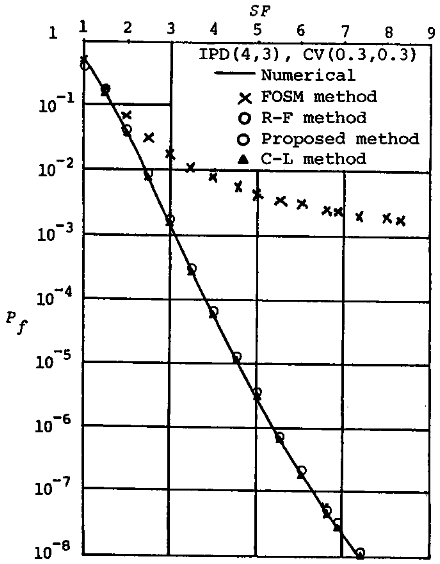


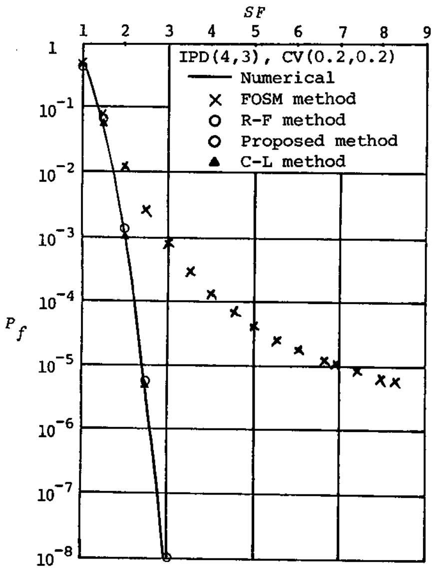
Fig. 2 One-member-one-load structure

$$Z = R - S \tag{21}$$

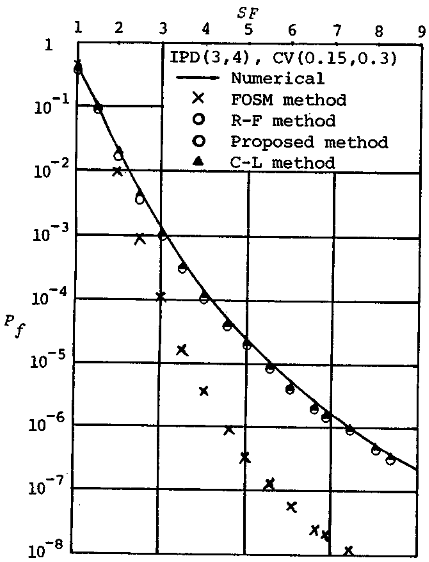
where R is the member strength, and S the applied load. The failure probabilities are calculated for the specified central safety factors $SF=\bar{R}\ \bar{S}$ and shown in Fig. 3. (a), (b), and (c).



(a) IPD (4, 3), CV (0.3, 0.3)



(b) IPD (4, 3), CV (0.2, 0.2)



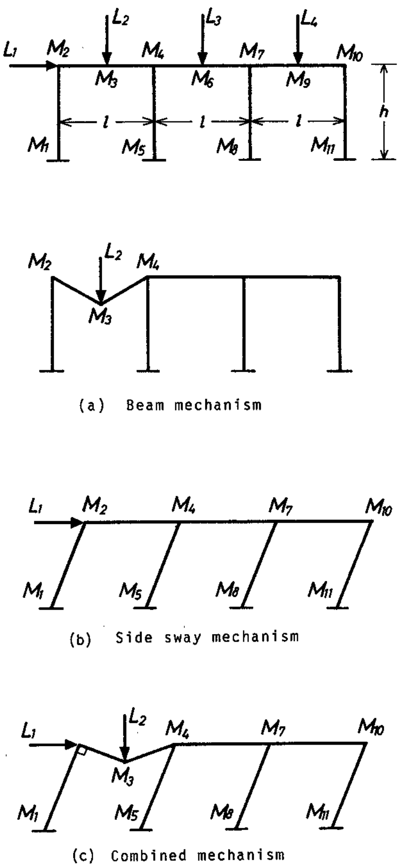
(c) IPD (3, 4), CV (0.15, 0.3)

Fig. 3 Failure probability P_f versus central safety factor SF for the one-member-one-load structure

The exact solutions are evaluated by a numerical integration and they are also plotted in the figures. The failure probabilities estimated by the simple FOSM method show great discrepancies as *SF* becomes large, while all the other methods, *i.e.*, the proposed method, R–F method, and C–L method give the values which are in fairly good agreement with the exact values.

4.2 Three bay frame structure

Failure probabilities are evaluated for the typical collapse modes of a three bay frame structure shown in Fig. 4. The results are listed in Table 1. The proposed method gives almost the same results as R–F method. However, C–L method yields a value different from the others. Further, the numerical experiments shows that C–L method fails to converge depending on the initial values or the types of distribution while the proposed method is versatile to apply to any types of limit state functions.



h=5m, l=10m

$\bar{L}_1$=50kN·m

$\bar{L}_2=\bar{L}_3=\bar{L}_4$=40kN

$\bar{M}_i$=101kN·m (i=1, 2, ..., 11)

(a) Beam mechanism

(b) Side sway mechanism

(c) Combined mechanism

Fig. 4 Collapse modes of a three bay frame structure

5. Reliability Analysis of Column Subjected to Combined Loads

By using the proposed method, reliability analysis is carried out for a column subjected to a combined load effect of a bending moment and an axial force as shown in Fig. 5. The critical section to fail is assumed to be the root of the column. The basic random variables are the horizontal load P_1, the vertical load P_2, and the yield stress Cy of the column. The form of the cross section is rectangular. Then the fully plastic bending moment capacity Mp and the fully plastic axial force capacity Np are given by

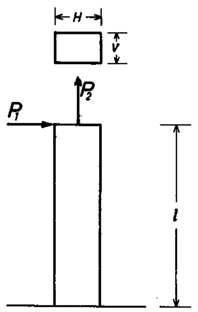


Fig. 5 A column subjected to combined loads

$$M_p = Z_p C_y, \ N_p = A_p C_y \tag{22}$$

where $Z_p = 1/4 \cdot VH^2$, and $A_p = VH$

The applied bending moment M and the axial force N are $M = P_1 \cdot l$ and $N = P_2$, respectively. The failure criterion under the combined load effect is often taken as

$$Z = 1 - \left|\frac{M}{M_p}\right| - \left(\frac{N}{N_p}\right)^2 \tag{23}$$

A linearized criterion is also proposed[8]:

$$Z = 1 - \left|\frac{M}{M_p}\right| - \left|\frac{N}{N_p}\right| \tag{24}$$

The effect of the failure criteria are examined on the resulting probabilities of failure. The mean and coefficient of variation of the yield stress are fixed to be $Cy = 245$ MPa and $CV_{Cy} = 0.1$ while the mean of the horizontal and vertical forces are changed as explained below with their coefficients of variation fixed to be 0.1. The central safety factor is defined here by

$$SF = \frac{1}{|\bar{M}/\bar{M}_p| + |\bar{N}/\bar{N}_p|} \tag{25}$$

The relative contribution of the bending moment and the axial force to the failure is evaluated by the ratio:

$$R = |\bar{M}/\bar{M}_p| / |\bar{N}/\bar{N}_p| \tag{26}$$

Table 1 Failure probabilities for the collapse modes of a three bay frame structure

IPD (3, 4) , CV (0.3, 0.3)

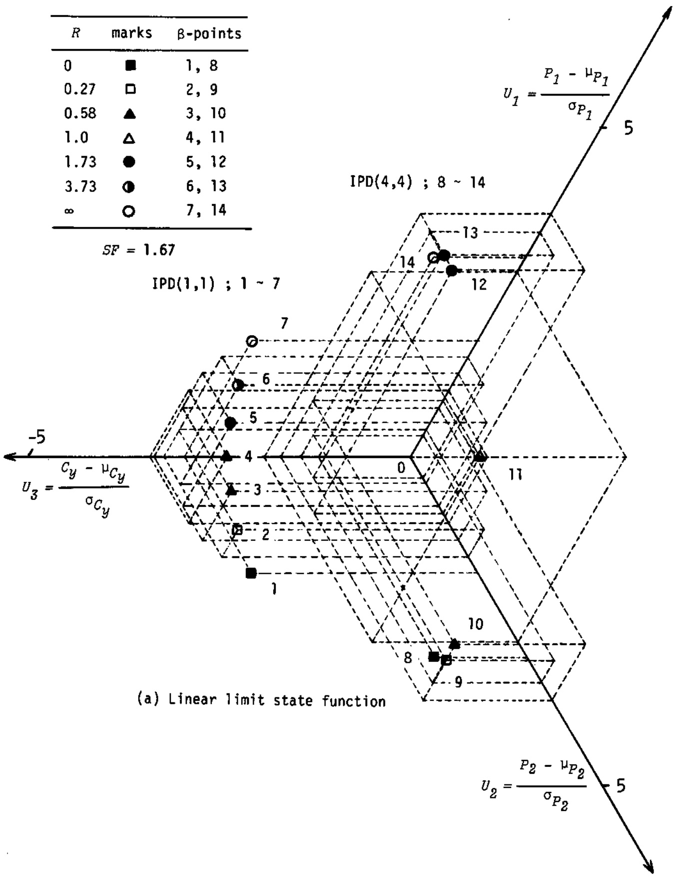
Limit state function	Method	Pf
(a) Beam mechanism	Proposed method	2.0718×10^{-2}
$Z=M_2+2M_3+M_4-(l/2)L_2$	R–F method	2.0718×10^{-2}
	C–L method	2.0828×10^{-2}
(b) Side sway mechanism	Proposed method	1.1440×10^{-4}
$Z=M_1+M_2+M_4+M_5+M_7+M_8+M_{10}+M_{11}$	R–F method	1.1440×10^{-4}
$-hL_1$	C–L method	fail to converge
(c) Combined mechanism	Proposed method	1.6565×10^{-4}
$Z=M_1+2M_3+2M_4+M_5+M_7+M_8+M_{10}+M_{11}$	R–F method	3.6607×10^{-4}
$-hL_1-(l/2)L_2$	C–L method	fail to converge

Table 2 Failure probabilities for the various values of *SF* and *R*

				IPD (1,1)		IPD (4,4)	
				Linear	Non-linear	Linear	Non-linear
SF	R	P_1(kN)	P_2(kN)	Pf	Pf	Pf	Pf
	0	0.0	1666.00	5.917×10^{-2}	5.917×10^{-2}	5.159×10^{-2}	5.159×10^{-2}
	0.27	0.885	1312.22	4.697×10^{-2}	9.119×10^{-3}	3.232×10^{-2}	6.794×10^{-3}
	0.58	1.525	1055.46	4.219×10^{-2}	1.944×10^{-3}	2.248×10^{-2}	8.479×10^{-4}
1.25	1.0	2.083	833.00	4.086×10^{-2}	6.761×10^{-4}	1.901×10^{-2}	8.311×10^{-5}
	1.73	2.640	610.15	4.219×10^{-2}	5.918×10^{-4}	2.248×10^{-2}	1.749×10^{-4}
	3.73	3.280	354.07	4.697×10^{-2}	2.527×10^{-3}	3.232×10^{-2}	2.179×10^{-3}
	∞	4.165	0.00	5.917×10^{-2}	5.917×10^{-2}	5.159×10^{-2}	5.159×10^{-2}
	0	0.0	1249.50	3.018×10^{-4}	3.018×10^{-4}	4.705×10^{-4}	4.705×10^{-4}
	0.27	0.661	984.90	1.639×10^{-4}	1.671×10^{-5}	9.100×10^{-5}	1.744×10^{-5}
	0.58	1.145	791.35	1.249×10^{-4}	2.080×10^{-6}	1.768×10^{-5}	5.016×10^{-7}
1.67	1.0	1.562	624.75	1.156×10^{-4}	5.746×10^{-7}	4.779×10^{-6}	6.369×10^{-9}
	1.73	1.978	458.15	1.249×10^{-4}	5.003×10^{-7}	1.768×10^{-5}	6.175×10^{-8}
	3.73	2.463	264.50	1.639×10^{-4}	3.217×10^{-6}	9.100×10^{-5}	4.850×10^{-6}
	∞	3.124	0.00	3.018×10^{-4}	3.018×10^{-4}	4.705×10^{-4}	4.705×10^{-4}
	0	0.0	833.00	1.267×10^{-8}	1.267×10^{-8}	4.024×10^{-8}	4.024×10^{-8}
	0.27	0.443	656.01	5.848×10^{-9}	5.341×10^{-10}	7.522×10^{-10}	1.384×10^{-10}
	0.58	0.765	526.85	4.242×10^{-9}	7.015×10^{-11}	1.065×10^{-11}	2.619×10^{-13}
2.50	1.0	1.041	416.50	3.882×10^{-9}	2.175×10^{-11}	1.419×10^{-13}	5.576×10^{-19}
	1.73	1.317	306.15	4.242×10^{-9}	1.898×10^{-11}	1.065×10^{-11}	1.901×10^{-14}
	3.73	1.640	176.99	5.848×10^{-9}	1.048×10^{-10}	7.522×10^{-10}	3.344×10^{-11}
	∞	2.083	0.00	1.267×10^{-8}	1.267×10^{-8}	4.024×10^{-8}	4.024×10^{-8}

$H = 0.10$ m, $V = 0.085$ m, $l = 10.0$ m
$C_y = 245$ MPa ; $CV_{P_1} = 0.1$, $CV_{P_2} = 0.1$, $CV_{C_y} = 0.1$

Table 2 lists the probabilities of failure for the various combinations of SF and R when all the basic random variables are distributed either normally or in Gumbel's form. As expected, it is seen that the numerical values due to the two criteria are identical when R is extremely large or small, *i.e.*, the bending moment or the axial force is dominant. In-between, the linear failure criterion yields a conservative estimation of the failure probability. Figs. 6 (a) and (b) plot in the U-space the β-points corresponding to the various values of R with SF = 1.67, which shows that the best linearization points are dependent on the probability distribution.



(a) Linear limit state function

Fig. 6 β-points in U-space

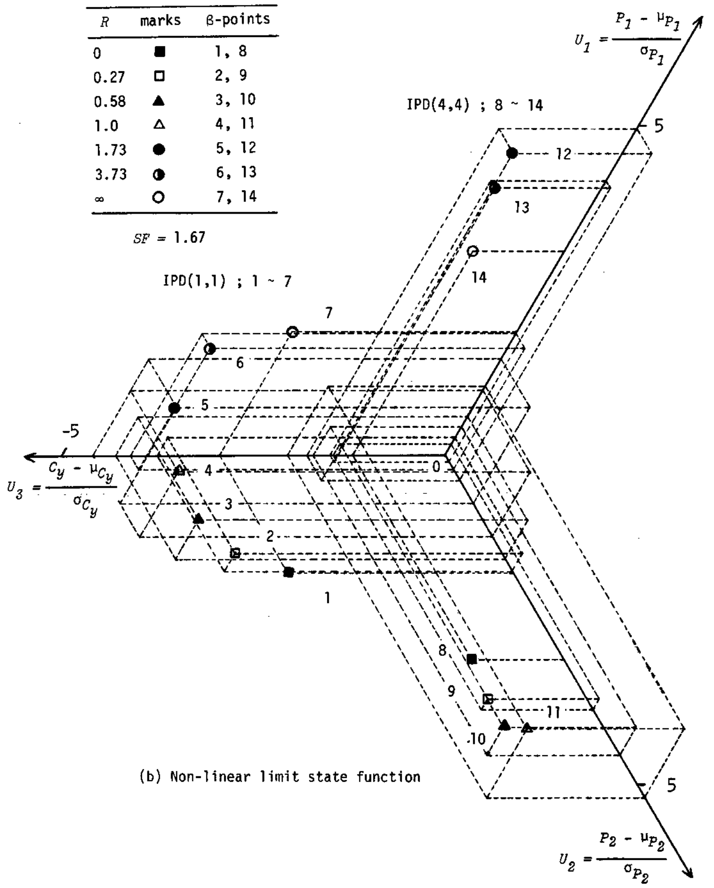


Fig. 6 β-points in U-space

6. Conclusion

The new method has been proposed for seaching β-point in the AFOSM method, by applying the extended Lagrangean function. It is compared with the methods by Rackwitz-Fiessler and Chen-Lind through numerical examples. It is shown that the three methods yield the similar results. Finally, the proposed method is successfully applied to the reliability analysis of a column subjected to a combined load effect of the bending moment and the axial force to examine the effect of the linear and non-linear failure criteria on the resulting failure probability. It is shown that the linear criterion gives a conservative estimate of failure probability particularly when the relative contributions of the bending moment and the axial force are competitive.

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