On the Periodic Solutions of Nonautonomous Piecewise Linear Systems with Asymmetric Characteristics

On the Periodic Solutions of Nonautonomous Piecewise Linear Systems with Asymmetric Characteristics

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This is a study of periodic solutions of a nonautonomous second order differential equation with piecewise linear restoring force having asymmetric characteristics, which has been studied very little, in the case of no damping.

In this report the simple periodic solutions like harmonic oscillation are analyzed, and the periodicity conditions including the initial conditions and also the stability conditions are obtained.

1. Introduction

The present paper deals with the periodic solutions of certain second order nonlinear differential equations which are not small perturbation of linear equations or of autonomous equations.

The differential equation

$$\ddot{x}+c\dot{x}+f(x)=F\cos\omega t\,, \tag{1}$$

in which $f(x)$ is nonlinear, occurs in several different kinds of physical problems, where here and throughout this paper dots over a quantity refer to differentiation with respect to the time. The obvious example is the pendulum with an external force applied. The problem of finding the forced oscillation of a single mass subjected to an elastic restoring force leads in general to equation (1) if the amplitude of the motion is not kept small. The series ferroresonance circuit containing iron core inductances also leads to equation (1). The problem of hunting of synchronous electrical machinery is still another example of physical problem which leads to the same equation.

In this paper piecewise linear systems subjected to the differential equation of type equation (1) were studied because certain calculations can be made explicitly. But a certain of the results will carry over to more general equations[1].

It is well-known that periodic solutions will be obtained by connecting the solutions in each interval smoothly at every corner point and giving the conditions of periodicity of equation[2], however the theory of the periodicity coditions including the initial conditions has not yet been established.

This paper will first give the periodicity conditions containing the initial conditions and the theory of the stability in relation to periodic solutions mentioned

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above.

Numerical results of regions in which periodic solutions, such as harmonic and subharmonic oscillations, occur and frequency response which means the relation between the amplitude of the oscillation and the driving frequency show many peculiarities that can not be explained by conventional linear theory.

Work on piecewise linear systems has been done by Maezawa[3]. His approach is via Fourier series. Other work on piecewise linear systems has been done by Loud[4]. His attention is directed to the branching phenomena of the symmetrical system.

2. Periodicity Conditions

In this chapter we will discuss the periodic solutions, type called ${}_1A$ shown in Fig. 2, of second order scalar differential equation of which restoring force is a piecewise linear function shown in Fig. 1, which are connected smoothly only twice in one period.

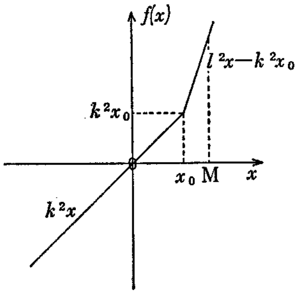


Fig. 1. Restoring force characteristics.

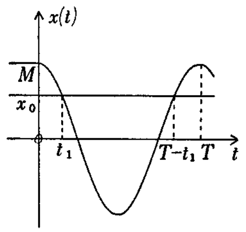


Fig. 2. Periodic solution of type ${}_1A$.

Consider the equation

$$\ddot{x}+f(x)=E\cos\omega t\,, \tag{2}$$

where $f(x)$ is a piecewise linear restoring term given by

$$\left.\begin{aligned} f(x)&=\begin{cases} l^2x-K^2x_0 & (x\geqq x_0)\,, \\ k^2x & (x\leqq x_0)\,, \end{cases} \\ l^2&=k^2+K^2. \end{aligned}\right\} \tag{3}$$

Since the equation (2) has no damping, the steady state periodic solutions are either in phase with the impressed force or 180° out of phase with it.

Then assume the initial conditions as follows,

$$\left.\begin{aligned} x(0)&=M(>x_0)\,, \\ \dot{x}(0)&=0\,. \end{aligned}\right\} \tag{4}$$

The conditions of (4) clearly indicate that the nonlinearity of the problem really has some effect on the solutions and also the solutions of equation (2) have symmetry with respect to $t=T/2$, where T means period of the solution and $T=2p\pi/\omega$ (p: positive integer).

Let $x^0(t)$ be periodic solution of equation (2), then conditions (5) hold:

$$\left.\begin{aligned} &x^0(t)=x^0(T-t)\,, \\ &\dot{x}^0(t)=-\dot{x}^0(T-t)\,. \end{aligned}\right\} \tag{5}$$

Throughout this paper the region $x \geqq x_0$ is called the domain I and the other region $x \leqq x_0$ is called the domain II. The equation governing the domain I is

$$\ddot{x}+l^2x-K^2x_0=E\cos\omega t\,, \tag{6}$$

and the second domain $x \leqq x_0$ is expressed by

$$\ddot{x}+k^2x=E\cos\omega t\,, \tag{7}$$

where $l\neq\omega$ and $k\neq\omega$.

To get conditions for the required periodic solutions, equation (6) must be solved under the conditions (4). Then the solution $x_1(t)$ is

$$x_1(t)=\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\cos lt+\frac{E}{l^2-\omega^2}\cos\omega t+\frac{K^2}{l^2}x_0 \tag{8}$$

and

$$\dot{x}_1(t)=-l\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\sin lt-\frac{\omega E}{l^2-\omega^2}\sin\omega t\,. \tag{9}$$

Suppose that $x_1(t)$ reaches x_0 first at $t=t_1$, then the following relations will be obtained:

$$x_1(t)>x_1\,, \quad 0\leqq t<t_1\,, \tag{10}$$

$$x_1(t_1)=\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\cos lt_1+\frac{E}{l^2-\omega^2}\cos\omega t_1+\frac{K^2}{l^2}x_0=x_0\,, \tag{11}$$

$$\dot{x}_1(t_1)=-l\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\sin lt_1-\frac{\omega E}{l^2-\omega^2}\sin\omega t_1\,, \tag{12}$$

$$(\;=-y_1)\,,$$

and in order that this solution is connected with a solution in the domain II, it must be that

$$\dot{x}_1(t_1)<0 \quad \textit{i.e.,} \quad y_1>0\,. \tag{13}$$

Therefore if the above solution $x_1(t)$ is connected with the solution of the second domain smoothly at $t=t_1$, from equation (7) the solution $x_2(t)$ in $x\leqq x_0$ is

$$\begin{aligned} x_2(t)=&\left(x_0-\frac{E}{k^2-\omega^2}\cos\omega t_1\right)\cos k(t-t_1)+\frac{E}{k^2-\omega^2}\cos\omega t \\ &+\frac{1}{k}\left(-y_1+\frac{\omega E}{k^2-\omega^2}\sin\omega t_1\right)\sin k(t-t_1)\,, \end{aligned} \tag{14}$$

and

$$\dot{x}_2(t) = -k\left(x_0 - \frac{E}{k^2-\omega^2}\cos\omega t_1\right)\sin k(t-t_1) - \frac{\omega E}{k^2-\omega^2}\sin\omega t + \left(-y_1 + \frac{\omega E}{k^2-\omega^2}\sin\omega t_1\right)\cos k(t-t_1)\,. \tag{15}$$

The necessary condition in order that the periodic solutions sought in equation (2) are ${}_1A$ type shown in Fig. 2 becomes

$$x_2(t) < x_0\,,\quad t_1 < t < T - t_1\,. \tag{16}$$

And moreover the necessary and sufficient conditions to get the periodic solutions by connecting again $x_2(t)$ at $t = T - t_1$ with the solution in $x \geqq x_0$ are derived from equation (5) and are

$$\left.\begin{aligned} &x_2(t_1) = x_2(T-t_1) = x_0\,,\\ &\dot{x}_2(t_1) = -\dot{x}_2(T-t_1) = -y_1\,. \end{aligned}\right\} \tag{17}$$

By setting in (14)

$$\left.\begin{aligned} &A = \sqrt{\left(x_0 - \frac{E}{k^2-\omega^2}\cos\omega t_1\right)^2 + \left\{\frac{1}{k}\left(-y_1 + \frac{\omega E}{k^2-\omega^2}\sin\omega t_1\right)\right\}^2}\,,\\ &\tan\varphi = \frac{-y_1 + \dfrac{\omega E}{k^2-\omega^2}\sin\omega t_1}{k\left(x_0 - \dfrac{E}{k^2-\omega^2}\cos\omega t_1\right)}\,, \end{aligned}\right\} \tag{18}$$

we obtain in place of equations (17) the following conditions:

$$\left.\begin{aligned} &A\cos\varphi = A\cos(kT - 2kt_1 - \varphi)\,,\\ &kA\sin\varphi = kA\sin(kT - 2kt_1 - \varphi)\,. \end{aligned}\right\} \tag{19}$$

There are two cases, based on different assumptions as to the nature of the amplitude A, which are of interest. These will be considered in the next articles.

(i) In case $A=0$

Assuming $A=0$, from equations (18) there result

$$x_0 = \frac{E}{k^2-\omega^2}\cos\omega t_1\,, \tag{20}$$

$$y_1 = \frac{\omega E}{k^2-\omega^2}\sin\omega t_1\,. \tag{21}$$

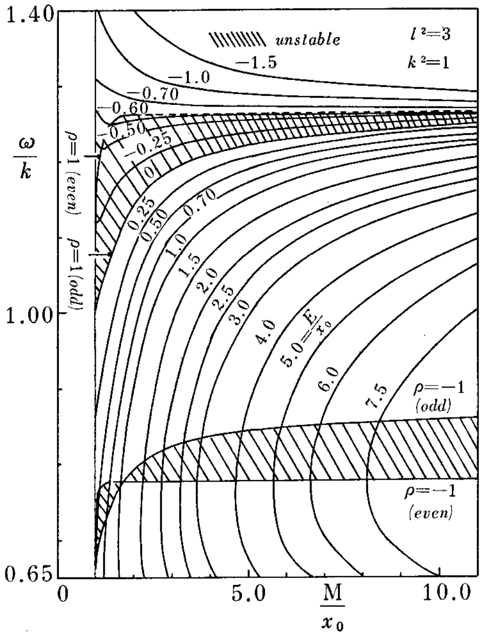
Substituting equations (20) and (21) into equations (11) and (12) there results

$$\omega\tan lt_1 = l\tan\omega t_1\,. \tag{22}$$

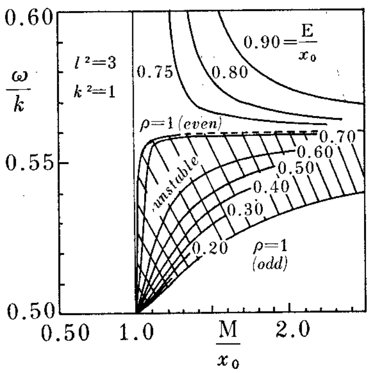
Then the conditions for the solution of equation (2) to be periodic under the initial conditions (4) are written as follows:

$$\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\cos lt_1+\frac{E}{l^2-\omega^2}\cos\omega t_1+\frac{K^2}{l^2}x_0=x_0\,,$$

$$x_0=\frac{E}{k^2-\omega^2}\cos\omega t_1,\qquad \omega\tan lt_1=l\tan\omega t_1\,.$$



(a)



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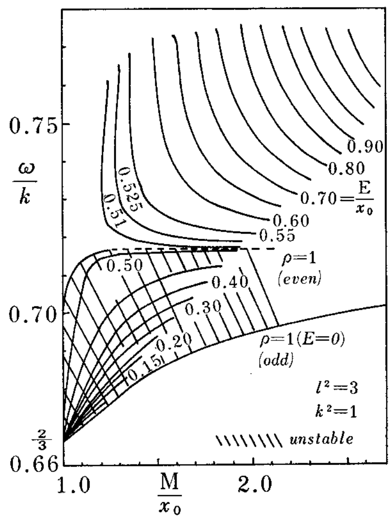


(c)

Fig. 3. Regions in which periodic solutions occur.
(a) Harmonic solutions. (b) Ultraharmonic solutions.
(c) Subharmonic solutions of order 3/2.

In this case due to the results of numerical analysis the harmonic oscillation domain is narrow and exists in the neighbourhood of corner point, so that an example is shown in case of $l^2=3$, $k^2=1$, $\omega=0.5$, $M/x_0=1.000038$ and $E/x_0=0.7500288$.

(ii) In case $A\neq 0$

Equations (19) become

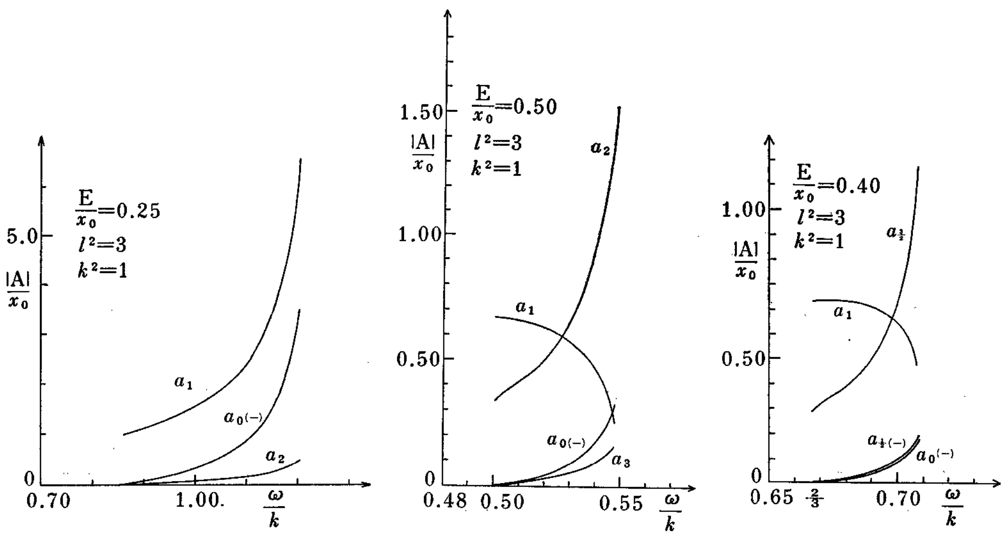
$$\left.\begin{aligned}\tan\varphi&=\tan k\left(\frac{T}{2}-t_1\right)\\&=\frac{-l\left(M-\dfrac{E}{l^2-\omega^2}-\dfrac{K^2}{l^2}x_0\right)\sin lt_1+\dfrac{\omega K^2E}{(k^2-\omega^2)(l^2-\omega^2)}\sin\omega t_1}{k\left(x_0-\dfrac{E}{k^2-\omega^2}\cos\omega t_1\right)}.\end{aligned}\right\}\quad(23)$$

In case $A\neq 0$ the conditions of periodicity are as follows:

$$\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\cos lt_1+\frac{E}{l^2-\omega^2}\cos\omega t_1+\frac{K^2}{l^2}x_0=x_0\,,$$

$$\tan k\left(\frac{T}{2}-t_1\right)=\frac{-l\left(M-\dfrac{E}{l^2-\omega^2}-\dfrac{K^2}{l^2}x_0\right)\sin lt_1+\dfrac{\omega K^2E}{(k^2-\omega^2)(l^2-\omega^2)}\sin\omega t_1}{k\left(x_0-\dfrac{E}{k^2-\omega^2}\cos\omega t_1\right)}.$$

These equations mean the relation among the four elements, initial value M, amplitude of the external force, E, angular frequency of the external force, ω, and transition



(a) (b) (c)

Fig. 4. Frequency responses.
(a) Harmonic solution. (b) Ultraharmonic solution.
(c) Subharmonic solution of order 3/2.

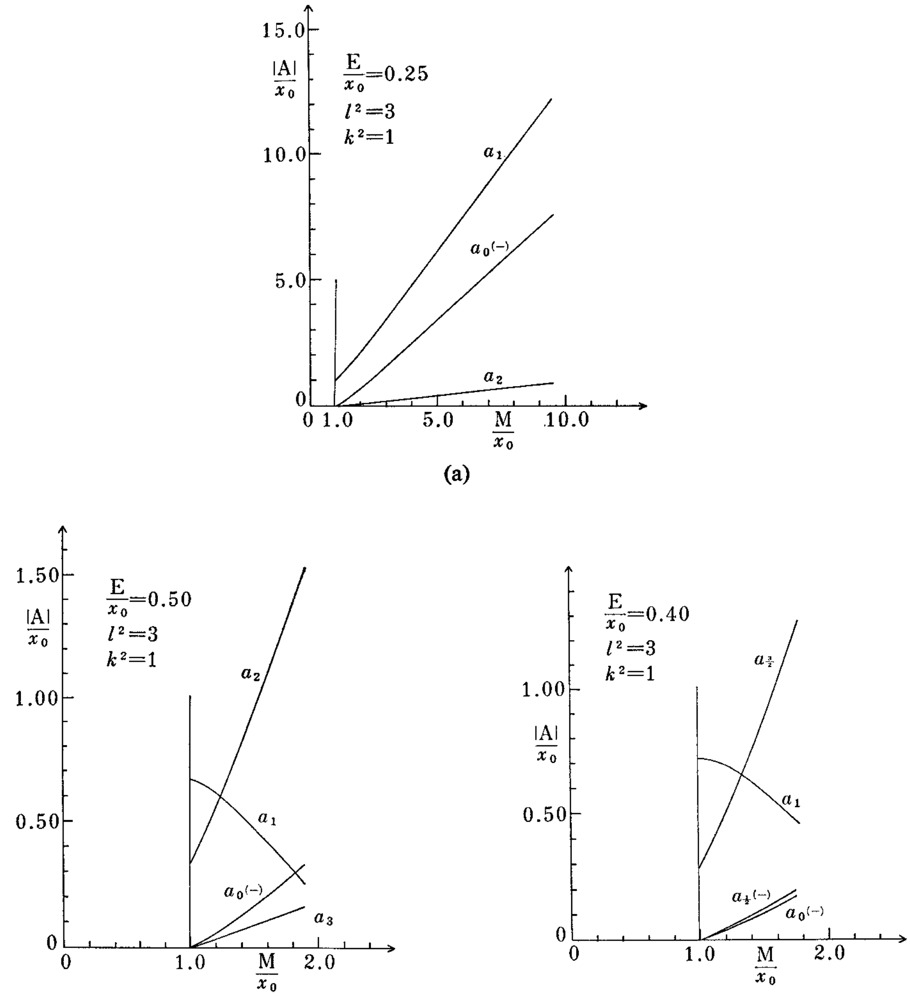


Fig. 5. Initial value responses.
(a) Harmonic solution. (b) Ultraharmonic solution.
(c) Subharmonic solution of order 3/2.

time t_1 (the time when the solution $x_1(t)$ reaches the corner point x_0 first). And if two of them are known, the remaining elements are obtained, that is to say, for given E and M angular frequency ω and transition time t_1 which lead to periodic solutions will be easily found. Also if E and ω are given, we have the initial value M and transition time t_1 to construct periodic solutions.

In Fig. 3 a part of the regions of periodic solutions is shown by the aid of numerical analysis, where $k^2=1$ and $l^2=3$. Frequency response examples are shown in Fig. 4 in case of $k^2=1$ and $l^2=3$. Fig. 5 illustrates initial value response between the initial value M and the amplitude of the periodic solution when $k^2=1$ and $l^2=3$.

3. Stability

Let $x^0(t)$ be the periodic solution of equation (2) with period $T=2p\pi/\omega$ under the initial conditions (4), then the discussion of the stability of the periodic solution of equation (2) depends on the variational equation associated with $x^0(t)$ (Fig. 6). The variational equation of equation (2) associated with the periodic solution $x^0(t)$ for the variation $y(t)$ is

$$\ddot{y}+a(t)y=0, \tag{24}$$

where $a(t)$ is an even and T-periodic function which is given by the formula

$$a(t)\equiv\frac{\partial f(x)}{\partial x}\bigg|_{x=x^0(t)}=\begin{cases} l^2 & (x^0(t)>x_0), \\ k^2 & (x^0(t)<x_0). \end{cases} \tag{25}$$

Therefore equation (24) means a Hill's equation. Then the conditions that equation (24) has periodic solutions are generally obtained by the Fourier series expansion and in this chapter the same procedure as in Chapter 2 is used since the coefficient $a(t)$ has the characteristics shown in Fig. 7.

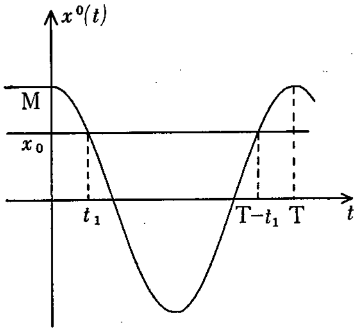


Fig. 6. Periodic solution $x^0(t)$.

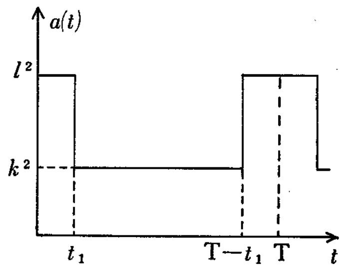


Fig. 7. Coefficient $a(t)$.

In what follows $\varphi(t)$ refers to even function and $\psi(t)$ odd function. Here we will give the each interval solutions of equation (24).

Case $0\leqq t<t_1$ $(x^0(t)>x_0)$

From equation (24) in this case we have

$$\ddot{y}+l^2y=0. \tag{26}$$

Let $\varphi_1(t)$ and $\psi_1(t)$ be solutions of equation (26) with the initial conditions $\varphi_1(0)=\dot{\psi}_1(0)=1$, $\dot{\varphi}_1(0)=\psi_1(0)=0$, then

$$\varphi_1(t)=\cos lt, \tag{27}$$

$$\psi_1(t)=\frac{1}{l}\sin lt. \tag{28}$$

Case $t_1 < t < T-t_1$ $(x^0(t) < x_0)$

From equation (24) in the same way as above

$$\ddot{y}+k^2y=0\,. \tag{29}$$

Let $\varphi_2(t)$ and $\psi_2(t)$ be solutions of equation (29) obtained by connecting smoothly $\varphi_1(t)$ and $\psi_1(t)$ at $t=t_1$, then we obtain

$$\varphi_2(t)=\cos lt_1 \cos k(t-t_1)-\frac{l}{k}\sin lt_1 \sin k(t-t_1)\;, \tag{30}$$

$$\dot{\varphi}_2(t)=-k\cos lt_1 \sin k(t-t_1)-l\sin lt_1 \cos k(t-t_1)\,, \tag{31}$$

$$\psi_2(t)=\frac{1}{l}\sin lt_1 \cos k(t-t_1)+\frac{1}{k}\cos lt_1 \sin k(t-t_1)\,, \tag{32}$$

$$\dot{\psi}_2(t)=-\frac{k}{l}\sin lt_1 \sin k(t-t_1)+\cos lt_1 \cos k(t-t_1)\,. \tag{33}$$

From above discussion we have following four cases for periodic solutions.

Case I $\varphi(t)$ T-periodic

The following condition can be obtained,

$$\dot{\varphi}_2\left(\frac{T}{2}\right)=0\,,$$

that is,

$$k\cos lt_1 \sin k\left(\frac{T}{2}-t_1\right)+l\sin lt_1 \cos k\left(\frac{T}{2}-t_1\right)=0\,. \tag{34}$$

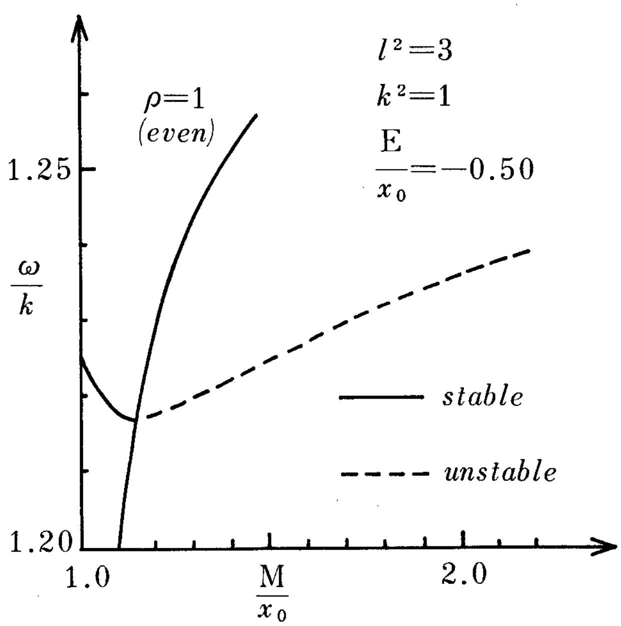


Fig. 8. Jump phenomena.

This equation implies the jump phenomena shown in Fig. 8.

Case II $\psi(t)$ T-periodic

In this case

$$\psi_2\left(\frac{T}{2}\right)=0\ ,$$

so that

$$k\sin lt_1\cos k\left(\frac{T}{2}-t_1\right)+l\cos lt_1\sin k\left(\frac{T}{2}-t_1\right)=0\ . \tag{35}$$

This comprises the relation $E=0$. It follows from equations (34) and (35) that if cases I and II occur together, there exist nonnegative numbers n_1 and n_2, of same parity, such that

$$\frac{n_1}{k}+\frac{n_2}{l}=\frac{2}{\omega}\ . \tag{36}$$

Case III $\varphi(t)$ $2T$-periodic

$$\varphi_2\left(\frac{T}{2}\right)=0\ ,$$

that is,

$$k\cos lt_1\cos k\left(\frac{T}{2}-t_1\right)-l\sin lt_1\sin k\left(\frac{T}{2}-t_1\right)=0\ . \tag{37}$$

Case IV $\psi(t)$ $2T$-periodic

$$\dot{\psi}_2\left(\frac{T}{2}\right)=0\ ,$$

$$-k\sin lt_1\sin k\left(\frac{T}{2}-t_1\right)+l\cos lt_1\cos k\left(\frac{T}{2}-t_1\right)=0\ . \tag{38}$$

If cases III and IV occur together, from equations (37) and (38) there exist nonnegative integer n_1 and n_2, of opposite parity, such that

$$\frac{n_1}{k}+\frac{n_2}{l}=\frac{2}{\omega}\ . \tag{39}$$

Let ρ_1 and ρ_2 be the characteristic multipliers of equation (24), then

$$\left.\begin{aligned}
&\rho_1\rho_2=1,\\
&\rho_1+\rho_2=2\left\{\varphi_2\left(\frac{T}{2}\right)\dot{\psi}_2\left(\frac{T}{2}\right)+\dot{\varphi}_2\left(\frac{T}{2}\right)\psi_2\left(\frac{T}{2}\right)\right\}\\
&\qquad=2\left\{\cos 2lt_1\ \cos 2k\left(\frac{T}{2}-t_1\right)-\frac{1}{2}\left(\frac{l}{k}+\frac{k}{l}\right)\sin 2lt_1\ \sin 2k\left(\frac{T}{2}-t_1\right)\right\}\ .
\end{aligned}\right\} \tag{40}$$

Furthermore the periodic solutions are stable when $|\rho_1+\rho_2|<2$ and unstable for

$|\rho_1+\rho_2|>2$. The condition $|\rho_1+\rho_2|=2$ means the border line of stable and unstable regions and has a connection with branching phenomena. The results obtained in this chapter are also shown in Fig. 3.

4. Periodicity Conditions for Special Case

In the equations (6) and (7) the conditions $\omega=l$ and $\omega=k$ lead the secular term in the each interval solution and so they say the periodic solutions do not occur[5].

But from the foregoing analyses following facts are obvious:

1) The periodicity conditions can be found.

2) According to the analogue and numerical results, the existence of the periodic solutions has been confirmed.

4.1 Case $\omega=l$

Assuming the special case $\omega=l$, equation (6) yields

$$\ddot{x}+l^2x-k^2x_0=E\cos lt. \tag{40}$$

Under the conditions (4) we get the solution $x_1(t)$ having secular term, $t\sin lt$, but in the same way for the periodic solutions as in Chapter 2 the conditions of periodicity become in general as follows:

$$\left(M-\frac{k^2}{l^2}x_0\right)\cos lt_1+\frac{E}{2l}t_1\sin lt_1+\frac{k^2}{l^2}x_0=x_0\,, \tag{42}$$

$$\tan k\left(\frac{T}{2}-t_1\right)=\frac{-l\left(M-\frac{k^2}{l^2}x_0\right)\sin lt_1+\frac{E}{2}t_1\cos lt_1+\frac{(k^2+l^2)E}{2l(k^2-l^2)}\sin lt_1}{k\left(x_0-\frac{E}{k^2-l^2}\cos lt_1\right)}\,. \tag{43}$$

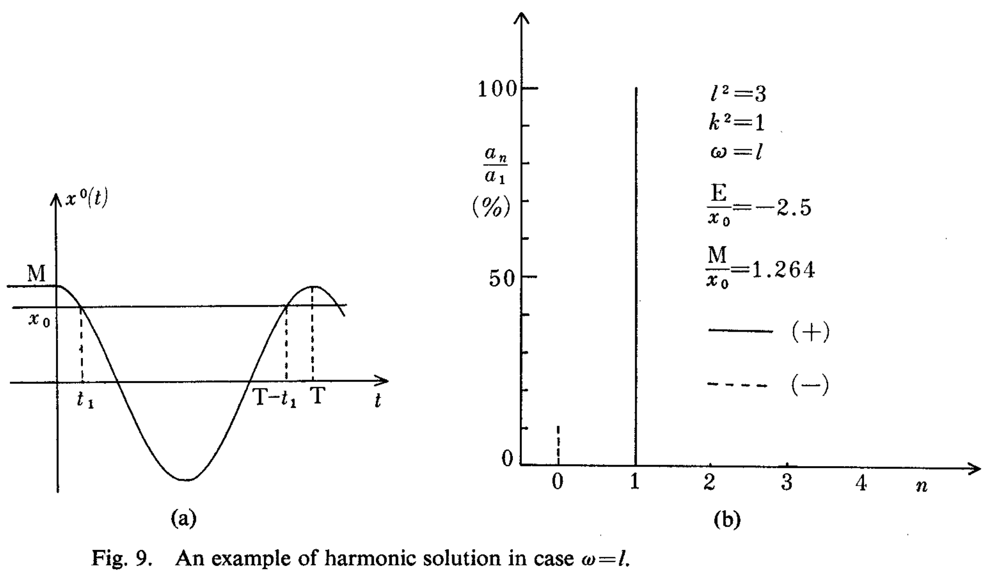


Fig. 9. An example of harmonic solution in case $\omega=l$.
(a) Harmonic solution by analogue computer. (b) Harmonic analysis.

As an example the harmonic solution obtained by using equations (42) and (43) is shown in Fig. 9, where $k^2=1$, $l^2=3$, $M/x_0=1.264$ and $E/x_0=-2.5$.

4.2 Case $\omega=k$

Using the conditions and the same procedure as the case $\omega=l$, the periodicity conditions are reduced to the forms:

$$\left(M-\frac{E}{l^2-k^2}-\frac{K^2}{l^2}x_0\right)\cos lt_1+\frac{E}{l^2-k^2}\cos kt_1+\frac{K^2}{l^2}x_0=x_0\,, \tag{44}$$

$$(-1)^p A_2 \sin(kt_1+\varphi_2)=\frac{p\pi E}{2k^2}\,, \tag{45}$$

where

$$y_1=l\left(M-\frac{E}{l^2-k^2}-\frac{K^2}{l^2}x_0\right)\sin lt_1+\frac{kE}{l^2-k^2}\sin kt_1\,,$$

$$A_2=\sqrt{\left(x_0-\frac{E}{2k}t_1\sin kt_1\right)^2+\left\{\frac{1}{k}\left(y_1+\frac{E}{2k}\sin kt_1+\frac{E}{2}t_1\cos kt_1\right)\right\}^2}\,,$$

$$\tan\varphi_2=-\frac{y_1+\dfrac{E}{2k}\sin kt_1+\dfrac{E}{2}t_1\cos kt_1}{k\left(x_0-\dfrac{E}{2k}t_1\sin kt_1\right)}$$

and period $T=2p\pi/\omega$. The results for $k^2=1$, $l^2=3$, $M/x_0=2.478$ and $E/x_0=1.0$ are shown in Fig. 10.

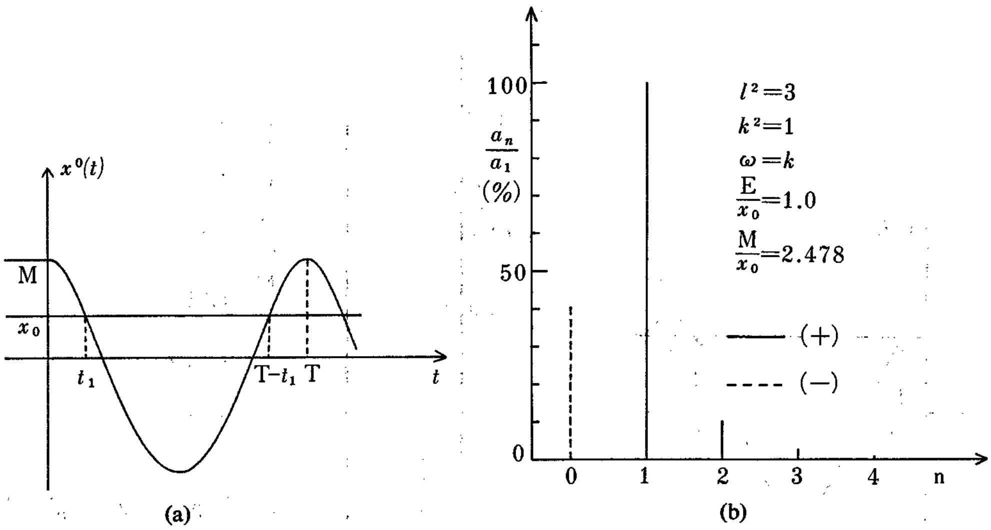


Fig. 10. An example of harmonic solution in case $\omega=k$.
(a) Harmonic solution by analogue computer.
(b) Harmonic analysis.

5. Conclusions

In this paper the periodicity conditions are dealt with in regard to the periodic solutions passing over the corner point twice in one period in asymmetric systems with piecewise linear restoring force, which have been studied very little. And the conditions of periodicity, general cases and special cases, including the initial conditions are found. It must be emphasized that from the conditions mentioned above periodic solutions, harmonic, ultraharmonic oscillations and so on, can be obtained easily by numerical calculations and the harmonic components of the periodic solutions are known exactly according to the initial conditions.

Hereafter the more general solutions than in this paper will be derived, that is to say, the periodic solutions will be distinguished in two cases according as the number of connection of solutions in half period and the branching phenomena in connection with subharmonics will be analyzed.

Finally it is noted that numerical calculations were performed by using TOSBAC—5600 Model 120 at the computer center, University of Osaka Prefecture.

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