

## On height functions on Jacobian surfaces

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# On height functions on Jacobian surfaces

By *Kentaro Yoshitomi*

## INTRODUCTION

First we recall some fundamental facts on the canonical height function on an abelian variety. For an algebraic number field  $K$ , let  $A$  be an abelian variety defined over  $K$ . Take a very ample divisor  $D$ . Then we have the projective embedding  $\phi_D: A \rightarrow \mathbf{P}^n$ . Using this embedding, we define the logarithmic height  $h_D$  on  $A$  by  $h \circ \phi_D$ , where  $h$  is the logarithmic height on  $\mathbf{P}^n$  (see [16], Chapter 3). The canonical height  $\hat{h}_D$  on  $A$  attached to  $D$  is defined by  $\hat{h}_D(z) = \lim_{n \rightarrow \infty} h_D(nz)/n^2$ . The function  $\hat{h}_D$  has a property that  $\hat{h}_D(z) \geq 0$  for any  $z \in A(\overline{K})$  and  $\hat{h}_D(z) = 0$  if and only if  $z$  is a torsion point. If  $D'$  is any ample divisor and  $mD'$  is very ample, then we define  $\hat{h}_{D'}$  by  $\frac{1}{m}\hat{h}_{mD'}$ . The height pairing  $\langle \cdot, \cdot \rangle_D$  is defined by  $\langle x, y \rangle_D = \hat{h}_D(x+y) - \hat{h}_D(x) - \hat{h}_D(y)$ . The regulator, which is an important factor of the Birch-Swinnerton-Dyer Conjecture, is defined via this height pairing. In general, it is very difficult to compute the canonical height directly by definition. Néron and Tate have shown that the canonical height decomposes into canonical local heights. In the case of elliptic curves, the archimedean local height is expressed using a theta function and the non-archimedean local height is expressed in a simple form (see [26], Chapter VI).

In this paper we compute canonical heights on Jacobian surfaces attached to the theta divisor and, as an example, we shall verify the Birch-Swinnerton-Dyer Conjecture numerically for certain Jacobian surface. We use Néron's formula [14], p.332, which asserts that the computation of the height pairing on Jacobian variety reduces to that of Néron's symbol. Néron's symbol is decomposed into Néron's local pairings first introduced by Néron [21], Chapter 2. Néron's local pairing is defined via Green's function at archimedean places and via intersection theory on an arithmetic surface at non-archimedean places; see [14].

The main result of this paper is the relation between the archimedean canonical local height and archimedean Néron's local pairing. In the case of elliptic curves,

there exists the following relation between the canonical local height and Néron's local pairing at an archimedean place. Let  $E : y^2 = 4x^3 - g_2x - g_3$  be an elliptic curve defined over an algebraic number field  $K$ . Let  $v$  be an archimedean place of  $K$ . Let  $E \simeq \mathbf{C}/\Lambda$  and  $\wp(z)$  be the Weierstrass  $\wp$ -function relative to  $\Lambda$ . Let  $k(z)$  be the Klein function, that is,  $k(z) = \Delta(\Lambda)^{1/12} e^{-\frac{1}{2}z\eta(z)} \sigma(z)$ , where  $\Delta(\Lambda) = g_2^3 - 27g_3^2$ ,  $\sigma$  is the Weierstrass  $\sigma$ -function, and  $\eta$  is the quasi-period map associated to  $\Lambda$  (see [26], p.41 and p.465). For  $z \in \mathbf{C}$ , we denote by  $\tilde{z}$  its image in  $\mathbf{C}/\Lambda$ . Then  $\hat{\lambda}_v(\tilde{z}) = -\log |k(z)|$  is the archimedean canonical local height on  $E - \{O\}$ , where  $O$  is the origin of  $E$ . Néron's local pairing is defined via Green's function, which is, in this case, also expressed in terms of the Klein function. For any  $P \in E(\bar{K})$ , if we take, as a uniformizer,  $\frac{y}{2x^2}$  at  $O$  and its translation at  $P$ , then we have

$$\langle (P) - (O), (P) - (O) \rangle_v = 2(\hat{\lambda}_v(\tilde{z}_P) + \frac{1}{12} \log |\Delta(\Lambda)|),$$

where  $P = (\wp(z_P), \wp'(z_P))$ .

We shall generalize the relation above for the case of hyperelliptic curves of genus 2. That is : For an algebraic number field  $K$ , let  $C$  be a hyperelliptic curve of genus 2 defined by  $y^2 = f(x) = x^5 + a_1x^4 + \cdots + a_5 \in K[x]$  and  $\mathcal{B}$  be the set of finite Weierstrass points. Let  $J$  be the Jacobian variety of  $C$  and  $\Theta$  be the theta divisor of  $J$ . For a divisor  $D$  of degree 0 on  $C$ , we denote its image in  $J = \text{Pic}^0(C)$  by  $\bar{D}$ . For  $P \in C(\mathbf{C})$ , we denote the hyperelliptic integral from  $\infty$  to  $P$  by  $u^P \in \mathbf{C}^2$ , which is defined up to the period lattice  $\Lambda$ . For  $z \in \mathbf{C}^2$ , we denote its image in  $J = \mathbf{C}^2/\Lambda$  by  $\tilde{z}$ . Let  $\phi$  be the function as in Proposition 1.10. Let  $v$  be an archimedean place of  $K$ ,  $\hat{\lambda}_v$  be the canonical local height on  $J - \Theta$  which is normalized as in Definition 2.1 with the fixed function  $\phi$  as above, and  $\langle \cdot, \cdot \rangle_v$  be Néron's local pairing explicitly defined as in (2.5). Then we have (Theorem 2.18):

**Main Theorem** For  $P_i(x_i, y_i) \in C(K)$ , ( $i = 1, 2$ ), let  $b = P_1 - P_2$  with  $\bar{b} \notin \Theta$ , and  $z_b = u^{P_1} - u^{P_2} \in \mathbf{C}^2$ . As the base of tangent space at  $P_i$ , we take  $2y_i \frac{\partial}{\partial x} = f'(x_i) \frac{\partial}{\partial y}$ .

Then we can take  $\frac{x - x_i}{2y_i}$  if  $P_i \notin \mathcal{B}$  and  $\frac{y - y_i}{f'(x_i)}$  if  $P_i \in \mathcal{B}$  as a uniformizer at  $P_i$ . In both cases, for an archimedean place  $v$ , if we take the uniformizer as above, the relation

$$\langle b, b \rangle_v = 2 \hat{\lambda}_v(\tilde{z}_b)$$

between Néron's local pairing and the canonical local height holds.

We can compute the canonical local height at archimedean places numerically. In the case of elliptic curves, one can achieve this by evaluating a rapidly convergent series, which is called Tate's series [25]. In Call and Silverman [4], they generalized

Tate's series for a class of varieties with a divisor and a morphism which satisfy certain conditions, including higher dimensional abelian varieties. Thus we can use their series to evaluate  $\hat{\lambda}_v$  for archimedean places  $v$ . We shall give concrete expression of this series. In Grant [12], defining equations of Jacobian surfaces and the addition theorem are formulated by using the theory of hyperelliptic  $\mathfrak{p}$  functions which goes back to an old book of Baker [1]. To construct the generalized Tate's series, we must first find appropriate domains in the Jacobian variety. We can take three domains which are obtained by partitioning the Jacobian surface by three translations of the theta divisor. Then we can construct the generalized Tate's series explicitly via hyperelliptic  $\mathfrak{p}$  functions and compute the archimedean canonical local height numerically. By virtue of Theorem 2.18, we can compute Néron's local pairing at archimedean places. At non-archimedean places, we compute Néron's local pairing using intersection theory on an arithmetic surface, and hence we can compute the canonical height.

In section 1, we shall review some facts on Jacobian surfaces and hyperelliptic  $\mathfrak{p}$ -functions. In section 2, we shall give the explicit formula of Green's function using naturally generalized Klein function (Proposition 2.9). Using this formula, we shall prove Main Theorem. In section 3, we shall construct the generalized Tate's series in our case, using hyperelliptic  $\mathfrak{p}$ -functions. Finally, in section 4, we shall give some examples. Especially, we shall check the Birch-Swinnerton-Dyer Conjecture numerically.

Some algebraic computations are executed using the mathematical computing system Maple V. The Tate's series is computed using GNU g++ Ver 2.7.2 and LiDIA library 1.2.<sup>1</sup>

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#### NOTATION AND TERMINOLOGY

Throughout this paper, we use the following notation. By an algebraic number field, we understand a finite algebraic extension of  $\mathbf{Q}$  in  $\mathbf{C}$ . For an algebraic number field  $K$ , let  $\Sigma_K^\infty$  denote the set of infinite places of  $K$ ,  $\Sigma_K^0$  the set of finite places of  $K$ , and  $\Sigma_K = \Sigma_K^\infty \cup \Sigma_K^0$ . For  $v \in \Sigma_K^0$ , let  $K_v$  be the completion of  $K$  at  $v$ , let  $\pi_v$  be an uniformizer at  $v$ ,  $O_v$  the ring of  $v$ -adic integers,  $k_v = O_v/\pi_v O_v$ ,  $q_v$  the number of elements of  $k_v$ , and  $p_v$  the residual characteristic. As usual, for  $a \in K$  and  $v \in \Sigma_K$ , define:

$$|a|_v = \begin{cases} |a| & \text{if } v \in \Sigma_K^\infty \text{ and } v \text{ is a real place,} \\ |a|^2 & \text{if } v \in \Sigma_K^\infty \text{ and } v \text{ is a complex place,} \\ q_v^{-\text{ord}_{\pi_v}(a)} & \text{if } v \in \Sigma_K^0. \end{cases}$$

<sup>1</sup>This library may be available from anonymous ftp site: <ftp://crypt1.cs.uni-sb.de/pub/LiDIA>.

As is well known, the product formula  $\prod_{v \in \Sigma_K} |a|_v = 1$  holds. We also use additive notation  $v(a) = -\log |a|_v$  for every  $v \in \Sigma_K$ .

For any finite set  $S$ , we denote by  $\#S$  the cardinality of  $S$ . For any divisor  $a$ , we denote the support of  $a$  by  $\text{supp}(a)$ . For divisors  $a$  and  $a'$ , we write  $a \sim a'$  if  $a$  is linearly equivalent to  $a'$ . For a complex number, a complex vector, or a complex matrix  $x$ , we denote by  $\bar{x}$  its complex conjugate. For  $z \in \mathbf{C}^2$ , we denote by  $\tilde{z}$  its image in  $\mathbf{C}^2/\Lambda$  (see § 1.2).

## 1. REVIEW ON JACOBIAN SURFACES

We assume that the characteristic of a ground field  $K$  is not equal to 2; moreover, except for the section 1.1, we assume that  $K$  is a subfield of  $\mathbf{C}$ . Let  $C$  be a hyperelliptic curve of genus 2 over  $K$ , defined by the equation

$$(1.1) \quad y^2 = f(x) := x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = \prod_{i=1}^5 (x - \beta_i).$$

We consider  $C$  as a projective non-singular curve and we denote by  $\infty$  the point at infinity. The double covering  $C \rightarrow \mathbf{P}^1$ ,  $P(x, y) \mapsto x$  is branched over 5 finite points and  $\infty$ . We denote the set of finite Weierstrass points by  $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$ , where  $B_i = (\beta_i, 0)$ , ( $i = 1, \dots, 5$ ). Let  $P^\iota$  be the image of  $P$  under the hyperelliptic involution with respect to this covering, that is  $P^\iota = (x, -y)$  when  $P = (x, y)$ .

**1.1. Algebraic Theory.** We review the algebraic definition of the Jacobian variety  $J = \text{Jac}(C)$ . Let  $\text{Div}_0(C)$  be the divisor group of degree zero and  $J$  is the abelian variety whose points represent  $\text{Pic}_0(C)$ . For any  $D \in \text{Div}_0(C)$ , denote by  $\bar{D}$  its image in  $\text{Pic}_0(C)$ . For  $m$  points  $P_1, P_2, \dots, P_m \in C$ , put  $D(P_1, P_2, \dots, P_m) = P_1 + P_2 + \dots + P_m - m\infty \in \text{Div}_0(C)$ .

**Definition 1.1.** For  $m > 0$ , we define

$$\begin{aligned} \text{Div}_0^m(C) &= \{D(P_1, P_2, \dots, P_m) \mid P_1, \dots, P_m \in C\} \subset \text{Div}_0(C), \\ \text{Div}_0^{+,m}(C) &= \left\{ D(P_1, P_2, \dots, P_m) \mid \begin{array}{l} P_i \neq \infty \text{ for every } i, \\ P_i \neq P_j^\iota \text{ whenever } i \neq j \end{array} \right\} \subset \text{Div}_0^m(C), \end{aligned}$$

and denote their images in  $\text{Pic}_0(C)$  by  $\overline{\text{Div}_0^m(C)}$  and  $\overline{\text{Div}_0^{+,m}(C)}$ .

Then  $J = \overline{\text{Div}_0^2(C)}$  (see [19], pp.3.28–3.31). If we put  $\Theta = \overline{\text{Div}_0^1(C)}$ , which is the theta divisor of  $J$ , then we have  $J - \Theta = \overline{\text{Div}_0^{+,2}(C)}$  (loc.cit.). Hence any points of  $J$  can be written as  $\overline{D(P_1, P_2)}$  with  $P_1 \neq P_2^\iota$ ,  $P_i \neq \infty$ , or  $\overline{D(P)}$ , which belongs to the

theta divisor. The zero element  $O_J$  of  $J$  is  $\overline{D(\infty)} = \overline{D(P, P^\iota)}$ . All 2-torsion points of  $J$  are given by  $\overline{D(B_i)}$ , ( $i = 1, \dots, 5$ ) and  $\overline{D(B_i, B_j)}$ , ( $i, j = 1, \dots, 5, i \neq j$ ). We abbreviate these to  $\overline{B_i}$  and  $\overline{B_{ij}}$ .

Reduction of any divisor of degree 0 to the form  $\overline{D(P_1, P_2)}$  is explicitly given as follows.

Let  $P_1, P_2, P_3$  be three points of  $C$  and  $P_i = (x_i, y_i)$ , ( $i = 1, 2, 3$ ). For simplicity, we assume that the points  $P_i$  are finite, distinct and  $P_i \neq P_j^\iota$  for  $i \neq j$ . Then we can find the polynomial  $V(x)$  of degree 2 satisfying the equations  $V(x_1) = y_1$ ,  $V(x_2) = y_2$ ,  $V(x_3) = y_3$ . We write  $V_{P_1, P_2, P_3}$  for this  $V$ .

Then we define a rational function  $\tilde{V}$  on  $C$  by

$$\tilde{V}(x, y) = \frac{y + V(x)}{(x - x_1)(x - x_2)(x - x_3)},$$

which has poles at  $P_i$ , ( $i = 1, 2, 3$ ) and has a simple zero at  $\infty$ . Hence either  $\tilde{V}$  has zeros of order 1 at two finite points  $P_4, P_5$  or  $\tilde{V}$  has a zero of order 2 at one point  $P_4 = P_5$ .

That is,  $V(x_4) = -y_4$  and  $V(x_5) = -y_5$ , or  $V(x_4) = -y_4$ ,  $V'(x_4) = -\frac{dy}{dx} \Big|_{\substack{x=x_4 \\ y=y_4}}$ . Since  $x_i$ , ( $i = 1, \dots, 5$ ) is the solutions  $f(x) - V^2(x) = 0$ , we can find  $x_4$  and  $x_5$  and by the equations  $y_4 = -V(x_4)$ ,  $y_5 = -V(x_5)$ , we can find the coordinates of  $P_4, P_5$  which satisfy  $P_1 + P_2 + P_3 \sim P_4 + P_5 + \infty$ , that is  $\overline{D(P_1, P_2, P_3)} = \overline{D(P_4, P_5)}$ . For any divisor of degree 0, we can reduce it using the procedure above recursively. For the reduction algorithm, see Cantor [6].

**1.2. Analytic Theory.** For convenience of the reader, we review analytic theory of hyperelliptic integrals following [1].

First we take a basis  $\gamma_1, \gamma_2, \gamma'_1, \gamma'_2$  of the first homology group  $H_1(C, \mathbf{Z})$  with intersection numbers  $\gamma_1 \cdot \gamma_2 = \gamma'_1 \cdot \gamma'_2 = 0$ ,  $\gamma_i \cdot \gamma'_j = \delta_{ij}$  (Kronecker's  $\delta$ ). We take a basis of the differentials of the first kind,  $\mu_1 = \frac{dx}{2y}$ ,  $\mu_2 = \frac{x dx}{2y}$ , and write  $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ . We denote their periods by

$$\omega_{ij} = \int_{\gamma_j} \mu_i, \quad \omega'_{ij} = \int_{\gamma'_j} \mu_i, \quad (i, j = 1, 2).$$

It is well known that  $\tau = \omega^{-1}\omega'$  belongs to the Siegel upper half space  $\mathfrak{h}_2$ . If we define the period lattice  $\Lambda = \omega \mathbf{Z}^2 \oplus \omega' \mathbf{Z}^2$ , then  $J = \mathbf{C}^2 / \Lambda$  is the Jacobian variety of  $C$ .

We define a hyperelliptic integral  $u^{P,P_0} = \int_{P_0}^P \mu \in \mathbf{C}^2$ , and  $u^P = u^{P,\infty}$ . Let  $u_i^{P,P_0}$  denote the  $i$ -th coordinate of  $u^{P,P_0}$ , ( $i = 1, 2$ ). The integral  $u^{P,P_0}$  is determined up to  $\Lambda$ . For any divisor  $b = \sum_P m_P P$ , we denote the corresponding integral  $\sum_P m_P u^P$  by  $z_b$ . For any vector  $z \in \mathbf{C}^2$ , we denote by  $\tilde{z}$  its image in  $J = \mathbf{C}^2/\Lambda$ .

The map  $(P_1, P_2) \mapsto u^{P_1} + u^{P_2}$  becomes a surjection from the symmetric 2-product of  $C$   $S^2(C)$  to  $J$ . The image of  $\{(P, \infty) \in S^2(C)\}$  under this map is the theta divisor of  $J$ . Put  $\tilde{\Theta} = (\mathbf{C}^2 \rightarrow J)^*\Theta$ , that is the pullback of  $\Theta$  in  $\mathbf{C}^2$ .

Next let  $\zeta_1, \zeta_2$  be the differentials of the second kind on  $C$  defined by

$$\zeta_1 = \frac{(3x^3 + 2a_1x^2 + a_2x)dx}{2y}, \quad \zeta_2 = \frac{x^2 dx}{2y},$$

and define their periods  $\eta = (\eta_{ij})$ ,  $\eta' = (\eta'_{ij})$  by

$$\eta_{ij} = \int_{\gamma_j} \zeta_i, \quad \eta'_{ij} = \int_{\gamma'_j} \zeta_i, \quad (i, j = 1, 2).$$

We define an  $\mathbf{R}$ -linear map  $\tilde{\eta}: \mathbf{C}^2 \rightarrow \mathbf{C}^2$  by

$$\tilde{\eta}(u) = \eta r + \eta' r', \quad \text{where } u = \omega r + \omega' r', r, r' \in \mathbf{R}^2.$$

Between the periods, the following relation holds

$$(1.2) \quad \eta' = \eta \tau + 2\pi i^t \omega^{-1}.$$

Furthermore, we have

$$(1.3) \quad \eta \omega^{-1} \quad \text{is symmetric, which is equivalent to } \eta^t \eta' = \eta'^t \eta.$$

**1.3.  $\sigma$ -function.** As in [18], Chapter 2, for  $\tau \in \mathfrak{h}_2$ , we define a theta function on  $\mathbf{C}^2$  by

$$\theta(z, \tau) = \sum_{n \in \mathbf{Z}^2} \exp[\pi i^t n \tau n + 2\pi i^t n z],$$

and for  $a, b \in \mathbf{Q}^2$ ,

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) &= \exp(\pi i^t a \tau a + 2\pi i^t a (z + b)) \theta(z + \tau a + b) \\ &= \sum_{n \in \mathbf{Z}^2} \exp[\pi i^t (n + a) \tau (n + a) + 2\pi i^t (n + a) (z + b)]. \end{aligned}$$

For  $m, n \in \mathbf{Z}^2$ , the factor of automorphy is given by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z + \tau m + n, \tau) = \exp(2\pi i ({}^t a n - {}^t b m - {}^t m z) - \pi i {}^t m \tau m) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau).$$

For the theta characteristic  $\delta = \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix}$ , with  $\delta' = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$  and  $\delta'' = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ , the function  $\theta[\delta](z, \tau)$  has a simple zero only at  $\tilde{\Theta}$ .

Now we define the hyperelliptic  $\sigma$ -function.

**Definition 1.2.** For  $u \in \mathbf{C}^2$ ,

$$\sigma(u) = \exp\left(-\frac{1}{2} {}^t u \eta \omega^{-1} u\right) \theta[\delta](\omega^{-1} u).$$

Here we shall list some fundamental properties of the  $\sigma$ -function.

**Lemma 1.3.** For  $p, p' \in \mathbf{Z}^2$  and  $l = \omega' p' + \omega p$ , let  $\alpha_l(u)$  be the automorphy factor of  $\sigma(u)$ , that is  $\sigma(u + l) = \alpha_l(u) \sigma(u)$ , then we have  $\alpha_l(u) = \exp(L_l u + C_l)$ , where

$$\begin{aligned} L_l &= -{}^t l \eta \omega^{-1} - 2\pi i {}^t p' \omega^{-1}, \\ C_l &= -\frac{1}{2} {}^t l \eta \omega^{-1} l - \pi i {}^t p' \tau p' + 2\pi i {}^t \delta' p - 2\pi i {}^t \delta'' p'. \end{aligned}$$

Let  $c = \frac{\partial}{\partial u_1} \sigma(u) \Big|_{u=0}$ . Then  $c \neq 0$  and  $\sigma(u)$  has, at  $u = 0$ , the Taylor expansion

$$\sigma(u) = c \left( u_1 + \frac{a_3}{6} u_1^3 - \frac{1}{3} u_2^3 + (\text{terms of degree } \geq 5) \right).$$

Define a polynomial of two variable  $F$  by

$$(1.4) \quad \begin{aligned} F(x_1, x_2) &= x_1^2 x_2^2 (x_1 + x_2) + 2a_1 x_1^2 x_2^2 + a_2 x_1 x_2 (x_1 + x_2) \\ &\quad + 2a_3 x_1 x_2 + a_4 (x_1 + x_2) + 2a_5, \end{aligned}$$

and define a double integral

$$R_{Q, Q_0}^{P, P_0} = \int_{P_0}^P \int_{Q_0}^Q \frac{F(x, z) + 2y s}{4(x-z)^2} \frac{dx}{y} \frac{dz}{s},$$

with  $s^2 = f(z)$ . Then the following proposition holds:

**Proposition 1.4** ([1], p35). We put  $u' = u^{P_1, A_1} + u^{P_2, A_2}$ ,  $u'' = u^{Q_1, A_1} + u^{Q_2, A_2}$ , with  $A_i \in \mathcal{B}$ . For  $P, Q \in \mathcal{C}$ ,  $A \in \mathcal{B}$ , we have

$$R_{P_1, Q_1}^{P, Q} + R_{P_2, Q_2}^{P, Q} = \log \frac{\sigma(u^{P, A} - u')}{\sigma(u^{P, A} - u'')} \Big/ \frac{\sigma(u^{Q, A} - u')}{\sigma(u^{Q, A} - u'')}.$$



Now we define hyperelliptic  $\mathfrak{p}$ -functions.

**Definition 1.5.** For  $i, j, \dots, k = 1, 2$  and  $u \in \mathbf{C}^2$ , we define

$$\zeta_i(u) = \frac{\partial}{\partial u_i} \log \sigma(u) \quad \text{and} \quad \mathfrak{p}_{ij\dots k}(u) = -\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \cdots \frac{\partial}{\partial u_k} \log \sigma(u).$$

We define  $\mathfrak{p}(u) = \mathfrak{p}_{11}(u) \mathfrak{p}_{22}(u) - \mathfrak{p}_{12}^2(u)$ .

Define a polynomial  $\psi$  by

$$(1.5) \quad \psi(x_1, x_2) = x_1^3 x_2 (3x_1 + x_2) + 4a_1 x_1^3 x_2 + a_2 x_1^2 (x_1 + 3x_2) \\ + 2a_3 x_1 (x_1 + x_2) + a_4 (3x_1 + x_2) + 4a_5,$$

and let  $F$  be one as (1.4). Let  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  be points on the curve  $C$  and we put  $u = u^{P_1} + u^{P_2}$ . Then we have

**Proposition 1.6.** (i) ([1],[12]) When  $P_1 \neq P_2$  and  $P_1 \neq P_2'$ ,

$$(1.6) \quad \mathfrak{p}_{11}(u) = \frac{F(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2}, \quad \mathfrak{p}_{12}(u) = -x_1 x_2, \quad \mathfrak{p}_{22}(u) = x_1 + x_2, \\ \mathfrak{p}_{111}(u) = 2 \frac{y_2 \psi(x_1, x_2) - y_1 \psi(x_2, x_1)}{(x_1 - x_2)^3}, \quad \mathfrak{p}_{112}(u) = 2 \frac{x_2^2 y_1 - x_1^2 y_2}{x_1 - x_2}, \\ \mathfrak{p}_{122}(u) = -2 \frac{x_2 y_1 - x_1 y_2}{x_1 - x_2}, \quad \mathfrak{p}_{222}(u) = 2 \frac{y_1 - y_2}{x_1 - x_2}.$$

(ii) When  $P_1 = P_2 = (x, y) \notin \mathcal{B}$  (equivalently  $y \neq 0$ ),

$$(1.7) \quad \mathfrak{p}_{11}(u) = 4x^3 + 2a_1 x^2 + a_2 x + \frac{f'^2(x) - 2f(x)f''(x)}{4y^2} \\ \mathfrak{p}_{12}(u) = -x^2, \quad \mathfrak{p}_{22}(u) = 2x, \\ \mathfrak{p}_{111}(u) = -(14x^2 + 8a_1 x + 2a_2)y - \frac{(2x^3 - f''(x))f'(x)}{2y} - \frac{f'^3(x)}{4y^3}, \\ \mathfrak{p}_{112}(u) = \frac{x^2 f'(x) - 4x f(x)}{y}, \quad \mathfrak{p}_{122}(u) = -\frac{x f'(x) - 2f(x)}{y}, \quad \mathfrak{p}_{222}(u) = \frac{f'(x)}{y}.$$

*Proof.* As for (i), see Baker [1]. The formulae (1.7) can be deduced from (1.6) by L'hôpital's rule.  $\square$

Immediately, by the proposition above, we have

**Corollary 1.7.** *If  $u = u^{P_1} + u^{P_2}$ , then*

$$(1.8) \quad \begin{aligned} x_1 + x_2 &= \mathfrak{p}_{22}(u), & x_1 x_2 &= -\mathfrak{p}_{12}(u), \\ y_1 + y_2 &= \mathfrak{p}_{122}(u) + \frac{1}{2} \mathfrak{p}_{22}(u) \mathfrak{p}_{222}(u), \\ y_1 y_2 &= \frac{1}{4} (\mathfrak{p}_{122}^2(u) - \mathfrak{p}_{112}(u) \mathfrak{p}_{222}(u)). \end{aligned}$$

For  $u, v \in \mathbf{C}^2 - \tilde{\Theta}$ , define

$$(1.9) \quad q(u, v) = -c^2 \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}.$$

Then the following formula holds [1],p.100:

$$(1.10) \quad q(u, v) = \mathfrak{p}_{11}(u) - \mathfrak{p}_{11}(v) + \mathfrak{p}_{12}(u) \mathfrak{p}_{22}(v) - \mathfrak{p}_{12}(v) \mathfrak{p}_{22}(u).$$

Using the fact that, at  $\infty$  of the curve, the coordinate functions  $x$  and  $y$  has a pole of order 2 and 5 respectively, we have:

**Lemma 1.8.** *Let  $P(x, y)$  and  $u = u^P$ . Then we have*

$$(1.11) \quad \begin{aligned} (\mathfrak{p}_{11}/\mathfrak{p}_{12})(u) &= (\mathfrak{p}_{12}/\mathfrak{p}_{22})(u) = -x, \\ (\mathfrak{p}_{111}/\mathfrak{p}_{112})(u) &= (\mathfrak{p}_{112}/\mathfrak{p}_{122})(u) = (\mathfrak{p}_{122}/\mathfrak{p}_{222})(u) = -x, \\ (\mathfrak{p}/\mathfrak{p}_{222})(u) &= -y. \end{aligned}$$

Let  $\sigma_i(u) = \frac{\partial \sigma}{\partial u_i}(u)$  for  $i = 1, 2$ . From this lemma, since  $\frac{\mathfrak{p}_{12}}{\mathfrak{p}_{22}}(u) = \frac{\sigma_1(u)}{\sigma_2(u)} = -\frac{1}{u_2^2} + \dots$  for  $u \in \tilde{\Theta}$ , we know that  $\sigma_2(u)$  is not 0 along  $\tilde{\Theta}$  except for  $\infty$ . For  $u \in \mathbf{C}^2 - \tilde{\Theta}$  and  $v \in \tilde{\Theta}$ , we define

$$(1.12) \quad Q(u, v) = -c^2 \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma_2^2(v)}.$$

**Proposition 1.9** (cf. [13],p.124). *Let  $P(x, y)$  and  $v = u^P$ , then*

$$Q(u, v) = -x^2 + \mathfrak{p}_{12}(u) + \mathfrak{p}_{22}(u)x.$$

Hence if  $P_i = (x_i, y_i)$  and  $u = u^{P_1} + u^{P_2}$ , then

$$(1.13) \quad Q(u, v) = -(x - x_1)(x - x_2)$$

*Proof.* First note that  $(\mathfrak{p}_{12}/\mathfrak{p}_{22})(u^P) = \sigma_1(u^P)/\sigma_2(u^P) = -x$  by (1.11). Multiplying (1.10) by  $\sigma(v)^2/\sigma_2^2(v)$  and taking the limit  $v$  to  $u^P$ , then we obtain the formulae above.  $\square$

The following proposition is the heart of the duplication theorem and the definition of the canonical local height.

**Proposition 1.10.** *For  $u \in \mathbf{C}^2$ ,*

$$-c^3 \frac{\sigma(2u)}{\sigma^4(u)} = \mathbf{p}_{111}(u) - \mathbf{p}_{12}(u) \mathbf{p}_{122}(u) + \mathbf{p}_{22}(u) \mathbf{p}_{112}(u).$$

We denote the right-hand side by  $\phi(u)$ .

*Proof.* By the definition of  $q(u, v)$ ,  $\frac{q(u, v)}{\sigma(u - v)} = -c^2 \frac{\sigma(u + v)}{\sigma^2(u) \sigma^2(v)}$ . Using the Taylor expansion of  $\sigma(u)$  at  $u = 0$ , we get  $\frac{\partial}{\partial v_1} \sigma(u - v)|_{v=u} = -c$ . Thus, by L'hôpital's rule,  $\lim_{v \rightarrow u} \frac{q(u, v)}{\sigma(u - v)} = -\frac{1}{c} \frac{\partial}{\partial v_1} q(u, v)|_{v=u}$ . By differentiating the right-hand side of (1.10) with respect to  $v_1$  and substituting  $v = u$  into the result, we conclude the assertion.  $\square$

**Corollary 1.11.** *Put  $\phi_{ij\dots k}(u) = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \dots \frac{\partial}{\partial u_k} \phi(u)$ . Then we have the duplication formulae:*

$$\begin{aligned} \mathbf{p}_{ij}(2u) &= \mathbf{p}_{ij}(u) - \frac{\phi_{ij}(u)}{4\phi(u)} + \frac{\phi_i(u)\phi_j(u)}{4\phi(u)^2}, \\ \mathbf{p}_{ijk}(2u) &= \frac{1}{2} \mathbf{p}_{ijk}(u) - \frac{\phi_{ijk}(u)}{8\phi(u)} + \\ &\quad \frac{\phi_{ij}(u)\phi_k(u) + \phi_{jk}(u)\phi_i(u) + \phi_{ki}(u)\phi_j(u)}{8\phi^2(u)} - \frac{\phi_i(u)\phi_j(u)\phi_k(u)}{4\phi^3(u)}. \end{aligned}$$

**1.4. Defining equations and arithmetic.** We review defining equations for the affine model using  $\mathbf{p}$ -functions([12]). We define coordinate functions  $X_{ij}$ ,  $X_{ijk}$ , and  $X$  as follows.

$$X_{ij} = \mathbf{p}_{ij}, \quad X_{ijk} = \frac{1}{2} \mathbf{p}_{ijk}, \quad \text{and} \quad X = \frac{1}{2} (\mathbf{p} + a_2 \mathbf{p}_{12} - a_4).$$

We write  $X_0^h$ ,  $X_{ij}^h$ ,  $X_{ijk}^h$ ,  $X^h$  for homogeneous coordinates of a points of  $\mathbf{P}^8$  with  $X_{ij} = X_{ij}^h/X_0^h$ ,  $X_{ijk} = X_{ijk}^h/X_0^h$ , and  $X = X^h/X_0^h$ . We write  $[X_0^h : X_{11}^h : X_{12}^h : X_{22}^h : X_{111}^h : X_{112}^h : X_{122}^h : X_{222}^h : X^h]$  for a set of coordinates. Then we have

**Theorem 1.12** ([12]). *Let  $E_i$ , ( $i = 1, \dots, 14$ ), be the equations  $f_i$  in [12], pp.103–107. The ideal  $I(J - \Theta)$  is generated by  $E_2, \dots, E_7$ .*

**Remark 1.13.** There is a misprint in [12], p.106. The equation  $f_{10}$  should be read

$$f_{10} = X_{112}^2 - X_{111} X_{122} + \dots.$$

**Theorem 1.14** ([12]). *Let  $E_i^h$  be the homogenization of  $E_i$ , ( $i = 1, \dots, 14$ ). Then their Jacobian matrix is of full rank at every point of  $J$  and the equations give a non-singular model in  $\mathbf{P}^8$ .*

The theta divisor on  $J \subset \mathbf{P}^8$  is given by  $X_0^h = 0$ . Substituting  $X_0^h = 0$  into the equations  $E_i^h$ , we have

$$\Theta \subset \{[0 : 0 : 0 : 0 : X_{111}^h : X_{112}^h : X_{122}^h : X_{222}^h : X^h]\}.$$

The following proposition follows from Lemma 1.11:

**Proposition 1.15.** *For  $P(x, y) \in C$ , the coordinates of  $\overline{D(P)} \in \Theta \subset \mathbf{P}^8$  is given by*

$$[0 : 0 : 0 : 0 : -x^3 : x^2 : -x : 1 : -y].$$

*Especially, the zero element  $O_J$  of  $J$  has the coordinates*

$$[0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0].$$

The additive formulae are described in Grant [12] and we do not reproduce here. We can obtain the formulae of the addition of points one of which is on the Theta divisor and the other of which is not on the Theta divisor using Cantor's algorithm [6] or taking the limit of the additive formulae. As for the duplication theorem, see Corollary 1.11. Also see [5] for the computation on Kummer surfaces.

## 2. ARCHIMEDEAN LOCAL HEIGHTS

**2.1. Canonical local heights.** In this section we review on the Néron-Tate local heights (the canonical local heights) on a Jacobian surface  $J$ . See [4], Section 2 for more details.

Let  $\Psi_2 : J \rightarrow J$  be the multiplication by 2 map. The theta divisor  $\Theta$  satisfies

$$\Psi_2^* \Theta \sim 4\Theta.$$

Now we define the canonical local height  $\hat{\lambda}_v : J - \Theta \rightarrow \mathbf{R}$  for  $v \in \Sigma_K$  as follows:

**Definition 2.1.** (1)  $\hat{\lambda}_v$  is a Weil local height function corresponding the divisor  $\Theta$ .  
 (2) Let  $\phi$  be a function such that  $\Psi_2^* \Theta = 4\Theta + \text{div}(\phi)$ , then

$$\hat{\lambda}_v(2z) = 4\hat{\lambda}_v(z) + v(\phi(z)).$$

**Remark 2.2.** As in [4], p.171,  $\hat{\lambda}_v$  is uniquely determined if we fix  $\phi$ , since we assume the equation of the second condition holds with  $v(\phi(z))$  in spite of  $v(a\phi(z))$ . After this, we shall fix  $\phi$  as in Proposition 2.3 below.

**Proposition 2.3.** *Let  $\phi$  be the function defined in Proposition 1.10. Then  $\Psi_2^* \Theta = 4\Theta + \operatorname{div}(\phi)$ .*

*Proof.* Since  $\sigma(u)$  has zero at  $\tilde{\Theta}$  of order 1, we have  $\Psi_2^* \Theta = 4\Theta + \operatorname{div}\left(\frac{\sigma \circ \Psi_2}{\sigma^4}\right)$ . By Proposition 1.10 we have the assertion.  $\square$

Now we define a modified  $\sigma$ -function  $k(u)$ , which is a natural generalization of the Klein function, by

$$(2.1) \quad k(u) = c^{-1} \exp\left(\frac{1}{2} {}^t u \tilde{\eta}(u)\right) \sigma(u).$$

**Proposition 2.4.** *The function  $|k(u)|$  on  $\mathbf{C}^2$  is periodic for  $\Lambda$ .*

*Proof.* First note that  ${}^t z \tilde{\eta}(w) = {}^t \tilde{\eta}(w) z$ , since this is a scalar. Let  $\exp(\kappa_l(u)) = k(u+l)/k(u)$  for  $l = \omega p + \omega' p'$ ,  $p, p' \in \mathbf{Z}^2$  and  $u = \omega r + \omega' r'$  with  $r, r' \in \mathbf{R}^2$ . Then, by Lemma 1.3,

$$\kappa_l(u) = ({}^t p \ {}^t p') M_r \begin{pmatrix} r \\ r' \end{pmatrix} + ({}^t p \ {}^t p') M_p \begin{pmatrix} p \\ p' \end{pmatrix}.$$

where

$$M_r = \begin{pmatrix} \frac{1}{2} ({}^t \eta \omega - {}^t \omega \eta) & \frac{1}{2} ({}^t \eta \omega' + {}^t \omega \eta' - 2 {}^t \omega \eta \tau) \\ \frac{1}{2} ({}^t \eta' \omega - {}^t \omega' \eta) & \frac{1}{2} ({}^t \eta' \omega' + {}^t \omega' \eta' - 2 {}^t \omega' \eta \tau) \end{pmatrix} - 2\pi i \begin{pmatrix} 0 & 0 \\ 1 & \tau \end{pmatrix},$$

and

$$M_p = \begin{pmatrix} 0 & \frac{1}{2} ({}^t \omega \eta' - {}^t \omega \eta \tau) \\ 0 & \frac{1}{2} ({}^t \omega' \eta' - {}^t \omega' \eta \tau - 2\pi i \tau) \end{pmatrix}.$$

By (1.3),  $\eta \omega^{-1} = {}^t(\eta \omega^{-1}) = {}^t \omega^{-1} {}^t \eta$ , thus,  ${}^t \eta \omega = {}^t \omega \eta$ . That is  $(M_r)_{11} = 0$ . By (1.2),  ${}^t \eta' \omega = {}^t(\eta \tau + 2\pi i {}^t \omega^{-1}) \omega = \tau {}^t \eta \omega + 2\pi i \mathbf{1}_2$  and  ${}^t \omega' \eta = {}^t({}^t \eta \omega') = {}^t({}^t \eta \omega \tau) = \tau {}^t({}^t \eta \omega) = \tau {}^t \omega \eta$ . As above,  ${}^t \eta \omega = {}^t \omega \eta$ , so  $(M_r)_{21} = -\pi i \mathbf{1}_2$ . Similarly,  $(M_r)_{12} = \pi i \mathbf{1}_2$ . Finally, since  $\tau {}^t \eta \omega' = \tau {}^t(\eta \omega^{-1} \omega) \omega' = {}^t \omega' \eta \tau$ , using (1.2), we get  $(M_r)_{22} = 0$ .

Also by (1.2), we conclude that  $M_p = \begin{pmatrix} 0 & \pi i \mathbf{1}_2 \\ 0 & 0 \end{pmatrix}$ .

Since any of  $p, p', r, r'$  belongs to  $\mathbf{R}^2$ ,  $\exp(\kappa_l(u))$  is of the form  $\exp(i \cdot \text{'real number'})$  and  $|\exp(\kappa_l(u))| = 1$ .  $\square$

**Corollary 2.5.** *For  $v \in \Sigma_K^\infty$  and  $u \in \mathbf{C}^2 - \tilde{\Theta}$ , if we take  $\phi$  as in Proposition 1.10, then*

$$(2.2) \quad \hat{\lambda}_v(\tilde{u}) = -\log |k(u)|_v.$$

*Proof.* By the proposition above, the right-hand side of (2.2) is well defined and clearly it is a Weil function for  $\Theta$ . Furthermore

$$\frac{k(2u)}{k^4(u)} = c^3 \frac{\sigma(2u)}{\sigma^4(u)} = -\phi(u),$$

thus the right-hand side of (2.2) satisfies the property (2) in the definition of  $\hat{\lambda}_v$ . By the uniqueness, the assertion follows.  $\square$

**2.2. Green's Function.** A meromorphic differential  $\varrho$  on the Riemann surface  $C(\mathbf{C})$  is said to be of the third kind if  $\text{ord}_x(\varrho) \geq -1$  for all  $x \in C(\mathbf{C})$ . For such  $\varrho$ , we define the divisor  $\text{Res}(\varrho)$  by  $\sum_{x \in C(\mathbf{C})} \text{res}_x(\varrho) x$ , which belongs to  $\text{Div}_0(C)$ . Conversely, by the

Riemann-Roch theorem, for any  $a \in \text{Div}_0(C)$ , there exists a differential of the third kind  $\varrho$  such that  $\text{Res}(\varrho) = a$ , and it is determined up to an addition of a differential of the first kind. We write  $\omega_a$  for this  $\varrho$ .

**Lemma 2.6.** *We can choose  $\varrho$  uniquely with pure imaginary periods.*

*Proof.* For any differential of the third kind  $\tilde{\varrho}$ , we define

$$\mathbf{r} = -2 \left( \text{Re} \left( \int_{\gamma_1} \tilde{\varrho} \right), \text{Re} \left( \int_{\gamma_2} \tilde{\varrho} \right), \text{Re} \left( \int_{\gamma'_1} \tilde{\varrho} \right), \text{Re} \left( \int_{\gamma'_2} \tilde{\varrho} \right) \right).$$

Then we can define complex numbers  $c_1, c_2$  by

$$(c_1, c_2, \bar{c}_1, \bar{c}_2) = \mathbf{r} \tilde{\Omega}^{-1},$$

where  $\tilde{\Omega} = \begin{pmatrix} \omega & \omega' \\ \bar{\omega} & \bar{\omega}' \end{pmatrix}$ . Then  $\varrho = \tilde{\varrho} + c_1 \mu_1 + c_2 \mu_2$  has pure imaginary periods.  $\square$

**Lemma 2.7.** *If  $a = D(P_1)$  with  $P_1(x_1, y_1) \in C$ , then  $\omega_a$  is explicitly given by*

$$\omega_a = \frac{y + y_1}{2y(x - x_1)} dx + \omega_h,$$

where  $\omega_h$  is any differential of the first kind.

*Proof.* Noting that  $\text{div}(dx) = \sum B_i - 3\infty$  and  $\text{div}(y) = \sum B_i - 5\infty$ , it is obvious that  $\omega_a$  has simple poles at only  $P_1$  and  $\infty$ . The function  $x - x_1$  is a uniformizer at  $P_1$  and  $\text{res}_{P_1}(\omega_a) = 1$ . By the residue theorem,  $\text{res}_{\infty}(\omega_a) = -1$  and  $\text{Res}(\omega_a) = P_1 - \infty = a$ .  $\square$

**Definition 2.8.** For any  $v \in \Sigma_K^\infty$  and for each  $a \in \text{Div}_0(C)$ , Green's function on  $C(\mathbf{C}) - |a|$  attached to  $a$  is a real valued harmonic function  $g_a$  such that

- (1)  $g_a - m_x \log |z|_v$  is harmonic near  $x$ , where  $z$  is a local parameter at  $x$  and  $m_x$  is the order of  $x$  in  $a$ .

- (2)  $g_a$  is a solution of a differential equation  $\partial\bar{\partial}g_a = -2\pi i\delta_a$ , where  $\delta_a$  is (1,1)-current which represents the evaluation of (0,0)-forms at  $a$ .

For  $a \in \text{Div}_0(C)$ , choose  $\omega_a$  so that it has pure imaginary periods (which exists by Lemma 2.6), then the differential equation  $\omega_a + \bar{\omega}_a = dg$  has a solution  $g$  and we can take  $g$  as  $g_a$  [14].

Now we have an explicit formula of Green's function.

**Proposition 2.9.** (1) When  $a = P_1 - \infty$ ,

$$g_a(P) \equiv \frac{1}{2} \log \left| \frac{k(2u^P - u^{P_1})}{k(2u^P)} \right|_v.$$

- (2) When  $a = P_1 + P_2 - 2\infty$ ,

$$\begin{aligned} g_a(P) &\equiv \frac{1}{2} \log \left| \frac{k(2u^P - u^{P_1})}{k(2u^P + u^{P_2})} (x - x_2)^2 \right|_v \\ &\equiv \frac{1}{2} \log \left| \frac{k(2u^P - u^{P_2})}{k(2u^P + u^{P_1})} (x - x_1)^2 \right|_v \\ &\equiv \frac{1}{2} \log \left| \frac{k(2u^P - u^{P_1}) k(2u^P - u^{P_2})}{k(2u^P)^2} \right|_v. \end{aligned}$$

In the both cases, the symbol  $\equiv$  means equality up to a constant.

*Proof.* (1) We shall prove that the function  $|\theta(z)|$  in Bost [2] is coincide with  $|k(z)|$  up to a constant multiple which depends only  $\tau$ . Put  $z = \omega r + \omega' r'$ ,  $r, r' \in \mathbf{R}^2$ . Using (1.2), we have

$$\begin{aligned} |k(z)| &= |c^{-1} \exp(\pi i({}^t r r' + {}^t r' \tau r'))| |\theta[\delta](\omega^{-1}z)| \\ &= |c^{-1}| \exp(-\pi {}^t r' \text{Im}(\tau) r') |\theta[\delta](\omega^{-1}z)|. \end{aligned}$$

If we put  $z_0 = \omega^{-1}z = r + \tau r' = x_0 + iy_0$  and  $Y = \text{Im}(\tau)$ , then

$$y_0 = \text{Im}(z_0) = \text{Im}(\tau r') = \text{Im}(\tau) r' = Y r'.$$

That is  $r' = Y^{-1}y_0$  and  ${}^t r' \text{Im}(\tau) r' = {}^t y_0 Y^{-1}y_0$ . On the other hand, for  $D$  a divisor of degree 1, we can deduce that  $|\theta|(D) = \det(Y)^{1/4} \exp(-\pi {}^t y_0 Y^{-1}y_0) |\theta[\delta](z_0)|$ , where  $[z] \in \mathbf{C}^2/\Lambda$  is the point corresponding to  $\bar{D} - \infty$  and  $z_0, y_0$  are as above (Note that  $\Delta = \delta' + \tau\delta''$  is a 2-torsion). Thus  $|k(z)|$  and  $|\theta|$  coincide up to the factor  $|c^{-1}| \det(Y)^{1/4}$ . By virtue of Bost's result [2], which is proved in the appendix of [3] we conclude the formula.

- (2) The third expression of the right hand side is obvious by (1). We can also deduce the first and second ones from Bost's result [2].  $\square$

**Remark 2.10.** We can give another proof of the proposition above by directly checking the differential equation  $\omega_a + \overline{\omega}_a = dg_a$  (cf. Lemma 2.7).

**Remark 2.11.** In the formula (2), using the equation (cf. (1.13))

$$\frac{\sigma(2u^P - u^{P_1})\sigma(2u^P + u^{P_1})}{\sigma^2(2u^P)} = M(x - x_1)^2, \quad \text{where } M = \frac{\sigma_2^2(u^{P_1})}{c^2},$$

we see that the first and the third expressions are equal up to a constant.

**2.3. Néron's local pairing.** We review Néron's local pairing following [14]. For any  $v \in \Sigma_K$ , let  $\text{Div}_0(C)_{/K_v}$  be the  $K_v$ -rational subgroup of  $\text{Div}_0(C)$  and let  $Z_0(C)_{/K_v}$  be the point-wise  $K_v$ -rational subgroup. Two divisors  $a, b \in \text{Div}_0(C)$  are called *relatively prime* if they have disjoint support, that is  $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ . For any rational function  $f$  on  $C$  and a divisor  $a = \sum_x a_x x \in \text{Div}(C)$ , write  $f(a) = \prod_x f(x)^{a_x}$  if  $\text{supp}(\text{div}(f)) \cap \text{supp}(a) = \emptyset$ . We define a modified value of  $f$  at  $x$  as follows: Fix a tangent vector  $\frac{\partial}{\partial t}$  at  $x$  on  $C$  and take an uniformizer  $z$  around  $x$  with  $\frac{\partial z}{\partial t} = 1$ . Then we define the modified value  $f[x]$  of  $f$  at  $x$  by  $f[x] = \frac{f}{z^m} \Big|_{z=0}$ , where  $m$  is the order of  $f$  at  $x$ . For any divisor  $a = \sum a_x x$ , we define  $f[a] = \prod f[x]^{a_x}$ .

**Proposition 2.12** ([14], p.328). *There is a unique pairing  $\langle a, b \rangle_v$  on relatively prime divisors  $a \in Z_0(C)_{/K_v}$ ,  $b \in \text{Div}_0(C)_{/K_v}$  with values in  $\mathbf{R}$  which satisfies the following properties:*

- (i)  $\langle a, b \rangle_v + \langle a, c \rangle_v = \langle a, b + c \rangle_v$ .
- (ii)  $\langle a, b \rangle_v = \langle b, a \rangle_v$  for any  $b \in Z_0(C)_{/K_v}$ .
- (iii)  $\langle a, \text{div}(g) \rangle_v = \log |g(a)|_v$  for any  $g \in K(C)^*$ .
- (iv) For fixed  $b$  and  $x_0 \in C(K_v) - \text{supp}(b)$ ,  $C(K_v) - \text{supp}(b) \ni x \mapsto \langle x - x_0, b \rangle_v \in \mathbf{R}$  is continuous.

This pairing is called Néron's local pairing. This pairing satisfies functoriality. That is, let  $C'$  be another curve and  $\Phi \in C \times C'$  be a correspondence rational over  $K_v$ , then we have  $\langle a, \Phi^* b \rangle_C = \langle \Phi_* a, b \rangle_{C'}$  for  $a \in \text{Div}_0(C)$ ,  $b \in \text{Div}_0(C')$  when the both sides are defined. If  $L_w$  be an extension of  $K_v$ ,

$$(2.3) \quad \langle a, b \rangle_w = [L_w : K_v] \langle a, b \rangle_v.$$

If  $\text{supp}(a) \cap \text{supp}(b) \neq \emptyset$ , we modify the pairing by

$$(2.4) \quad \langle a, b \rangle_v = \log |g[a]|_v + \langle a, b' \rangle_v,$$

where  $b = b' + \text{div}(g)$  such that  $\text{supp}(a) \cap \text{supp}(b') = \emptyset$ .



For  $a \in Z_0(C)_{/K_v}$  and  $b \in \text{Div}_0(C)_{/K_v}$  with  $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ , the pairing is explicitly defined as follows.

For the archimedean place  $v$ , the pairing is explicitly given by

$$(2.5) \quad \langle a, b \rangle_v = g_a(b),$$

where  $g_a$  is Green's function attached to  $a$  (§ 2.2).

**Remark 2.13.** In Proposition 1.4, substituting  $P_1, P_2, P_3, P_3, P_5, P_5$  for  $P, Q, P_1, P_2, Q_1, Q_2$  respectively, we have

$$2 R_{P_3, P_5}^{P_1, P_2} = \log \frac{\sigma(u^1 - 2u^3)}{\sigma(u^1 - 2u^5)} \Big/ \frac{\sigma(u^2 - 2u^3)}{\sigma(u^2 - 2u^5)}.$$

Using (1.6), we get  $\frac{d}{dx} \frac{d}{dz} \log \sigma(u^P - u^Q) = \frac{\mathfrak{p}_{11}(u^P - u^Q)}{4ys}$  where  $P = (x, y), Q = (z, s)$ .

Hence we have

$$R_{P_3, P_5}^{P_1, P_2} = \int_{P_2}^{P_1} \int_{P_5}^{P_3} \frac{d}{dx} \frac{d}{dz} \log \sigma(u^P - u^Q) dx dz.$$

Thus we get

$$\log \frac{\sigma(u^1 - 2u^3)}{\sigma(u^1 - 2u^5)} \Big/ \frac{\sigma(u^2 - 2u^3)}{\sigma(u^2 - 2u^5)} = 2 \log \frac{\sigma(u^1 - u^3)}{\sigma(u^1 - u^5)} \Big/ \frac{\sigma(u^2 - u^3)}{\sigma(u^2 - u^5)},$$

and this implies the symmetry of the pairing.

For the non-archimedean place  $v$ , the pairing is explicitly given by

$$\langle a, b \rangle_v = -(A \cdot B) \log q_v,$$

where  $A \cdot B$  is the intersection number of  $A$  and  $B$  (See [11], Chapter 7, 20 or [26], Chapter IV, Section 7 for the definition of the intersection number). Here rational divisors  $A, B \in \text{Div}(\mathcal{C}) \otimes \mathbf{Q}$  are extensions of divisors  $a, b$  in a regular model  $\mathcal{C}$  of  $C$  over  $O_v$  that satisfy  $(A \cdot \mathcal{F}) = (B \cdot \mathcal{F}) = 0$  for any fibral irreducible divisor  $\mathcal{F}$  of  $\mathcal{C}/O_v$ .

The following lemma is useful to compute the ‘‘correction term’’ (cf. [7]).

**Lemma 2.14.** *Let  $\mathcal{C}/O_v$  be a regular arithmetic surface. Let  $\sum_{i=0}^n m_i C_i$  be the special fiber, where  $C_i$  is an irreducible divisor, and  $\sigma$  be a horizontal divisor of degree 0. We assume that  $m_0 = 1$ . Let  $M$  be a matrix given by  $M_{ij} = (C_i \cdot C_j)$ , for  $1 \leq i, j \leq n$ . Define rational numbers  $a_i$  ( $i = 1, \dots, n$ ) by*

$$(a_1, \dots, a_n) = -((\sigma \cdot C_1), \dots, (\sigma \cdot C_n)) M^{-1}.$$

*Then we have  $(\sigma + \sum_{i=1}^n a_i C_i) \cdot C_j = 0$  for any  $j = 0, \dots, n$ .*

Finally Néron's formula is  $\langle \bar{a}, \bar{b} \rangle = \sum_{v \in \Sigma_K} \langle a, b \rangle_v$ ; see Néron [21], pp.295–296. Here  $\langle \bar{a}, \bar{b} \rangle$  is the height pairing on  $J \times J$  satisfying

$$\langle \bar{a}, \bar{b} \rangle = \hat{h}(\bar{a} + \bar{b}) - \hat{h}(\bar{a}) - \hat{h}(\bar{b}),$$

where  $\hat{h}$  is the canonical height  $\hat{h}_\Theta$  attached to  $\Theta$  (see Introduction). (We identify  $J$  and  $\hat{J}$  by  $J \rightarrow \hat{J}$ ,  $a \mapsto (\text{class of } (\Theta - a) - \Theta)$ ). If  $L$  is a finite extension of  $K$ , we have

$$\langle \alpha, \beta \rangle_L = [L : K] \langle \alpha, \beta \rangle_K .$$

**2.4. The canonical local height and Néron's local pairing.** Let  $P_1, P_2$  be  $K$ -rational points on  $C$ . Take  $P_3, P_4, P_5, P_6$  which satisfy  $P_1 + P_3 + P_4 \sim P_2 + P_5 + P_6$ . Define polynomials  $G_1 = V_{P_1, P_3, P_4}$  and  $G_2 = V_{P_2, P_5, P_6}$  (see § 1.1). For simplicity, we write  $u^*$  for  $u^{P^*}$ .

Let  $\overline{D(P_1, P_3, P_4)} = \overline{D(P_2, P_5, P_6)} = \overline{D(P_{11}, P_{12})}$ ,  $P_{1j} = (x_{1j}, y_{1j})$ . Then  $G_1$  is characterized by  $G_1(x_1) = y_1$ ,  $G_1(x_{11}) = -y_{11}$ ,  $G_1(x_{12}) = -y_{12}$  and  $G_2$  is characterized in the similar way.

First we can prove the following lemma by direct computation.

**Lemma 2.15.** *Let  $G_1, G_2$  as above. Then we have the following relation.*

$$q(u^1 - u^2, u^{11} + u^{12}) = \frac{(y_2 + G_1(x_2))(y_1 + G_2(x_1))}{(x_1 - x_2)^2}$$

**Remark 2.16.** As the referee notes, the following formula holds:

$$q(u^1 + u^2, u^3 + u^4) = \frac{\det \begin{pmatrix} y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \\ y_4 & x_4^2 & x_4 & 1 \end{pmatrix} \det \begin{pmatrix} y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ -y_3 & x_3^2 & x_3 & 1 \\ -y_4 & x_4^2 & x_4 & 1 \end{pmatrix}}{(x_1 - x_2)^2 (x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)^2}.$$

We can prove this formula by considering of zeros and poles, or by direct computation. On the other hand, by Cramer's rule, we have

$$y - G_1(x) = \frac{\det \begin{pmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_{11}^2 & x_{11} & 1 & -y_{11} \\ x_{12}^2 & x_{12} & 1 & -y_{12} \\ x^2 & x & 1 & y \end{pmatrix}}{(x_1 - x_{11})(x_1 - x_{12})(x_{11} - x_{12})}.$$

The lemma above is immediately deduced from these equations.

**Proposition 2.17.** *Let  $P_3, P_4, P_5, P_6$  and  $G_1, G_2$  as above. Then we have*

$$\begin{aligned} & q(u^1 - u^2, u^3 + u^4 - u^2) (= q(u^1 - u^2, u^5 + u^6 - u^1)) \\ &= \frac{(x_1 - x_3)(x_1 - x_4)(x_2 - x_5)(x_2 - x_6)(y_2 + G_1(x_2))(y_1 + G_2(x_1))}{(x_2 - x_3)(x_2 - x_4)(x_1 - x_5)(x_1 - x_6)(x_1 - x_2)^2}. \end{aligned}$$

*Proof.* As the referee notes, we have the following equation:

$$q(u^1 - u^2, u^3 + u^4 - u^2) = q(u^1 - u^2, u^{11} + u^{12}) \frac{Q(u^3 + u^4, u^1)Q(u^5 + u^6, u^2)}{Q(u^3 + u^4, u^2)Q(u^5 + u^6, u^1)}.$$

This is checked by the definition of  $q(u, v)$  and  $Q(u, v)$  ((1.9) and (1.12)). Putting (1.13) and Lemma 2.15 together, the assertion follows.  $\square$

**Theorem 2.18.** *For  $P_i(x_i, y_i) \in C(K)$ ,  $i = 1, 2$ , let  $b = P_1 - P_2$  with  $\bar{b} \notin \Theta$ , and  $z_b = u^1 - u^2 \in \mathbf{C}^2$ . As the base of tangent space at  $P_i$ , we take  $2y_i \frac{\partial}{\partial x} = f'(x_i) \frac{\partial}{\partial y}$ .*

*Then we can take  $\frac{x - x_i}{2y_i}$  if  $P_i \notin \mathcal{B}$  and  $\frac{y - y_i}{f'(x_i)}$  if  $P_i \in \mathcal{B}$  as a uniformizer at  $P_i$ . In both cases, for an archimedean place  $v$ , if we take the uniformizer as above, the relation*

$$(2.6) \quad \langle b, b \rangle_v = 2 \hat{\lambda}_v(\tilde{z}_b)$$

*between Néron's local pairing and the canonical local height holds.*

*Proof.* For any divisor  $b' \in \text{Div}_0(C)$  which is linearly equivalent to  $b$ , let  $G = G_{b,b'}$  be a rational function such that  $b = b' + \text{div}(G_{b,b'})$ . Then we have

$$\langle b, b \rangle_v = \langle b, b' \rangle_v + \log |G[b]|_v, \quad \langle b, b' \rangle_v = g_b(b'),$$

where  $G[b]$  means the modified value of  $G$  at  $b$  (see (2.4)). As in the proof of Proposition 2.17, we can take  $b'$  in the form  $b' = P_5 + P_6 - P_3 - P_4$  and  $P_1 + P_3 + P_4 \sim P_{11} + P_{12} + \infty$ .

If we define

$$\begin{aligned} \tilde{G}_1(P) &= \frac{y + G_1(x)}{(x - x_1)(x - x_3)(x - x_4)}, \\ \tilde{G}_2(P) &= \frac{y + G_2(x)}{(x - x_2)(x - x_5)(x - x_6)}, \end{aligned}$$

then,

$$\begin{aligned} \text{div}(\tilde{G}_1) &= P_{11} + P_{12} + \infty - P_1 - P_3 - P_4, \\ \text{div}(\tilde{G}_2) &= P_{11} + P_{12} + \infty - P_2 - P_5 - P_6. \end{aligned}$$

Thus if we define  $G = \tilde{G}_2/\tilde{G}_1$ , then  $\text{div}(G) = b - b'$ , that is we can take  $G$  as  $G_{b,b'}$ .

When both  $P_i$ , ( $i = 1, 2$ ) are not Weierstrass points, we may take the local parameter at  $P_i$  as mentioned in the theorem, and we have

$$G[b] = \frac{(x_1 - x_3)(x_1 - x_4)(x_2 - x_5)(x_2 - x_6)(y_2 + G_1(x_2))(y_1 + G_2(x_1))}{-(x_2 - x_3)(x_2 - x_4)(x_1 - x_5)(x_1 - x_6)(x_1 - x_2)^2}.$$

By Proposition 2.9, we have

$$\begin{aligned} \langle b, b' \rangle_v &= g_b(P_5) + g_b(P_6) - g_b(P_3) - g_b(P_4) \\ &= \frac{1}{2} \log \left| \frac{k(2u^5 - u^1)k(2u^6 - u^1)k(2u^3 - u^2)k(2u^4 - u^2)}{k(2u^5 - u^2)k(2u^6 - u^2)k(2u^3 - u^1)k(2u^4 - u^1)} \right|_v. \end{aligned}$$

Now we may assume

$$(2.7) \quad u^1 + u^3 + u^4 = u^2 + u^5 + u^6.$$

Put  $V_{i,j} = {}^t u^i \tilde{\eta}(u^j)$ . Then the formula in the log  $|\quad|$  is of the form  $\exp(E) \Sigma$ , where

$$\begin{aligned} E &= -(V_{1,5} - V_{2,5} + V_{1,6} - V_{2,6}) + (V_{1,3} + V_{1,4} - V_{2,3} - V_{2,3}) \\ &\quad - (V_{5,1} + V_{6,1} - V_{5,2} - V_{6,2}) + (V_{3,1} + V_{4,1} - V_{3,2} - V_{4,2}), \end{aligned}$$

and

$$\Sigma = \frac{\sigma(2u^5 - u^1)\sigma(2u^6 - u^1)\sigma(2u^3 - u^2)\sigma(2u^4 - u^2)}{\sigma(2u^5 - u^2)\sigma(2u^6 - u^2)\sigma(2u^3 - u^1)\sigma(2u^4 - u^1)}.$$

By the assumption (2.7), we have

$$E = -2 {}^t z_b \tilde{\eta}(z_b).$$

On the other hand, by (1.10), we have

$$\Sigma = \Sigma_1 \times \frac{\sigma^2(u^5 + u^6 - u^1)\sigma^2(u^3 + u^4 - u^2)}{\sigma^2(u^5 + u^6 - u^2)\sigma^2(u^3 + u^4 - u^1)},$$

where

$$\Sigma_1 = \frac{q(u^5 + u^6 - u^1, u^5 - u^6)q(u^3 + u^4 - u^2, u^3 - u^4)}{q(u^5 + u^6 - u^2, u^5 - u^6)q(u^3 + u^4 - u^1, u^3 - u^4)}.$$

Using (1.10) again, we have

$$\begin{aligned} &\sigma^2(u^5 + u^6 - u^2)\sigma^2(u^3 + u^4 - u^1) \\ &= \frac{1}{c^4} q^2(u^3 + u^4 - u^2, u^1 - u^2) \sigma^4(u^3 + u^4 - u^2) \sigma^4(z_b), \end{aligned}$$

and by the assumption (2.7), we get  $\Sigma = \Sigma_1 \Sigma_2 (c^{-1} \sigma(z_b))^{-4}$ , where

$$\Sigma_2 = \frac{1}{q^2(u^3 + u^4 - u^2, u^1 - u^2)}.$$

Here we prove the following lemma.

**Lemma 2.19.** *Let  $\Sigma_1$  be as above, then  $\Sigma_1 = 1$ .*

*Proof.* Let  $i, j, k \in \{1, \dots, 6\}$  be distinct indices. We put  $v^+ = u^i + u^j$ ,  $v^- = u^i - u^j$ , and  $u = v^+ - u^k$ . By (1.10),  $q(u, v^+) = 0$  and, by (1.6),  $\mathfrak{p}_{12}(v^-) = \mathfrak{p}_{12}(v^+)$  and  $\mathfrak{p}_{22}(v^-) = \mathfrak{p}_{22}(v^+)$ . Hence we have

$$\begin{aligned} q(u, v^-) &= \mathfrak{p}_{11}(u) - \mathfrak{p}_{11}(v^-) + \mathfrak{p}_{12}(u) \mathfrak{p}_{22}(v^-) - \mathfrak{p}_{12}(v^-) \mathfrak{p}_{22}(u) - q(u, v^+) \\ &= \mathfrak{p}_{11}(v^+) - \mathfrak{p}_{11}(v^-) = \frac{-4y_i y_j}{(x_i - x_j)^2}, \end{aligned}$$

and this does not depend on the index  $k$ , thereby completing the proof.  $\square$

Finally, by Proposition 2.17, we have

$$\langle b, b \rangle_v = \frac{1}{2} \log |\Sigma_2 k^{-4}(z_b)|_v + \frac{1}{2} \log |G[b]|_v^2 = -2 \log |k(z_b)|_v.$$

As for the case where  $P_i \in \mathcal{B}$ , we can prove the equation similarly, noticing that

$$\lim_{P \rightarrow P_i} \frac{x - x_i}{y - y_i} = \frac{2y_i}{f'(x_i)}.$$

In any case, by Corollary 2.5, we have  $\langle b, b \rangle_v = 2 \hat{\lambda}_v(\tilde{z}_b)$ . This completes the proof.  $\square$

By Proposition 2.17 and the proof of Theorem 2.18, we have

**Corollary 2.20.** *Let  $b = P_1 - P_2 \in Z_0(C)_{K_v}$ ,  $b' = P_5 + P_6 - P_3 - P_4 \in \text{Div}_0(C)_{K_v}$  and  $u^i$  be as above. Then we have*

$$\langle b, b \rangle_v = \langle b, b' \rangle_v + \log |q(u^1 - u^2, u^3 + u^4 - u^2)|_v.$$

### 3. TATE'S SERIES

In this section, we shall give concrete expression of Tate's series for the canonical local height.

**3.1. Generalities.** We review Tate's series [4]. In general let  $V$  be a non-singular projective variety,  $\Psi$  be a morphism  $V \rightarrow V$ , and  $\Theta$  be a divisor  $\Theta \in \text{Div}(V) \otimes \mathbf{R}$ , with  $\Psi^* \Theta = \alpha \Theta + \text{div}(\phi)$ , for some real number  $\alpha > 1$  and a function  $\phi$ . Let  $t_1, \dots, t_r$ ,  $t_i \in K(V)^* \otimes \mathbf{R}$  be functions with  $\text{div}(t_i) = \Theta - D_i$  satisfying  $\bigcap_i \text{supp}(D_i) = \emptyset$ . For

each  $i = 1, \dots, r$ , we define functions  $w_i = \phi \cdot t_i^\alpha$ ,  $z_i = \frac{\phi \cdot t_i^\alpha}{t_i \circ \Psi}$ , and for  $i, j = 1, \dots, r$ ,

we define  $s_{ij} = \frac{z_j w_i}{w_j}$ . For any ample divisor  $D$ , we define a distance function  $\lambda_D$  as

in [4], pp.191–192. Then we have

**Theorem 3.1** ([4]). *Let  $P \in V(\overline{K_v}) - \text{supp}(\Theta)$  be given. Define a sequence of indices  $i_0, i_1, \dots, i_n, \dots$ , by*

$$\lambda_{\text{supp}(D_{i_n})}(\Psi^n P) = \min_{1 \leq i \leq r} \lambda_{\text{supp}(D_i)}(\Psi^n P).$$

*Define a sequence of real numbers  $c_n$  as*

$$c_n = -v(s_{i_n i_{n+1}}(\Psi^n P)), \quad n = 0, 1, 2, \dots,$$

*which is bounded independently of  $n$  and  $P$ . Then*

$$\hat{\lambda}_\Theta(P) = v(t_{i_0}(P)) + \sum_{n=0}^{N-1} \alpha^{-n-1} c_n + O(\alpha^{-N}),$$

*where the constant of  $O(\alpha^{-N})$  is independent of both  $P$  and  $N$ .*

**3.2. The case of Jacobian surfaces.** Now we apply the above to the case of Jacobian surfaces. That is  $V = J$ ,  $\Theta$  is the theta divisor,  $\Psi = \Psi_2$ , that is the multiplication by 2 map, and  $\alpha = 4$ .

For  $P \in J$ , we denote by  $T_P$  the translation map  $J \rightarrow J$ ,  $D \mapsto D + P$ .

**Proposition 3.2.** *Let  $D_1 = T_{B_1}^* \Theta$ ,  $D_2 = T_{B_2}^* \Theta$ , and  $D_3 = T_{B_{13}}^* \Theta$ . Then  $D_i$  is irreducible and  $\bigcap D_i = \emptyset$ .*

*Proof.* The first assertion is obvious since  $\Theta$  is irreducible. Any point in  $D_1$  can be written  $\overline{D(P, B_1)}$ . If this point belongs to  $D_2$ , then, for some  $Q \in C$ ,  $P + B_1 \sim Q + B_2$ . If  $P \neq B_1 (= B_1')$  and  $P \neq \infty$ , by the uniqueness,  $P = B_2$  and  $Q = B_1$ , hence the point is  $\overline{D(P, B_1)} = \overline{B_{12}}$ . If  $P = B_1$ , then  $Q = B_2$  and  $\overline{D(P, B_1)} = O_J$ . The case  $P = \infty$  does not occur. Thus  $D_1 \cap D_2 = \{O_J, \overline{B_{12}}\}$ , hence we have to prove that both  $O_J$  and  $\overline{B_{12}}$  do not belong to  $D_3$ . If  $O_J \in D_3$ , that is for some  $P \in C$ ,  $B_1 + B_3 \sim P + \infty$ . Since  $\overline{B_{13}} \notin \Theta$ , this case does not occur. If  $\overline{B_{12}} \in D_3$ , then  $B_2 \sim B_3$ , which leads to contradiction.  $\square$

**Proposition 3.3.** *Let  $t_i$  be the elements of  $K(J)^* \otimes \mathbf{R}$  corresponding to the divisors  $D_i$  of Proposition 3.2 (see § 3.1). We can take  $t_i$  as follows:*

$$\begin{aligned} t_1 &= (\mathfrak{p}_{12} + \beta_1 \mathfrak{p}_{22} - \beta_1^2)^{-1/2}, \\ t_2 &= (\mathfrak{p}_{12} + \beta_2 \mathfrak{p}_{22} - \beta_2^2)^{-1/2}, \\ t_3 &= (\mathfrak{p}_{11} + (\beta_1 + \beta_3) \mathfrak{p}_{12} + \beta_1 \beta_3 \mathfrak{p}_{22} + A_{13})^{-1/2}, \end{aligned}$$

*where  $A_{13} = (\beta_1 + \beta_3)(\beta_1^2 + \beta_1 \beta_3 + \beta_3^2) + a_1(\beta_1 + \beta_3)^2 + a_2(\beta_1 + \beta_3) + a_3$ .*

*Proof.* First, as for  $t_1, t_2$ , we can show that the function  $\mathfrak{p}_{12}(u) + \beta_i \mathfrak{p}_{22}(u) - \beta_i^2$  vanishes only when  $u = u^{B_i} + u^P$ , since if we put  $u = u^{P_1} + u^{P_2}$ , then this function is equal to  $(x_1 - \beta_i)(x_2 - \beta_i)$ . The function has poles at  $\Theta$  of order 2, thus  $\text{div}(t_i^2) = 2T_{B_i}^* \Theta - 2\Theta$ , that is,  $\text{div}(t_i) = D_i - \Theta$ .

To prove the formula for  $t_3$ , we use the following lemma.

**Lemma 3.4.** *We fix  $v_0 = u^{P_1} + u^{P_2} \notin \tilde{\Theta}$ ,  $P_1, P_2 \in C$ . For  $u \in \mathbf{C}^2 - \tilde{\Theta}$ , define a rational function  $q_{P_1, P_2}$  on  $J$  by*

$$q_{P_1, P_2}(\tilde{u}) = \mathfrak{p}_{11}(u) + \mathfrak{p}_{22}(v_0) \mathfrak{p}_{12}(u) - \mathfrak{p}_{12}(v_0) \mathfrak{p}_{22}(u) - \mathfrak{p}_{11}(v_0).$$

Then

$$\text{div}(q_{P_1, P_2}) = T_{P_1, P_2}^* \Theta + T_{P_1', P_2'}^* \Theta - 2\Theta.$$

*Proof.* By (1.10), the function  $q_{P_1, P_2}$  vanishes at  $T_{P_1, P_2}^* \Theta$  and  $T_{P_1', P_2'}^* \Theta$ , has poles at  $\Theta$  of order 2 and has no poles at elsewhere. Thus the lemma follows.  $\square$

*Proof of Proposition 3.3.* By Lemma 3.4, we can take  $q_{B_1, B_3}^{1/2}$  as  $t_3$ . Using the fact that  $f(\beta_i) = 0$ , we have  $F(\beta_1, \beta_3) + A_{13}(\beta_1 - \beta_3)^2 = 0$ , and from (1.6), the assertion follows.  $\square$

Finally, for  $u \in \mathbf{C}^2$  and  $\tilde{u} \in J$ , as a function measuring the distance of  $\tilde{u}$  and  $\Theta$ , we take

$$\lambda_{\Theta}(\tilde{u}) = \max(\log |\mathfrak{p}_{ij}(u)|, \log |\mathfrak{p}_{ijk}(u)|, \log |\mathfrak{p}(u)|).$$

If  $\mathfrak{p}_I(u) = 0$  for some index  $I$ , we regard the value  $\log |\mathfrak{p}_I(u)|$  as  $-\infty$  and may ignore it.

#### 4. EXAMPLES

In this section, we give some examples. Throughout this section, we denote by  $N_T$  the number of terms of the summation of Tate's series of Theorem 3.1. For the archimedean place  $v = v_{\infty}$  of  $\mathbf{Q}$ , we write  $\hat{\lambda}_{\infty}$  for  $\hat{\lambda}_v$ . For the symbols  $I_{a-b-c}$ ,  $I_{a-b-c}^*$  etc., see [17], also [20].

**Example 4.1.** Let  $C: y^2 = f(x) = x^5 - x + \frac{1}{4}$ . The curve  $C$  has the model over  $\mathbf{Z}$ ,  $\mathcal{C}: y^2 + y = x^5 - x$ . This arithmetic surface has singular fiber at  $p = 139 (= p_1)$ ,  $449 (= p_2)$ , but it is regular at any point on the surface. In fact, we can prove the singular fibers  $\mathcal{C}_{p_1}$  and  $\mathcal{C}_{p_2}$  are both of genus 1 with one normal singularity and they are of type  $I_{1-0-0}$ .

Let  $\mathcal{C}_\eta$  be the generic fiber of  $\mathcal{C}$  and  $\alpha : C \rightarrow \mathcal{C}_\eta$  be an isomorphism  $(x, y) \mapsto (x, y - \frac{1}{2})$ . Take points  $P_1(1, \frac{1}{2})$  and  $P_2(-1, \frac{1}{2})$  on  $C$  and put  $b = P_1 - P_2$ . Then for  $N_T \geq 50$ ,

$$\hat{\lambda}_\infty(z_b) = 0.347955759656624049028090018047 \dots$$

Take points  $P_3(0, \frac{1}{2})$  and  $P_4(-1, -\frac{1}{2})$  on  $C$ . Let  $P_5, P_6$ , with  $\alpha(P_5) = (x_5, y_5)$  and  $\alpha(P_6) = (x_6, y_6)$  be the points which satisfy  $b \sim b'$  for  $b' = P_5 + P_6 - P_3 - P_4$ . Then, by the addition theorem,

$$\begin{aligned} x_5 + x_6 &= 93/11^2, & x_5 x_6 &= -68/11^2, \\ y_5 + y_6 &= 3 \cdot 61 \cdot 1031/11^5, & y_5 y_6 &= 2^2 \cdot 3^2 \cdot 17 \cdot 73/11^5. \end{aligned}$$

We write  $\tilde{P}$  for the section corresponding to the point  $\alpha(P) \in \mathcal{C}_\eta$ . Then  $\tilde{P}_i$  and  $\tilde{P}_j$  do not intersect for  $i = 1, 2$  and  $j = 3, 4$ . Also  $\tilde{P}_1$  intersects neither  $\tilde{P}_5$  nor  $\tilde{P}_6$ . One of  $\tilde{P}_5$  and  $\tilde{P}_6$  intersects  $\tilde{P}_2$  with multiplicity 1 on the fiber  $\mathcal{C}_{73}$ . Thus we have  $\langle b, b' \rangle_v = \log 73$  if  $p_v = 73$  and  $\langle b, b' \rangle_v = 0$  for other finite places  $v$ . Since  $q(u^1 - u^2, u^3 + u^4 - u^2) = -73/4$ , by Corollary 2.20, we have

$$\begin{aligned} \langle \bar{b}, \bar{b} \rangle &= 2 \hat{\lambda}_\infty(z_b) + 2 \log 2 \\ &= 2.0822058804331387168906442790105599508865 \dots \end{aligned}$$

**Remark 4.1.** In the example above, if we take  $P_3 = (2, \frac{11}{2})$ ,  $P_4 = (-1, -\frac{1}{2})$ , then  $\tilde{G}[b] = -31/4$ . For the place  $v$  with  $p_v = 31$ ,  $\langle b, b' \rangle_v = \log 31$ , and for the other places  $v$ ,  $\langle b, b' \rangle_v = 0$ . Hence we obtain the same result for  $\langle \bar{b}, \bar{b} \rangle$  and the global height is surely independent of  $P_3, P_4$ . In this way, we can check the computation of Néron's symbol.

**Example 4.2.** Let  $N = 23$  and  $X_0(N)$  be the modular curve. It has the canonical model [10],p.416:

$$y^2 = f(x) = x^6 - 14x^5 + 57x^4 - 106x^3 + 90x^2 - 16x - 19.$$

Let  $\chi$  be the quadratic character corresponding to the quadratic field  $\mathbf{Q}(\sqrt{-7})$ , let  $X_0(N)_\chi$  be the twisted modular curve which is given by

$$(4.1) \quad -7y^2 = f(x) = x^6 - 14x^5 + 57x^4 - 106x^3 + 90x^2 - 16x - 19,$$

and denote this by  $C$ . Let  $J$  be the Jacobian variety of  $C$ . We want to verify the Birch-Swinnerton-Dyer Conjecture for  $J$ .

Now we recall the Birch-Swinnerton-Dyer Conjecture. Let  $A$  be an abelian variety defined over  $\mathbf{Q}$ , let  $A'$  be the dual abelian variety of  $A$ , let  $V_\infty$  be the volume of real periods  $\text{Vol}(A(\mathbf{R}))$ , let  $S$  be the finite set of bad primes, let  $V_S$  be  $\text{Vol}(\prod_{p \in S} A(\mathbf{Q}_p))$ ,



let  $\text{III}$  be the Tate-Shafarevich group of  $A$ , let  $A(\mathbf{Q})_{tors}$  be the torsion part of the Mordell-Weil group of  $A$ , and let  $r$  be the Mordell-Weil rank of  $A$ , which conjecturally equals the order of the Hasse-Weil zeta function  $L(s, A)$  at  $s = 1$ . Let  $\alpha_i$ ,  $1 \leq i \leq r$  be a system of generators of  $A(\mathbf{Q}) \otimes \mathbf{Q}$  and  $R = \det(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i, j \leq r}$  be the regulator of  $A$ . Then the conjecture is as follows [15], p.51, Conjecture 2.8.2:

$$(4.2) \quad \lim_{s \rightarrow 1} (s-1)^{-r} L(s, A) = \frac{R V_\infty V_S \#\text{III}}{\#A(\mathbf{Q})_{tors} \#A'(\mathbf{Q})_{tors}}.$$

Since we do not have methods to compute the order of  $\text{III}$ , we want to check

$$(4.3) \quad \frac{\lim_{s \rightarrow 1} (s-1)^{-r} L(s, A) \#A(\mathbf{Q})_{tors} \#A'(\mathbf{Q})_{tors}}{R V_\infty V_S} \in \mathbf{Q}.$$

Let  $S_2(N)$  be the space of cusp forms of weight 2 with respect to  $\Gamma_0(N)$ . The space  $S_2(23)$  is 2-dimensional. Let  $g \in S_2(23)$  be the one of the eigen cusp forms which has the Fourier expansion  $g(q) = a_1 + a_2q + \dots$ , with  $a_1 = 1$ ,  $a_2 = \frac{-1 + \sqrt{5}}{2}$  (cf. [9]). It is well known that the coefficients  $a_n$  belong to  $K = \mathbf{Q}(\sqrt{5})$  and  $g, g^\sigma$  are basis of  $S_2(23)$  where  $\sigma$  is the generator of  $\text{Gal}(K/\mathbf{Q})$ . Let  $g_\chi$  be a cusp form given by  $\sum_{n \geq 1} \chi(n) a_n q^n$ , which belongs to  $S_2(23 \cdot 7^2)$ . Then the Hasse-Weil  $\zeta$ -function  $L(s, J)$  equals  $L(s, g_\chi) L(s, g_\chi^\sigma)$ . Since the signs of the functional equations are  $-1$ , both of  $L(s, g_\chi)$  and  $L(s, g_\chi^\sigma)$  have odd analytic rank (analytic rank means the order at  $s = 1$ ). In fact, they are of analytic rank 1, that is the first derivatives of them do not vanish at  $s = 1$ . We check this by computing the special value of the derivatives of the  $L$ -functions using the following proposition.

**Proposition 4.2** ([8], p.31, Prop.2.13.1). *For  $g = \sum_{n=1}^{\infty} a_n q^n \in S_2(N)$ ,*

$$L'(g, 1) = 2 \sum_{n=1}^{\infty} \frac{a_n}{n} G_1\left(\frac{2\pi n}{\sqrt{N}}\right) \quad \text{where} \quad G_1(x) = \int_1^{\infty} e^{-xy} \frac{dy}{y}.$$

By this method, we have

$$\begin{aligned} L'(g_\chi, 1) &= 3.3236701591276114211249090245717594419417 \\ &\quad 548256170127399799836304033108 \dots, \\ L'(g_\chi^\sigma, 1) &= 1.2235733780550577014994167260813838530875 \\ &\quad 469109100787909011075184313338 \dots. \end{aligned}$$

Thus, by the conjecture, the Mordell-Weil rank of  $J$  should be 2. On the other hand, we have four rational points of  $C$ ;  $P_1(1, 1)$ ,  $P_2(3, 5)$ , and their images of the

hyperelliptic involution:  $P_1^\iota, P_2^\iota$ . We define  $b_1 = P_1 - P_2$ ,  $b_2 = P_1 - P_2^\iota$ , and  $b_3 = P_1 - P_1^\iota$ ,  $b_i \in Z_0(C)_{\mathbf{Q}}$ . Put  $\alpha_i = \frac{b_i}{\bar{b}_i} \in J(\mathbf{Q})$ , ( $i = 1, 2, 3$ ), then we have  $\alpha_3 = \alpha_1 + \alpha_2$ . If  $R' := \det \left( \langle \alpha_i, \alpha_j \rangle_{1 \leq i, j \leq 2} \right)$  is not 0, then  $\alpha_1, \alpha_2$  are independent and  $R'$  is the regulator up to a multiple of an integer.

Next we compute the archimedean local height. We have the factorization

$$f(x) = (x^3 - 3x^2 + 2x + 1)(x^3 - 11x^2 + 22x - 19).$$

Take the real root  $x_0$  of  $x^3 - 3x^2 + 2x + 1$ , then we have an isomorphism over  $\mathbf{Q}(x_0)$

$$(x, y) \mapsto \left( \frac{f'(x_0)}{x - x_0}, \frac{f'(x_0)^2 y}{(x - x_0)^3} \right),$$

the image of which is the curve given by  $y^2 = x^5 + \dots \in \mathbf{Q}(x_0)[x]$ . Using this equation, we compute the Tate's series, and the results are as follows ( $N_T \geq 150$ ):

$$\begin{aligned} \hat{\lambda}_\infty(\alpha_1) &= 8.7417108302483296767154557179790709120077 \\ &\quad 0880444567048579023157642390338444909942 \dots, \\ \hat{\lambda}_\infty(\alpha_2) &= 8.5824393360065566735121235093036839525837 \\ &\quad 4396997136609666242902462638802297360634 \dots, \\ \hat{\lambda}_\infty(\alpha_3) &= 8.7561543364583716258929769839951270330761 \\ &\quad 3410174036934238900919308262429139876024 \dots \end{aligned}$$

Take points  $P_3(5 - \sqrt{13}, 10 - 3\sqrt{13})$ ,  $P_4(5 + \sqrt{13}, 10 + 3\sqrt{13})$  on  $C$ . Then we obtain points  $P_j$ , ( $j = 5, \dots, 10$ ) on  $C$  such that  $b_i$  is linearly equivalent to  $b'_i = P_{2i+3} + P_{2i+4} - P_3 - P_4$ . Let  $G_i = G_{b_i, b'_i}$  be functions satisfying  $\text{div}(G_i) = b_i - b'_i$ . We have, by the addition theorem,

$$\begin{aligned} P_5 &= (3\sqrt{-1}, 2 + 21\sqrt{-1}), \\ P_6 &= (-3\sqrt{-1}, 2 - 21\sqrt{-1}), \\ P_7 &= \left( \frac{1015 + 3\sqrt{32009}}{256}, \frac{-70569557 - 419001\sqrt{32009}}{4194304} \right), \\ P_8 &= \left( \frac{1015 - 3\sqrt{32009}}{256}, \frac{-70569557 + 419001\sqrt{32009}}{4194304} \right), \\ P_9 &= \left( \frac{437 + \sqrt{147206}}{107}, \frac{27721445 + 67635\sqrt{147206}}{1225043} \right), \end{aligned}$$

$$P_{10} = \left( \frac{437 - \sqrt{147206}}{107}, \frac{27721445 - 67635\sqrt{147206}}{1225043} \right).$$

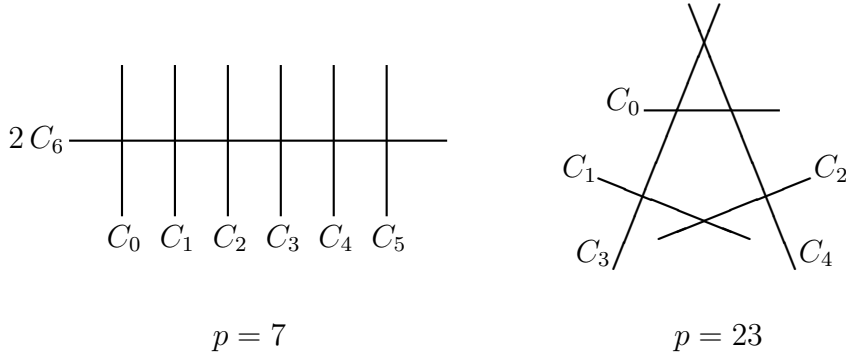
All of these points are not same as  $P_1$  and  $P_2$ . Thus we compute the real archimedean Néron's local pairing  $r_i := \langle b_i, b'_i \rangle_\infty = 2\hat{\lambda}_\infty(\alpha_i) - \log |G_i[b_i]|$ . We have

$$\begin{aligned} r_1 &= -0.3779682038474793239566516214064928511906 \\ &\quad 5187702973806955564168003000827460539180 \dots, \\ r_2 &= -0.1995352620720013629776611485766647308863 \\ &\quad 6166002662262541504391361101778520199515 \dots, \\ r_3 &= -2.3135175421748408234366903964991804162103 \\ &\quad 7008163231687118098438209046679595805722 \dots \end{aligned}$$

Next we consider Néron's local pairings at non-archimedean places. At  $p = 2$ , the model (4.1) over  $\mathbf{Z}$  is not normal, thus we must blow up by  $y = (x^3 + x^2 + 1) + 2Y$ . Then we have

$$\langle b_1, b'_1 \rangle_2 = -\log 2, \quad \langle b_2, b'_2 \rangle_2 = 0, \quad \langle b_3, b'_3 \rangle_2 = -\log 2.$$

Let  $k = k_i$  be a biquadratic field which is generated by the coordinates of the points of the support of  $b'_i$ . Following [17], we have: At  $p = 7$ , the fiber of the minimal regular model is of type  $I_{0-0-0}^*$ , and if  $p = 7$  is ramified in  $k$ ,  $C \times O_k$  has good fiber over the primes lying over 7; at  $p = 23$ , if  $p$  is unramified in the field  $k$ , then the fiber of the minimal regular model is of type  $I_{1-2-3}$ , and if ramified, ramification index is 2 and the type is  $I_{2-4-6}$ . In our case, both of primes 7 and 23 are unramified in  $k$ . We figure the fibres at each prime:



The fibres  $C_i$  are  $(-2)$ -curves, except for  $C_6$  at  $p = 7$  and  $C_3, C_4$  at  $p = 23$ , all of which are  $(-3)$ -curves. At  $p = 7$ , we can decide which fiber the section hits by looking up the  $x$  coordinates. At  $p = 23$ , the sections corresponding to the  $P_i$  do not hit  $C_0, C_1, C_2$ , and we can decide which fiber they hit, by checking whether  $(x + 2)(x + 5)(x + 9)$  equals  $4y \pmod{23}$  or  $-4y \pmod{23}$ .

**Remark 4.3.** At  $\mathfrak{P}|23$ , if the reduction of  $x$ -coordinate is  $-2$ , then the section hits  $C_0$ . If the reduction is  $-5$ , putting  $\xi = \frac{x+5}{y} \pmod{\mathfrak{P}}$ , and  $\eta = \frac{23}{y} \pmod{\mathfrak{P}}$ , then; if  $11\xi - 5\eta = 4$ , hit  $C_1$ , else hit  $C_2$ , when  $C_3$  is the fiber  $(x+2)(x+5)(x+9) = 4y$ . This can be proved by chasing the procedures of blowing-ups.

Thus, using Lemma 2.14, we have

$$\begin{aligned} \langle b_1, b'_1 \rangle_7 &= \langle b_2, b'_2 \rangle_7 = \langle b_3, b'_3 \rangle_7 = 0 \\ \langle b_1, b'_1 \rangle_{23} &= -\frac{6}{11} \log 23, \quad \langle b_2, b'_2 \rangle_{23} = 0, \quad \langle b_3, b'_3 \rangle_{23} = -\frac{6}{11} \log 23. \end{aligned}$$

It is easy to compute the intersection numbers at other primes. Summing up over all finite places, we have

$$\begin{aligned} \sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_1, b'_1 \rangle_v &= -\log 2 + 3 \log 3 - \frac{6}{11} \log 23, \\ \sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_2, b'_2 \rangle_v &= -3 \log 3 + \log 887, \\ \sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_3, b'_3 \rangle_v &= -\log 2 + \log 3 + \log 179 - \frac{6}{11} \log 23. \end{aligned}$$

Put  $h_i = \langle \alpha_i, \alpha_i \rangle$  for  $i = 1, 2, 3$ . Summing up the above with the archimedean part, we have

$$\begin{aligned} h_1 &= 0.5144519092719137003718049687337098118895 \\ &\quad 6676307438099043759656003472103072539585 \dots, \\ h_2 &= 3.2924728542332490532396517934684869119793 \\ &\quad 9026185015676151351622678083750406914366 \dots, \\ h_3 &= 1.5690637994490878142790801498173730131778 \\ &\quad 0086100745083786934319002486632428241500 \dots \end{aligned}$$

Since  $\langle \alpha_1, \alpha_2 \rangle = (h_3 - h_1 - h_2)/2$ , we have the regulator up to a multiple of an integer

$$\begin{aligned} R' &= 0.4418135224747459009837796585512486028911 \\ &\quad 9027232784016701161394069098323593670992 \dots \end{aligned}$$

Next we compute the real periods using the method in Cremona [8]. We compute imaginary periods of  $\Gamma_0(23)$ , and multiply it by  $\sqrt{-7}$  to get the real period of  $J$  up to a multiple of a rational number.

Let  $(c : d)$  be the M-symbol [8],p.8. Let  $\gamma_1^- = \frac{1}{2}(3 : 1) - \frac{1}{2}(-3 : 1)$ , and  $\gamma_2^- = \frac{1}{2}(2 : 1) - \frac{1}{2}(3 : 1) + (4 : 1) - \frac{1}{2}(-3 : 1)$ , then these are basis of the anti-holomorphic homology space  $H^-(N)$ . Define  $V'_\infty$  be  $\sqrt{7}^2 \langle \gamma_1^-, g \rangle \langle \gamma_1^-, g^\sigma \rangle$ . Then this is the real period of  $J$  up to a multiple of a rational number. Computing by direct method(see [8],p.25,Prop.2.10.1),

$$V'_\infty = 10.2506719848116009699526519174413057653616 \\ 5631926561357938968846741216424637142414 \dots$$

The number of connected components of Néron model of  $J/\mathbf{Z}_7$  is 16 and that of  $J/\mathbf{Z}_{23}$  is 11(cf. [17]). Hence we have  $V_S = 11 \cdot 16$ . Let  $\tilde{J}_p$  be the reduction of  $J$  at a prime  $p$ . By the formula [5], p.80, (8.2.5), we have  $\#\tilde{J}_3(\mathbf{F}_3) = 21$  and  $\#\tilde{J}_{11}(\mathbf{F}_{11}) = 221$ . Since  $C$  and  $J$  has good reduction at  $p = 3$  and 11, and  $J(\mathbf{Q})_{tors}$  is injectively mapped to  $\tilde{J}_3(\mathbf{F}_3)$  and  $\tilde{J}_{11}(\mathbf{F}_{11})$ . Thus we have  $\#J(\mathbf{Q})_{tors} = 1$ . If we put

$$T := \frac{\frac{1}{2}L'(g_\chi, 1)L'(g_\chi^\sigma, 1)\#J(\mathbf{Q})_{tors}^2}{R'V'_\infty V_S},$$

then

$$392 \cdot T = 0.9999999999 \dots \text{('9' repeats at least 50 times)}.$$

Thus we can guess  $T = \frac{1}{392}$  and is a rational number as desired.

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