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著者	Shirao Yoshiaki, Nagahara Toshikuni, Kido Masao
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A Study of $\frac{2n+1}{4}$-Harmonics (n=0, 1, 2, ...) in the Neighborhood of Branching Point in the Nonautonomous Piecewise Linear Systems with Unsymmetrical Restoring Force

Yoshiaki SHIRAO*, Toshikuni NAGAHARA* and Masao KIDO*

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This is a study of $\frac{2n+1}{4}$-harmonics in the neighborhood of branching point in the nonautonomous second order differential equation with piecewise linear restoring force having unsymmetrical characteristics in the case of the undamped systems.

This report dealt with the qualitative behavior of the branching of the trajectories of $\frac{2n+1}{4}$-harmonics (n=0, 1, 2, ...) from that of harmonics.

1. Introduction

It is well known[1),2),3),4)] that nonlinear systems can possess periodic solutions which are called subharmonic oscillations in which the smallest period of the motion may be any integral multiple of the period of the external force.

However, the qualitative behavior of the solutions whose fundamental frequency is a fraction $\frac{1}{4}$ of the driving frequency has been studied very little in piecewise linear systems with unsymmetrical restoring force in the neighborhood of the branching point, at which the subharmonic oscillations will turn to the harmonic oscillations.

It is the purpose of this paper to outline the fundamental nature of the solutions having period $4T_0$ (T_0: period of external force) in the neighborhood of branching point.

2. Periodicity Conditions

In order to present the periodicity conditions[5)] as to the solutions like harmonic solutions shown in Fig. 1, consider the differential equation

$$\ddot{x}+f(x)=E\cos\omega t \tag{1}$$

where $f(x)$ is a piecewise linear restoring force (shown in Fig. 2) given by

$$f(x)=\begin{cases} l^2x-K^2x_0 & (x\geqq x_0) \\ k^2x & (x\leqq x_0) \end{cases} \tag{2}$$

$$l^2=k^2+K^2$$

in which, k, l, K and x_0 are positive constants. In this paper dots over a quantity refer to differentiation with respect to the time t.

Initially in this equation, assume

* Department of Electrical Engineering, College of Engineering.

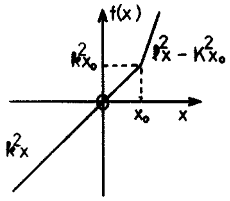

Fig. 1 Restoring force characteristics

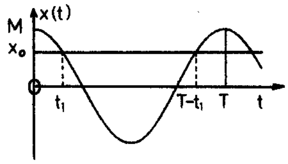

Fig. 2 Periodic solution of type ${}_1A$

$$\left.\begin{aligned} &x(0)=M \quad (>x_0) \\ &\dot{x}(0)=0 \end{aligned}\right\} \tag{3}$$

Then, the condition of x's continuity requires that

$$\left(M-\frac{E}{l^2-\omega^2}-\frac{K^2}{l^2}x_0\right)\cos lt_1+\frac{E}{l^2-\omega^2}\cos\omega t_1=\frac{k^2}{l^2}x_0 \tag{4}$$

$$\tan k\left(\frac{T_0}{2}-t_1\right)=\frac{-l\left(M-\dfrac{E}{l^2-\omega^2}-\dfrac{K^2}{l^2}x_0\right)\sin lt_1+\dfrac{\omega K^2E}{(l^2-\omega^2)(k^2-\omega^2)}\sin\omega t_1}{k\left(x_0-\dfrac{E}{k^2-\omega^2}\cos\omega t_1\right)} \tag{5}$$

where $T_0=\dfrac{2\pi}{\omega}$ and t_1 is the time when the solution reaches the first corner point x_0.

It is clear that periodicity conditions (4) and (5) depend on four parameters M, E, ω and t_1. Rearranging Eqs. (4) and (5), and using the other parameters (ω and t_1), M and E which lead to periodic solutions are expressed as follows:

$$\begin{aligned}\frac{M}{x_0}=&\frac{1}{l^2(k^2-\omega^2)(l^2-\omega^2)\,\Delta\cos k\left(\frac{T_0}{2}-t_1\right)}\Bigg[kK^2(l^2-\omega^2)\cos lt_1\cos\omega t_1\sin k\left(\frac{T_0}{2}-t_1\right)\\ &+lK^2(k^2-\omega^2)\sin lt_1\cos\omega t_1\cos k\left(\frac{T_0}{2}-t_1\right)+\omega K^4\cos lt_1\sin\omega t_1\cos k\left(\frac{T_0}{2}-t_1\right)\\ &+kl(k^2-\omega^2)\left\{k\sin lt_1\cos k\left(\frac{T_0}{2}-t_1\right)+l\cos lt_1\sin k\left(\frac{T_0}{2}-t_1\right)\right\}\\ &+\omega kK^2\left\{\omega\cos\omega t_1\sin k\left(\frac{T_0}{2}-t_1\right)+k\sin\omega t_1\cos k\left(\frac{T_0}{2}-t_1\right)\right\}\Bigg]\end{aligned} \tag{6}$$

$$\frac{E}{x_0}=\frac{k}{l\Delta\cos k\left(\frac{T_0}{2}-t_1\right)}\left\{l\cos lt_1\sin k\left(\frac{T_0}{2}-t_1\right)+k\sin lt_1\cos k\left(\frac{T_0}{2}-t_1\right)\right\} \tag{7}$$

$$\frac{d}{dt_1}\left(\frac{E}{x_0}\right)=\frac{1}{\Delta\cos k\left(\frac{T_0}{2}-t_1\right)}\left(\frac{y_1}{x_0}\right)\left\{k\cos lt_1\sin k\left(\frac{T_0}{2}-t_1\right)+l\sin lt_1\cos k\left(\frac{T_0}{2}-t_1\right)\right\} \tag{8}$$

$$\frac{y_1}{x_0}=\frac{kl}{\Delta l^2(l^2-\omega^2)(k^2-\omega^2)\cos k\left(\frac{T_0}{2}-t_1\right)}\left\{\omega k(l^2-\omega^2)\sin lt_1\sin\omega t_1\cos k\left(\frac{T_0}{2}-t_1\right)\right.$$

$$+\omega^2 K^2 \sin lt_1 \cos\omega t_1 \sin k\left(\frac{T_0}{2}-t_1\right)+\omega l(k^2-\omega^2)\cos lt_1 \sin\omega t_1 \sin k\left(\frac{T_0}{2}-t_1\right)\Big\} \tag{9}$$

$$\Delta(l^2-\omega^2)(k^2-\omega^2)\cos k\left(\frac{T_0}{2}-t_1\right)=\omega K^2 \cos lt_1 \sin\omega t_1 \cos k\left(\frac{T_0}{2}-t_1\right)$$
$$+l(k^2-\omega^2)\sin lt_1 \cos\omega t_1 \cos k\left(\frac{T_0}{2}-t_1\right)+k(l^2-\omega^2)\cos lt_1 \cos\omega t_1 \sin k\left(\frac{T_0}{2}-t_1\right) \tag{10}$$

3. Branching Condition

If $x^0(t)$ is the fundamental harmonic solution type ${}_1A$ obtainable by using the periodicity conditions defined in section 2, the stability of $x^0(t)$ can be studied by the first-order variation equation for the solution in Eq. (1). Now, if y is the variation, then the first-order variation equation is given as

$$\ddot{y}+a(t)y=0 \tag{11}$$

where

$$a(t)\equiv\frac{\partial f}{\partial x}\bigg|_{x=x^0(t)}=\begin{cases} l^2 & (x^0(t)>x_0) \\ k^2 & (x^0(t)<x_0)\end{cases} \tag{12}$$

and Eq. (11) means a Hill's equation. The reason for this lies apparently in the fact that $a(t)$ has the characteristics (shown in Fig. 3) of

$$\left.\begin{aligned} a(t)&=a(-t) \\ a(t)&=a(t+T_0)\end{aligned}\right\} \tag{13}$$

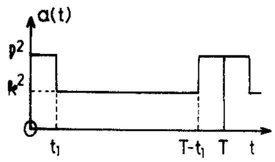

Fig. 3 Coefficient $a(t)$ of periodic solution type ${}_1A$

It is known from Floquet theory as to normal solutions that

$$y(t+T_0)=\rho y(t) \tag{14}$$

and for

$$\rho=e^{j\frac{2m\pi}{n}} \qquad (m,\ n \text{ relatively prime integers}) \tag{15}$$

there exist periodic solutions of period nT_0. The condition $\rho=e^{j\frac{m\pi}{2}}$ means that $y(t+4T_0)=y(t)$, and

$$\left.\begin{aligned} \rho_1\rho_2&=1 \\ \rho_1+\rho_2&=0\end{aligned}\right\} \tag{16}$$

if ρ_1 and ρ_2 are the roots of the characteristic equation of Eq. (11). In this report the case of $n=4$ will be considered similar to the previous reports.[6),7)] Let $\varphi(t)$ and $\psi(t)$

be fundamental solutions of Eq. (11) which satisfy the following conditions at $t=0$:

$$\left.\begin{array}{l}\varphi(0)=1,\ \dot{\varphi}(0)=0\\ \psi(0)=0,\ \dot{\psi}(0)=1\end{array}\right\} \tag{17}$$

Then $\varphi(t)$ is an even function but $\psi(t)$ is an odd function. Using ρ_1 and ρ_2 expressed in terms of $\varphi(t)$ and $\psi(t)$, we get

$$\rho_1+\rho_2=\varphi(T_0)+\dot{\psi}(T_0)=0 \tag{18}$$

and from the expicit formulas for $\varphi(t)$ and $\psi(t)$

$$\varphi(T_0)=\dot{\psi}(T_0)=\cos 2lt_1 \cos 2k\left(\frac{T_0}{2}-t_1\right)-\frac{1}{2}\left(\frac{l}{k}+\frac{k}{l}\right)\sin 2lt_1 \sin 2k\left(\frac{T_0}{2}-t_1\right)=0 \tag{19}$$

From this expression we can readily obtain the following theorem:
Theorem 1.
The solutions of period $4T_0$ of Eq. (11) are admitted under the condition (20) if Eq. (19) holds.

$$\frac{k}{\omega}<\frac{2s+1}{4}<\frac{l}{\omega} \qquad (s=0,\ 1,\ 2,\ \ldots\ldots) \tag{20}$$

Let the following decreasing sequence:

$$P_i=\frac{\omega}{4}\left\{\frac{i}{l}+\frac{2s+1-i}{k}\right\} \qquad (i=0,\ 1,\ 2,\ \ldots\ldots,\ 2s+1) \tag{21}$$

where $P_0>1$ and $P_{2s+1}<1$ under the condition (20), and

$$P_j>1>P_{j+1} \tag{22}$$

Then Eq. (19) holds in the following interval:

$$\frac{j\pi}{2l}<t_1<\frac{(j+1)\pi}{2l} \tag{23}$$

The proof can be done in the following ways when the even/odd characteristics of j and s are used. Assume both j and s are even (the other possibilities are similar); $2k\left(\frac{T_0}{2}-t_1\right)$ then becomes

$$\frac{(2s-j)\pi}{2}<2k\left(\frac{T_0}{2}-t_1\right)<\frac{(2s+1-j)\pi}{2} \tag{24}$$

when $j\equiv 0 \pmod 4$, it is in the first quadrant;
when $j\equiv 2 \pmod 4$, it is in the third quadrant:

$$\text{Similarly for } j\frac{\pi}{2}<2lt_1<(j+1)\frac{\pi}{2} \tag{25}$$

when $j\equiv 0 \pmod 4$, it is in the first quadrant;
when $j\equiv 2 \pmod 4$, it is in the third quadrant.
Here let

$$h(t)=\cos 2lt \cos 2k\left(\frac{T_0}{2}-t\right)-\frac{1}{2}\left(\frac{l}{k}+\frac{k}{l}\right)\sin lt \ \sin 2k\left(\frac{T_0}{2}-t\right) \tag{26}$$

then

$$\dot{h}(t)=-\frac{K^2}{kl}\left(l\cos 2lt \ \sin 2k\left(\frac{T_0}{2}-t\right)+kK^2\sin 2lt \ \cos 2k\left(\frac{T_0}{2}-t\right)\right)<0 \tag{27}$$

$$\ddot{h}(t)=\frac{2K^4}{kl}\sin 2lt \ \sin 2k\left(\frac{T_0}{2}-t\right)>0 \tag{28}$$

This expression is generally valid for the interval of time between $\frac{j\pi}{2l}$ and $\frac{(j+1)\pi}{2l}$. It can also be shown that there corresponds a definite value of t for any practical case.

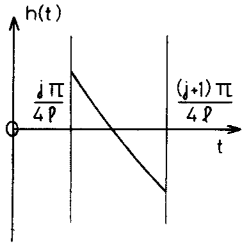

Fig. 4 $h(t)$ in the interval $\frac{j\pi}{4l}<t<\frac{(j+1)\pi}{4l}$

4. Branching Phenomena

In the neighborhood of the ${}_1A$ type fundamental periodic solution obtained by satisfying the initial conditions (3) of Eq. (1) in section 2, there exists the case for the periodic solutions with period $4T_0$ of the variation equation as shown in section 3.

In such a case the solution to Eq. (1) with $x=M$ and $\dot{x}=N$ at $t=0$ is written by $x(t; M, N, E)$ and the functions are defined as follows:

$$\left.\begin{aligned} F(M, N, E)&\equiv x(4T_0; M, N, E)-x(0; M, N, E) \\ G(M, N, E)&\equiv \dot{x}(4T_0; M, N, E)-\dot{x}(0; M, N, E) \end{aligned}\right\} \tag{29}$$

Now it is evident that solution $x(t; M, N, E)$ has a period $4T_0$ when

$$F(M, N, E)=G(M, N, E)=0 \tag{30}$$

Moreover figures in which the points in the (M, N, E) space satisfy $F=G=0$ give a set of curves. In fact, the curve

$$M=M(t_1),\ N=0,\ E=E(t_1) \tag{31}$$

given by Eqs. (6) and (7) is a portion of the point set for $F=G=0$.

Let $(M_0, 0, E_0)$ be the point satisfying Eq. (31) when Eq. (19) holds. The first-order partial derivatives may be written

$$\begin{aligned} F_M(M_0, 0, E_0)&=\varphi(4T_0)-1=0 \\ F_N(M_0, 0, E_0)&=\psi(4T_0)=0 \\ F_E(M_0, 0, E_0)&=x_E(4T_0)=0 \end{aligned} \tag{32}$$

$$\left.\begin{aligned} G_M(M_0, 0, E_0) &= \dot{\varphi}(4T_0)=0 \\ G_N(M_0, 0, E_0) &= \dot{\psi}(4T_0)-1=0 \\ G_E(M_0, 0, E_0) &= \dot{x}_E(4T_0)=0 \end{aligned}\right\} \tag{33}$$

where, $\varphi(t)$ and $\psi(t)$ are the solutions of Eq. (11) and $x_E(t)$ is obtained by differentiating $x(t;\ M,\ N,\ E)$ with respect to E and evaluating the result at the point $(M_0, 0, E_0)$. This is the solution of the following equation:

$$\left.\begin{aligned} &\ddot{y}+a(t)y=\cos\omega t \\ &y(0)=\dot{y}(0)=0 \end{aligned}\right\} \tag{34}$$

From the fact the value of the first-order partial derivatives of the functions F and G is zero at the point $(M_0, 0, E_0)$, it is necessary to take into account the second-order partial derivatives of F and G for $F=G=0$, which could be found by the formulas

$$\left.\begin{aligned} &F_{MM}(M_0, 0, E_0)=0 \\ &F_{ME}(M_0, 0, E_0)=0 \\ &F_{MN}(M_0, 0, E_0)=0 \\ &F_{NN}(M_0, 0, E_0)=0 \\ &F_{NE}(M_0, 0, E_0)=x_{NE}(4T_0;\ M_0, 0, E_0) \\ &=-\frac{4(l^2-k^2)}{\dot{x}^0(t_1)}(\psi^2(t_1)+\dot{\psi}^2(T_0)\varphi^2(t_1))\,(x_E(t_1)+x_E(T_0)\varphi(t_1))\neq 0 \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} &G_{MM}(M_0, 0, E_0)=0 \\ &G_{MN}(M_0, 0, E_0)=0 \\ &G_{NN}(M_0, 0, E_0)=0 \\ &G_{NE}(M_0, 0, E_0)=0 \\ &G_{ME}(M_0, 0, E_0)=\dot{x}_{ME}(4T_0;\ M, 0, E_0) \\ &=\frac{4(l^2-k^2)}{\dot{x}^0(t_1)}(\varphi^2(t_1)+\dot{\varphi}^2(T_0)\psi^2(t_1))\,(x_E(t_1)+x_E(T_0)\varphi(t_1))\neq 0 \\ &G_{EE}(M_0, 0, E_0)=\dot{x}_{EE}(4T_0;\ M_0, 0, E_0) \\ &=-\frac{8(l^2-k^2)x_E(2T_0)}{\dot{x}^0(t_1)}(\varphi^2(t_1)-\varphi^2(T_0)\psi^2(t_1))\,(x_E(t_1)+x_E(T_0)\varphi(t_1)) \end{aligned}\right\} \tag{36}$$

Eqs. (35) and (36) may be derived in the same manner as were the equations of previous reports[6),7)] and so they need not be derived here. It should only be pointed out that $F=G=0$ has two branches in the neighborhood of the point (M_0, E_0): one is bounded on $E=E_0$ (37) and the other on the following equations:

$$N=0 \text{ and } G_{ME}(M_0, 0, E_0)\,(M-M_0)+\frac{1}{2}G_{EE}(M_0, 0, E_0)\,(E-E_0)=0 \tag{38}$$

The solution curve for which the latter branch is bounded on

$$N=0,\ M-M_0=-\frac{G_{EE}(M_0, 0, E_0)}{2G_{ME}(M_0, 0, E_0)}\,(E-E_0) \tag{38$'$}$$

satisfies $x^0(t)=x^0(t+T_0)$ and corresponds to the fundamental harmonic solution as determined by Eqs. (6) and (7). The solution bounded on the former branch $E=E_0$ can be shown to be

$$x^0(t)=\frac{a_0}{2}+a_1\cos\omega t+a_2\cos 2\omega t+\ldots\ldots$$
$$+a_{\frac{1}{4}}\cos\frac{1}{4}\omega t+a_{\frac{3}{4}}\cos\frac{3}{4}\omega t+\ldots\ldots \tag{39}$$

Finally, by using the periodicity conditions discussed so far, we obtain the results, as shown in Figs. 5(a)–(d), from a numerical analysis in the neighborhood of the branching point. The results of calculation based upon this method show good agreement with those found with the theoretical analysis.

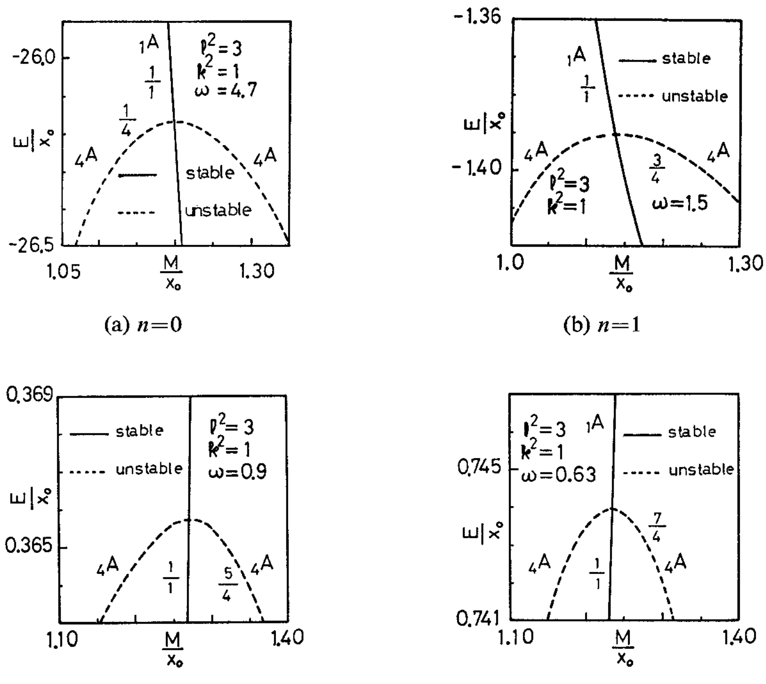

Fig. 5 Branching phenomena of solutions of order $\frac{2n+1}{4}$ ($n=0, 1, 2, \ldots$) from harmonic solution in M-E plane

5. Conclusions

In the foregoing sections the behavior of the trajectories of solutions in the neighborhood of a branching point, has been discussed, largely from the point of view of its first variation equation, having periodic solutions of period $4T_0$.

The results are summarized as follows:

(i) For the piecewise linear system, the periodicity conditions are obtained.

(ii) For the piecewise linear system, the condition that the solutions of the first variation equation has periodic solutions of period $4T_0$ can be written by a simple

equation.
(iii) Branching phenomena of the solutions of the fraction harmonics of order $(2n+1)/4$ ($n=0, 1, 2, \ldots$) from the harmonic solutions will occur under the following condition:

$$\frac{k}{\omega} < \frac{2n+1}{4} < \frac{l}{\omega}$$

It is impossible to summarize completely the branching phenomena in this report. It is hoped that this report will serve merely to point out problems that require further investigation, for example the branching phenomena from the fractional harmonics of order $(2n+1)/2$. This paper was partly published in Japanese in Trans. of IECE of Japan, vol. J64-A, 10, 870 (1981). The computations are performed on the ACOS-700 at the computer center, University of Osaka Prefecture.

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