Theoretical Treatise on the “Lightning Stroke”

Theoretical Treatise on the "Lightning Stroke"

Tokichi TSUJI* and Masao KIDO*

(Received January 31, 1955)

Lightning phenomena have been studied theoretically by various authorities for a long time. Bewley's study was one of the most famous and the most excellent of all, but his analysis for the lightning phenomena could not give rigorous interpretations by the direct application of the theoretical results already gained.

The present report deals with lightning problems by a newly established analytical method, which is quite different from those already published. By our own method we can discuss the phenomena more exactly and systematically.

Introduction

Let us consider the direct stroke to the ground at first. In practical cases, it is a few times that the lightning strikes the ground directly, and many times it strikes the things on the ground, for example, lightning rod, transmission line tower and so on.

However we will research the direct stroke to the ground as the basis of our theory, in which we will study the stroke to the tower and the lightning rod later. In other words, it can be said that our present report is the basic theory of the lightning phenomena.

Chapter I. Theory for the Lightning Stroke

§1. General Differential Equations

To obtain fundamental differential equations for the lightning stroke we make a start from following assumptions.

(1) The lightning bolt is assumed to be a "simple vertical cylinder".[1]

(2) The vertical cylinder has a definite diameter.

(3) The ground surface is in zero potential, because the resistance of the ground is very smaller than that of the space except the lightning bolt.

(4) The leakage of the stroke is neglected in the interests of mathematical simplicity.

(5) The branching of the stroke is ignored.

Now, we take points A and B, as shown in Fig. 1.1, which have height, x and $x+dx$, from the ground surface, respectively. "r" and "l" denote the resistance and inductance of the vertical cylinder per unit length at A, respectively. We assume that the positive direction of x is the same as the increase of potential E. Then we shall have the equation,

$$\frac{\partial E}{\partial x}=l\frac{\partial I}{\partial t}+rI, \qquad (1.1)$$

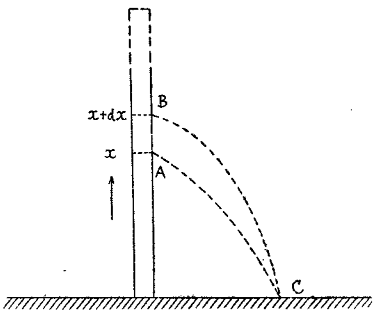

Fig. 1.1

* Department of Electrical Engineering, College of Engineering.

due to application of the law of induction to points A, B and ground surface C.

Next, according to the above-mentioned assumptions we must recognize the perfect conducting ground which is statically in zero potential. Therefore, the condition of current continuity on the conductor requires

$$\frac{\partial I}{\partial x}=c\frac{\partial E}{\partial t}. \tag{1.2}$$

(1.1) and (1.2) are general equations whose solutions, subject to the boundary conditions, yield the explicit equations of the lightning transients. "c" and "l" in these equations, which denote the capacitance and inductance per unit length at the point x above ground respectively, must be functions of x. It must be said that some authorities[1)2)3)] discussed l and c as constants with mistakes. As we shall see, l and c of the vertical conductor are of much important, but it is very difficult to know these values very exactly.

(i) Capacitance (c)

We shall, for convenience sake, calculate the value of c by making use of the formula, which gives the rigorous value of the whole capacitance of the vertical cylinder and is acknowledged as both theoretically and experimentally exact.

Origin of the coordinate is taken at the intersection of the ground level and a center line of the vertical conductor, as illustrated in Fig. 1.2, then the conductor and the ground surface may be taken as equivalent to itself and its image. The whole capacitance "C" of the conductor may be found, as well known,[2)]

$$C=\frac{x\times 10^{-9}}{18\log\frac{x}{a}},\quad a=\sqrt{3}\rho, \tag{1.3}$$

where $x\gg\rho$, $\delta\simeq 0$.

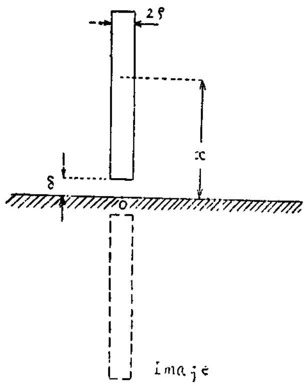

Fig. 1.2

A diameter of the lightning bolt, that is a vertical cylinder, must rest on assumed values, but is believed to be about 6 cm, very small in compared with the length of the bolt. The value of C, when the height x of the vertical cylinder is very small, is neglegible small in comparison with the value of C when x is very large. And the capacitance c at the height of x per unit length must be functions of x. Then

$$C=\int_{\text{very small}\geqq a}^{x} c\,dx,$$

and

$$c\simeq\frac{dC}{dx}.$$

Making use of equation (1.3), we find

$$c\simeq\frac{\left(\log\frac{x}{a}-1\right)\times 10^{-9}}{18\left(\log\frac{x}{a}\right)^{2}}\text{ farads/meter.} \tag{1.4}$$

This equation denotes approximate equality, when $x \gg a$. We can prove equation (1.4) without appreciable errors, by assuming that the charge density changes with height x.

(ii) Inductance (l)

Considering the conduction current to be at the periphery of the bolt, the return path is that of the displacement current in the electrostatic field, which is rectangular as indicated in Fig. 1.3. And the inductance of the vertical cylinder contributed by the superimposed fields of the conduction and displacement currents is well known formula,[2) 3)]

$$L = 2x\left(\log \frac{x}{\sqrt{\varepsilon \rho}}\right) \times 10^{-7}.$$

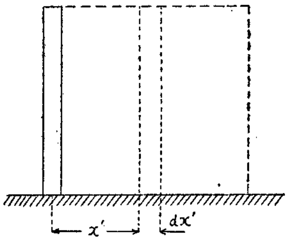

Fig. 1.3

From equation (1.3), this becomes

$$L \simeq 2x\left(\log \frac{x}{a}\right) \times 10^{-7}. \tag{1.5}$$

Therefore, the value of L, when the height x of the vertical cylinder is very small, is small enough compared with the value of L when x is very large, and the inductance l at the height x per unit length must be functions of x. Then

$$L = \int_{\text{very small} \geqq a}^{x} l\,dx,$$

and

$$l \simeq \frac{dL}{dx}.$$

Differentiating equation (1.5) with respect to x, results

$$l \simeq 2\left(\log \frac{x}{a} + 1\right) \times 10^{-7} \text{ henries/meter.} \tag{1.6}$$

Curves for c and l have been plottted in Fig. 1.4.

Calculating the product of l and c, if $a = \sqrt{3}\,\rho$, we get

$$lc = \frac{\left(\log \frac{x}{a}\right)^2 - 1}{9\left(\log \frac{x}{a}\right)^2} \times 10^{-16} \simeq \frac{10^{-16}}{9},$$

$$v = \frac{1}{\sqrt{lc}} \simeq \text{velocity of light.} \tag{1.7}$$

Thus it is seen that traveling waves on the lightning bolt travel at the velocity of light. And it illustrates the exactness of equations (1.4) and (1.6), that the above-mentioned fact identifies the general law of the electromagnetic wave, in the region of $x \gg a$.

(iii) Resistance (r)

The resistance per unit length must be constant inspite of height x, because a radius of the lightning bolt is assumed to be definite. A numerical value of r is believed to be from 0.5 to 5 ohms per meter.[2) 4)]

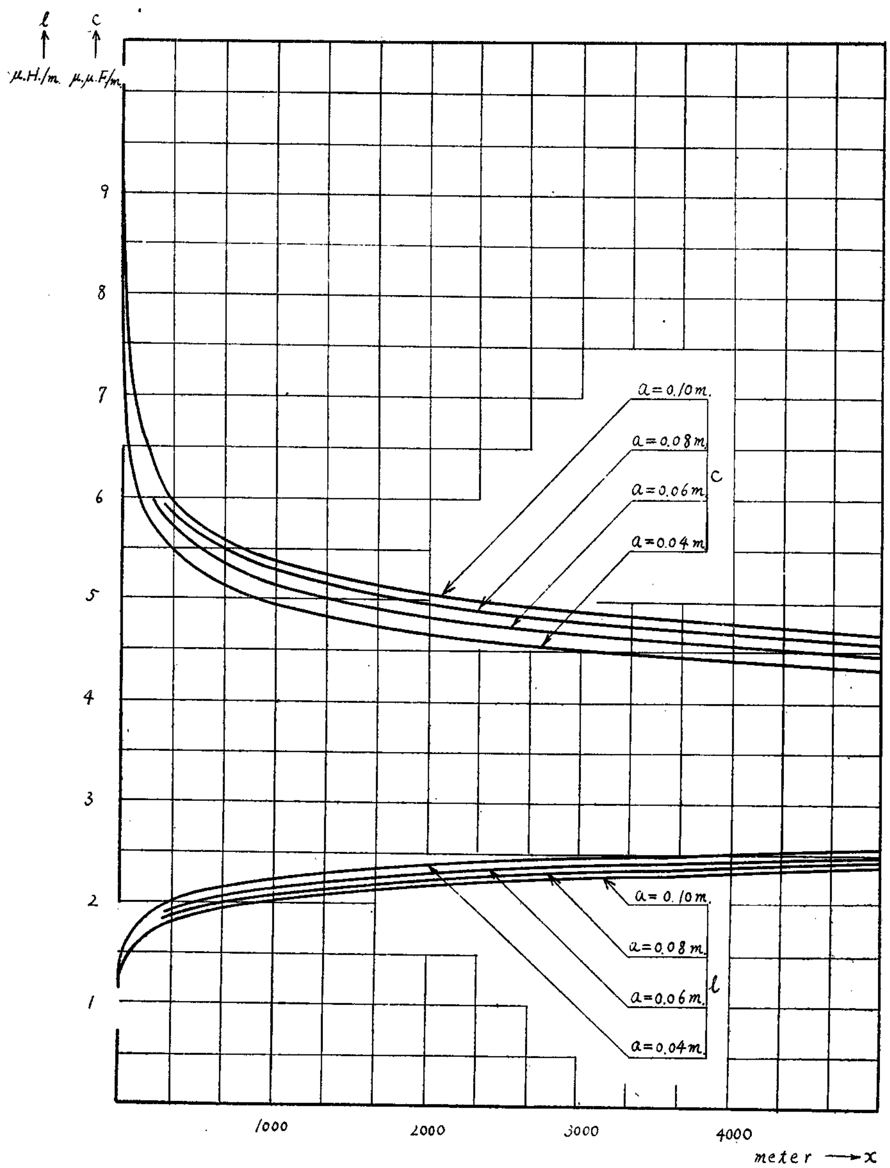

Fig. 1.4

§ 2. Solution of General Differential Equations

It is very difficult to determine the distribution of potential and current along the vertical conductor by solving differential equations (1.1) and (1.2) for the capacitance and inductance already obtained in § 1. Then we assume that the next relationships exist,

$$\left.\begin{aligned} &c = c_0 \varepsilon^{-\alpha x}, \\ &c_0, \alpha > 0, \\ &r, l = \text{constants}, \end{aligned}\right\} \tag{1.8}$$

because there is not much changes in the inductance for values of large x. But we can meet the curve of c expressed by (1. 8) with the plotted curve as shown in Fig. 1. 4 by selecting suitable constants of c_0 and l.

This assumption shall bring us the details of the "Ripples" that previous theory could not conclude in the wave front analysis of lightning waves. Examinations of the cathode-ray oscillo-grams of traveling waves caused by lightning, even if the waves were not multilated by flashover or modified by reflections, disclosed that the most of them were oscillatory and complicated from the wave front to the wave crest, for example illustrated in Fig. 1. 5.

In fact, we shall, by our own way, analysis the "Ripples" of the lightning wave and determine the oscillation period, distortion constant and etc.

Transforming equations (1. 1), (1. 2) and (1. 8), the fundamental relationships are described as follows:

$$\left.\begin{aligned} \frac{de}{dx} &= (ls+r)i\,, \\ \frac{di}{dx} &= c_0\varepsilon^{-\alpha x}se\,, \end{aligned}\right\} \qquad (1.9)$$

where $E(x, 0) = 0\,,\ \ I(x, 0) = 0\,.$

Now, changing the variable to

$$y = \varepsilon^{-\alpha x}\,, \qquad (1.10)$$

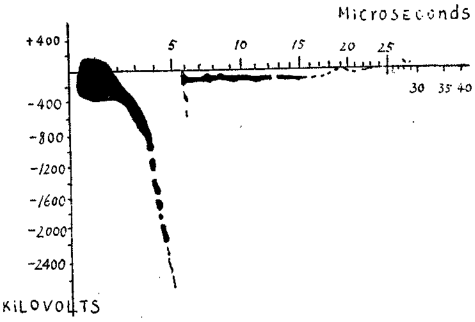

Fig. 1. 5 Typical natural lightning wave

and eliminating i from equation (1. 9), there results Bessel's equation of the first order,

$$\frac{d^2e}{dy^2}+\frac{1}{y}\frac{de}{dy}-\frac{c_0s}{\alpha^2}(ls+r)\frac{e}{y}=0\,, \qquad (1.11)$$

the solution to which is

$$e = B_1I_0(\xi\varepsilon^{-\frac{\alpha x}{2}})-B_2K_0(\xi\varepsilon^{-\frac{\alpha x}{2}})\,, \qquad (1.12)$$

$$\xi = \frac{2\sqrt{c_0l}}{\alpha}\sqrt{s\left(s+\frac{r}{l}\right)}\,, \qquad (1.13)$$

where B_1 and B_2 are constants to be determined from the boundary conditions, and I_0, K_0 are modified Bessel and Neumann functions, respectively.

Suppose that the lightning strikes the earth surface and the voltage of the cloud applied at $x=h$ is $F(t)$ or $f(s)$ in Heaviside function. Then

$$B_1I_0(\xi) = B_2K_0(\xi)\,.$$

Herefrom the integration constants are

$$B_1 = \frac{K_0(\xi)\cdot f(s)}{I_0(\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha h}{2}})}\,,$$

$$B_2 = \frac{I_0(\xi)\cdot f(s)}{I_0(\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha h}{2}})}\,,$$

and the distribution of the voltage on the lightning bolt is written,

$$e=\frac{I_0(\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha x}{2}})}{I_0(\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha h}{2}})}\cdot f(s)\,. \tag{1.14}$$

Similarly,

$$i=\frac{-\alpha\xi\varepsilon^{-\frac{\alpha x}{2}}}{2(ls+r)}\left\{\frac{I_1(\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot K_0(\xi)+I_0(\xi)\cdot K_1(\xi\varepsilon^{-\frac{\alpha x}{2}})}{I_0(\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha h}{2}})}\right\}\cdot f(s)\,. \tag{1.15}$$

For the grounded point, putting $x=0$ in (1. 15), we get

$$i_{x=0}=\frac{-\alpha\cdot f(s)}{2(ls+r)\{I_0(\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha h}{2}})\}}\,. \tag{1.16}$$

Equations (1. 14), (1. 15) and (1. 16) show that lightning waves travel along the lightning bolt or vertical cylinder from the cloud to the earth.

Chapter II. Wave Front of the Lightning Wave

In previous chapter, we established the basis of our lightning stroke theory. We will change these formulas, that is (1. 14) (1. 15) (1. 16), into other forms which are practically useful. We will apply the first term of the asymptotic expansions of I_0 and K_0 to equations (1. 14), (1. 15) and (1. 16), in according to the difficulty of direct transformation in operational calculus.

Suitable asymptotic expansions are

$$\left.\begin{aligned} I_\delta(\xi)&=\frac{\varepsilon^{\xi}}{\sqrt{2\pi\xi}}\left\{1-\frac{4\delta^2-1}{1!\,(8\xi)}+\frac{(4\delta^2-1)(4\delta^2-3)}{2!\,(8\xi)^2}-\cdots\cdots\right\},\\ K_\delta(\xi)&=\sqrt{\frac{\pi}{2\xi}}\cdot\varepsilon^{-\xi}\left\{1+\frac{4\delta^2-1}{1!\,(8\xi)}+\frac{(4\delta^2-1)(4\delta^2-3)}{2!\,(8\xi)^2}+\cdots\cdots\right\},\end{aligned}\right\} \tag{2.1}$$

where ξ satisfies the relationship,

$$\xi=\frac{2\sqrt{c_0 l}}{\alpha}\sqrt{s\left(s+\frac{r}{l}\right)}\gg 1\,. \tag{2.2}$$

As an example, take

$$c_0=10\ \mu.\mu.\ \text{farads/meter},\quad l=2.5\ \mu.\ \text{henries/meter},\quad \alpha=5\times10^{-4}/\text{meter},$$

then

$$2\sqrt{c_0 l}/\alpha=2\times10^{-5}\,.$$

Thus, we can not apply the above-mentioned asymptotic expansions to equations (1. 14), (1. 15) and (1. 16) except the domain in which s is large enough and consequently general solutions of such transient phenomena will not be reduced by means of the application of this method.

However, if s tends to the neighbourhood of infinity, it may be taken the form,

$$I_\delta(\xi)=\frac{\varepsilon^{\xi}}{\sqrt{2\pi\xi}},\quad K_\delta(\xi)=\sqrt{\frac{\pi}{2\xi}}\cdot\varepsilon^{-\xi}\,. \tag{2.3}$$

That is to say, we are able to discuss the physical characteristics of the wave front traveling along the vertical conductor by substituting (2. 3) in (1. 4), (1. 15) and (1. 16). Let us consider and explain the wave front of lightning waves in this chapter.

§1. Velocity of Propagation of the Wave Front

It was already described that the lightning wave propagates towards the earth surface from the cloud along the vertical cylinder or the lightning bolt. Firstly we shall calculate the velocity of propagation.

Substitute in equations (1. 4) and (1. 5), we have finally,

$$e_{front}=f\varepsilon^{-\frac{\alpha}{4}(h-x)-(\theta'-\theta)\xi}, \tag{2.4}$$

$$i_{front}=f\sqrt{\frac{c_0\varepsilon^{-\alpha x}}{l}}\cdot\sqrt{\frac{s}{s+\frac{r}{l}}}\cdot\varepsilon^{-\frac{\alpha}{4}(h-x)-(\theta'-\theta)\xi}, \tag{2.5}$$

in which

$$\theta=1-\varepsilon^{-\frac{\alpha x}{2}},\quad \theta'=1-\varepsilon^{-\frac{\alpha h}{2}}. \tag{2.6}$$

From these equations an interesting result is drawn: a traveling wave will reach the point x at time

$$t=\frac{2\sqrt{c_0 l}}{\alpha}(\varepsilon^{-\frac{\alpha x}{2}}-\varepsilon^{-\frac{\alpha h}{2}}). \tag{2.7}$$

Now, representing the velocity of propagation to be v, there results

$$v=\frac{1}{\sqrt{lc_0\varepsilon^{-\alpha x}}}=\frac{1}{\sqrt{l\,c_x}}. \tag{2.8}$$

The traveling wave along the vertical conductor will propagate from the cloud to the earth with the velocity given by equation (2. 8). "c_x" is the capacitance per unit length at heigh x above the ground level, in equation (2. 8).

§2. Wave Front Impedance, Z_{front}

From equations (1. 14) and (1. 15), impedance "Z" becomes,

$$Z=-\frac{2(ls+r)}{\alpha\xi\varepsilon^{-\frac{\alpha x}{2}}}\cdot\left\{\frac{I_0(\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot K_0(\xi)-I_0(\xi)\cdot K_0(\xi\varepsilon^{-\frac{\alpha x}{2}})}{I_1(\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot K_0(\xi)+I_0(\xi)\cdot K_1(\xi\varepsilon^{-\frac{\alpha h}{2}})}\right\}.$$

Expanding asymptotically, we obtain,

$$Z=-\frac{2(ls+r)}{\alpha\xi\varepsilon^{-\frac{\alpha x}{2}}}\left[\frac{\varepsilon^{-\xi\theta}-\varepsilon^{\xi\theta}}{\varepsilon^{-\xi\theta}+\varepsilon^{\xi\theta}}+\frac{1}{8\xi}\left\{(\varepsilon^{-\frac{\alpha h}{2}}-1)+\cdots\cdots\right\}+\cdots\cdots\right].$$

Similarly with §1, assuming that s and ξ are large enough, according to the procedure operational calculus, there results

$$Z_{front}=\lim_{S\to Large\ enough}\sqrt{\frac{ls+r}{c_0\varepsilon^{-\alpha x}s}}\simeq\sqrt{\frac{l}{c_0\varepsilon^{-\alpha x}}}=\sqrt{\frac{l}{c_x}}, \tag{2.9}$$

where Z_{front} denotes the wave front impedance.

§ 3. Attenuation and Distortion of the Wave Front traveling along the Vertical Conductor

As a traveling wave moves along the vertical conductor, it suffers various changes. In order to know changes of the wave, transforming (2.4) and (2.5), there becomes

$$\left.\begin{aligned} E_{front}(x,t) &= \pm A\varepsilon^{-\frac{\alpha}{4}(h-x)}\left\{\varepsilon^{-\frac{r}{2l}t_1}+\frac{rt_1}{2l}\int_{t_1}^{t}\varepsilon^{-\frac{r}{2l}\tau}\frac{I_1\left(\frac{r}{2l}\sqrt{\tau^2-t_1^2}\right)}{\sqrt{\tau^2-t_1^2}}d\tau\right\}\cdot H(t-t_1)\,,\\ I_{front}(x,t) &= \pm A\sqrt{\frac{c_0\varepsilon^{-\alpha x}}{l}}\cdot\varepsilon^{-\frac{\alpha}{4}(h-x)-\frac{rt}{2l}}\cdot I_0\left(\frac{r}{2l}\sqrt{t^2-t_1^2}\right)\cdot H(t-t_1)\,, \end{aligned}\right\} \quad (2.10)$$

in which

$$\left.\begin{aligned} t_1 &= \frac{2\sqrt{c_0 l}}{\alpha}\left(\varepsilon^{-\frac{\alpha x}{2}}-\varepsilon^{-\frac{\alpha h}{2}}\right),\\ H(t-t_1) &= 0\,,\quad \text{when}\quad t<t_1\,,\\ &= 1\,,\quad \text{when}\quad t\geqq t_1\,,\\ F(t) &= \pm A\cdot H(t)\,. \end{aligned}\right\} \quad (2.11)$$

" t_1 " means the time in which a wave of the cloud potential reaches the point whose height is x above ground.

Thus, it may be concluded that formulas of the voltage and current distribution along the lightning bolt have identical forms with the solution[6] to the " telegraph equation " having distribution constants r, l and c, for mathematical expressions. On the other hand, wave front velocity or time of propagation and wave front impedance will differ only from those in the telegraph equation. Eqations (2.10) are the general formulas representing distortions and attenuations of the wave front traveling along the vertical conductor, however it must be remembered that solutions (2.10) could not be rigorously adapted to the phenomena in crest part and tail of the traveling wave.

Making use of t_1 in equation (2.11), values at the tip of the wave become

$$\left.\begin{aligned} E_f(x,t_1) &= \pm A\varepsilon^{-\frac{\alpha}{4}(h-x)-\frac{rt_1}{2l}}\,,\\ I_f(x,t_1) &= \pm A\sqrt{\frac{c_0\varepsilon^{-\alpha x}}{l}}\cdot\varepsilon^{-\frac{\alpha}{4}(h-x)-\frac{rt_1}{2l}}\,. \end{aligned}\right\} \quad (2.12)$$

Comparison of (2.10) and (2.12) shows us that attenuation and distortion of the wave front are due to the factor

$$\frac{\alpha}{4}(h-x)+\frac{rt_1}{2l}\,.$$

This attenuation factor does not appear in the lines with definite constants, but only in the line with exponential taper, such as equivalent circuit representing lightning strokes.

When a voltage wave approaching along a vertical conductor reaches the ground level, it will be reflected backward as a negative wave along the same conductor, so that the approaching wave being cancelled out. Therefore,

$$E_{front}(x,t)=0\,,\quad E_f(x,t_2)=0\,.$$

On the other hand, a current wave will give rise to

$$I_{front}(x, t) = \pm 2A\sqrt{\frac{c_0}{l}} \cdot \varepsilon^{-\frac{\alpha h}{4} - \frac{rt}{2l}} \cdot I_0\left(\frac{r}{2l}\sqrt{t^2 - t_2^2}\right) \cdot H(t - t_2) , \tag{2.13}$$

$$\left.\begin{aligned} I_f(x, t_2) &= \pm 2A\sqrt{\frac{c_0}{l}} \cdot \varepsilon^{-\frac{\alpha h}{4} - \frac{rt_2}{2l}} , \\ t_2 &= \frac{2\sqrt{c_0 l}}{\alpha}(1 - \varepsilon^{-\frac{\alpha h}{2}}) , \end{aligned}\right\} \tag{2.14}$$

reflecting backward along the same conductor. But $I_{front}(x, t)$ shows the value of the tip of the current wave. These results will identify with those transformed directly from solution (1.16).

At the present paragraph we dealt with newly established analysis to the wave front of lightning surges, and in our conclusions, as a traveling wave moves along the vertical conductor it suffers various changes:

(a) the wave front decreases in magnitude or is attenuated,

(b) the wave changes its shape, that is, becomes more elongated or smoothed out.

§ 4. Newly-reduced Formulas

In 1930, Reinhold Rüdenberg proposed that a condition at the tip of a lightning bolt should be[7)]

$$I = 2\pi\varepsilon_0 v V = 5.56\, v V , \tag{2.15}$$

where I and V denoted the current and voltage at the tip of the lightning bolt, respectively, and v was velocity of propagation of its tip. However in his discription many mistakes were found, for examples;

(1) he assumed a shape of its tip to be semi-spherical and didn't consider the other part,

(2) effect of the earth was ignored,

(3) a diameter of the spherical part was assumed to be so large that it couldn't be neglected in compared with the length of the bolt.

Next, we will reduce a formula in which above-mentioned errors are collected. From (2.8) and (2.9), for e and i at the wave front,

$$i = e \cdot c_x \cdot v .$$

Herefrom

$$I = E \cdot c_x \cdot v . \tag{2.16}$$

In foregoing analysis, c_x was regarded as the exponential change and the wave was considered to be traveling along an already established conductor. But it is possible to reduce the same equation with (2.16), even if following assumptions are taken:

(i) the lightning bolt is raising up with passage of time,

(ii) value of c_x is given by equation (1.4),

(iii) the shape of its tip is assumed to be semi-spherical.

In this paragraph, we will introduce formulas reduced from above-mentioned assumptions by our own way. At first, let us consider a shape of the tip to be semi-spherical and the lightning bolt to be cylindrical. Capacitance of the lightning bolt per unit length at height x will be equal to that of the cylinder with the same length, as shown in Fig. 2.1, because a radius of the lightning bolt is very smaller than the height of the lightning bolt. Therefore, the capacitance per unit length can be written, by the application of our theory in Chapter I,

$$c_x = \frac{\log \frac{x}{a} - 1}{18\left(\log \frac{x}{a}\right)^2} \times 10^{-9} \quad \text{farads/meter.} \tag{2.17}$$

V and I represent the voltage and current at the section whose height is x from the earth, respectively, and v denotes the velocity of a lightning stroke. Then, charge dQ stored in the part dx becomes

$$dQ = c_x V\, dx\,. \tag{2.18}$$

Assumed that the tip moves the part dx during time dt,

$$dQ = I\, dt\,, \tag{2.19}$$

$$v = \frac{dx}{dt}\,. \tag{2.20}$$

Fig. 2.1

By (2.17), (2.18), (2.19) and (2.20), we obtain

$$I = \frac{\left(\log \frac{x}{a}\right) - 1}{18\left(\log \frac{x}{a}\right)^2} \cdot v V \times 10^{-9} = v \cdot c_x \cdot V\,. \tag{2.21}$$

Equation (2.21) has the identical form with (2.16), and is rewritten,

$$V = \frac{1}{v} \cdot g(x) \cdot I\,. \tag{2.22}$$

These are relations at the tip of the lightning bolt, which were reduced by us. In equation (2.21), provided,

$$v = \frac{1}{\sqrt{l_x c_x}}\,,$$

$$c_x = \frac{\log \frac{x}{a} - 1}{18\left(\log \frac{x}{a}\right)^2} \times 10^{-9}\,,$$

$$l_x = 2\left(\log \frac{x}{a} + 1\right) \times 10^{-7}\,,$$

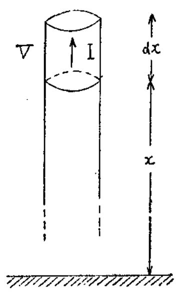

Fig. 2.2

then, without appreciable errors,

$$v = [c]\,, \quad \text{velocity of light.}$$

Thus we get finally

$$I = [c] \cdot c_x \cdot V = \frac{V}{[c] l_x}\,. \tag{2.23}$$

In foregoing analysis, we clarified the state of the wave front assuming that a wave of the cloud traveled along an already established conductor. However, we obtained the same results even if it was assumed that the lightning bolt was going up to the cloud with time. If the plarity of the cloud is different, the same formulas will be reduced by assuming that the lightning is coming down from cloud to earth. We shall propose the above-mentioned formulas which will take place of the Rüdenberg's formula.

Differences between Rüdenberg's formula and our formulas are illustrated in Fig. 2. 3. Our formulas are able to be used for discussion of the phenomena generally except in the neighbourhood of the ground level, because a diameter of the lightning bolt is from 0.04 meter to 0.10 meter by various surveys and researches.

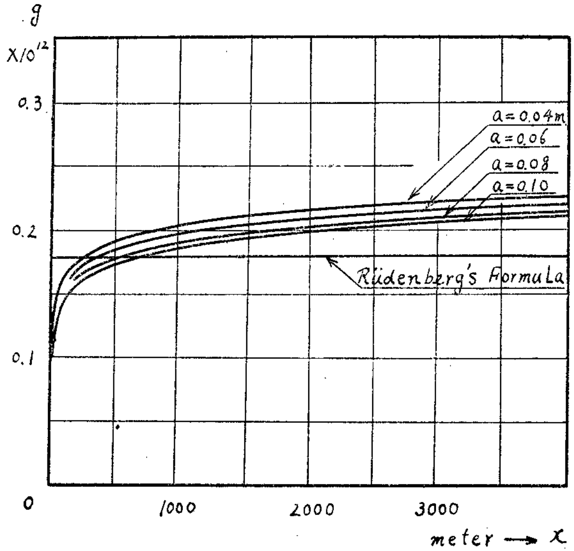

Fig. 2. 3

Chapter III. State in the Neighbourhood of the Steady State (or Crest Value)

We have already studied the state of the wave front in details, and we will examine the state in the neighbourhood of the steady state or crest value in this chapter. In order to explain the "Ripples", which we described in Chapter I, we shall research fluctuations of lightning waves chiefly.

§1. Wave Forms are oscillatory in the Neighbourhood of the Steady State

Rewritting solutions (1. 14), (1. 15) and (1. 16) by making use of $J_0(j\xi\varepsilon^{-\frac{\alpha x}{2}})$ and $Y_0(j\xi\varepsilon^{-\frac{\alpha x}{2}})$, we have

$$\left.\begin{aligned}
e &= \frac{J_0(j\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha x}{2}})}{J_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})}\cdot f(s)\,,\\
i &= \frac{\alpha j\xi\varepsilon^{-\frac{\alpha x}{2}}}{2(ls+r)}\left\{\frac{J_1(j\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_1(j\xi\varepsilon^{-\frac{\alpha x}{2}})}{J_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})}\right\}\cdot f(s)\,,\\
i_{x=0} &= \frac{\alpha f(s)}{\pi(ls+r)\{J_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\}}\,.
\end{aligned}\right\}\qquad (3.1)$$

These operational equations can be solved by the Heaviside expansion theorem. For its purpose, we must solve the following equation with respect to s or ξ.

$$\left.\begin{aligned}(ls+r)\{J_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\}=0\,.\\ \xi=\frac{2\sqrt{c_0 l}}{\alpha}\sqrt{s\left(s+\frac{r}{l}\right)}\,.\end{aligned}\right\}\qquad(3.2)$$

Putting in second factor of equation (3.2),

$$\left.\begin{aligned}j\xi\varepsilon^{-\frac{\alpha h}{2}}&=\phi\,,\\ \varepsilon^{\frac{\alpha h}{2}}&=k>1\,,\end{aligned}\right\}\qquad(3.3)$$

we have

$$ls+r=0\,,$$

or

$$J_0(\phi)\cdot Y_0(\phi k)-J_0(\phi k)\cdot Y_0(\phi)=0\,.$$

According to McMahon's research, roots of the last equation are obtained as follows[8],

$$\phi_n=\frac{n\pi}{k-1}+\frac{\frac{-1}{8k}}{\frac{n\pi}{k-1}}+\frac{\frac{100(k^3-1)}{3(8k)^3(k-1)}-\left(\frac{-1}{8k}\right)^2}{\left(\frac{n\pi}{k-1}\right)^3}+\cdots\cdots,\qquad(3.4)$$

in which positive roots are placed as the order of their magnitude. Therefore, we have the roots of equation (3.2) as follows,

$$\left.\begin{aligned}s_{\pm n}&=-\frac{r}{2l}\pm\frac{1}{2}\sqrt{\frac{r^2}{l^2}-\frac{\phi_n^2\alpha^2k^2}{c_0 l}}\,,\\ s_0&=-\frac{r}{l}\,.\end{aligned}\right\}\qquad(3.5)$$

Applying the Heaviside expansion theorem to (3.1), and assuming that a potential of the cloud is of rectangular wave form with height ($\pm A$), the current to the ground is given by

$$I_{x=0}=\pm A\left\{\frac{1}{rh}+\sum_{s_0,\,s_{\pm n}}\frac{\exp(st)}{sP'(s)}\right\},\qquad(3.6)$$

in which

$$P'(s)=\frac{\pi}{\alpha}\cdot\frac{d}{ds}\left[(ls+r)\{J_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha h}{2}})\}\right].\qquad(3.7)$$

From equation (3.6) it is evident that the value at the steady state is ($\pm A/rh$).

Similarly, the voltage and current at the point x along the vertical conductor will be written,

$$\left.\begin{aligned}E&=\pm A\left\{\frac{x}{h}+\sum_{s_0,\,s_{\pm n}}\frac{Q_1(s)}{sP'(s)}\cdot\exp(st)\right\},\\ I&=\pm A\left\{\frac{1}{rh}+\sum_{s_0,\,s_{\pm n}}\frac{Q_2(s)}{sP'(s)}\cdot\exp(st)\right\},\end{aligned}\right\}\qquad(3.8)$$

where

$$Q_1(s)=\frac{\pi}{\alpha}(ls+r)\{J_0(j\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_0(j\xi\varepsilon^{-\frac{\alpha x}{2}})\}\,,$$

$$Q_2(s)=\frac{\pi}{2}j\xi\varepsilon^{-\frac{\alpha x}{2}}\{J_1(j\xi\varepsilon^{-\frac{\alpha x}{2}})\cdot Y_0(j\xi)-J_0(j\xi)\cdot Y_1(j\xi\varepsilon^{-\frac{\alpha x}{2}})\}\,.$$

At the steady state the current along the lightning bolt has a definite value "$\pm A/hr$", while the potential is proportional to the height, namely $(\pm Ax/hr)$.

From the above results it is obvious that E, I, and $I_{x=0}$ consist of the infinite sum of exponential functions and will change exponentially with time as soon as the circuit of the stroke has been made.

If s is imaginary with negative real part, the solution corresponding to (3.6) involves terms as follows,

$$\varepsilon^{-\lambda_1 t}(b_1 \cos \lambda_2 t - b_2 \sin \lambda_2 t)\,. \qquad (3.9)$$

From (3.4)

$$\lambda_1 = \frac{r}{2l}\,, \quad \lambda_2 = \frac{1}{2}\sqrt{\frac{\phi_n^2 \alpha^2 k^2}{c_0 l} - \frac{r^2}{l^2}}\,, \qquad n = 1, 2, 3, \cdots\cdots .$$

Obviously equation (3.9) takes a form of damped oscillations with their attenuation constant "λ_1", phase constant "λ_2" and their amplitude $2\sqrt{b_1^2+b_2^2}$, as plotted in Fig. 3.1. Therefore the current and voltage along the vertical conductor are made up of superpositions of damped oscillated waves with different phase constant and amplitude, and exponential waves with different phase constant and amplitude, whose numbers are infinite. Owing to this fact, so-called "Ripples" exist between the wave front and the steady state.

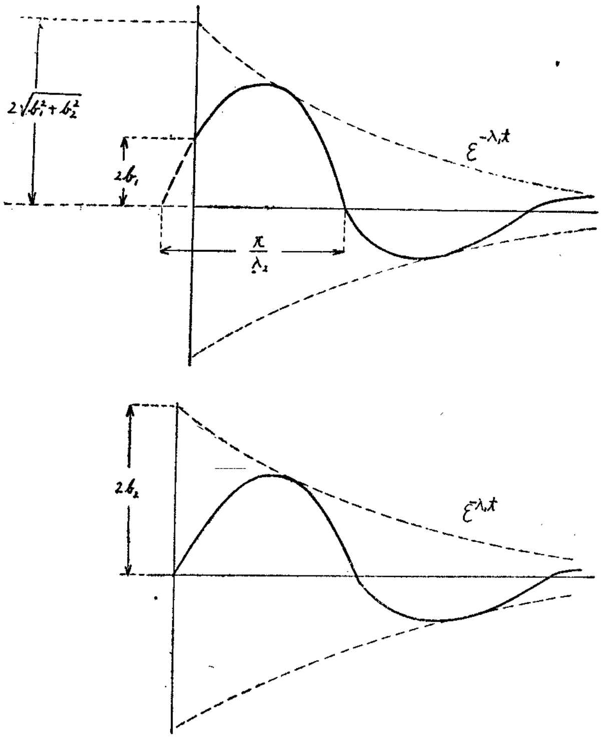

Fig. 3.1

§ 2. Details of "Ripples"

Equations (3.6) and (3.8) illustrate that the current and potential are oscillatory with the same oscillation period, because imaginary roots in various values of s are the same in these equation. Thus in present report, we will discuss the oscillations of the current to the ground.

At first, we get by substituting (3.5) in equation (3.7),

$$s_0 P'(s_0) = -rh\,,$$

and

$$s_{\pm n}P'(s_{\pm n}) = \pm\frac{\pi l\phi_n k}{2\alpha}\sqrt{\frac{r^2}{l^2}-\frac{\phi_n^2\alpha^2k^2}{c_0 l}}$$
$$\times[\{J_0(\phi_n)\cdot Y_1(\phi_n k)-J_1(\phi_n k)\cdot Y_0(\phi_n)\}-k^{-1}\{J_0(\phi_n k)\cdot Y_1(\phi_n)-J_1(\phi_n)\cdot Y_0(\phi_n k)\}].$$

For values of ϕ_n which satisfy the relationship

$$\frac{\phi_n^2\alpha^2k^2}{c_0 l} > \frac{r^2}{l^2}, \tag{3.10}$$

there will be different type oscillations corresponding to each values of ϕ_n. By some calculations we shall find next relationships in equation (3.9),

$$\left.\begin{aligned}
&b_1 = 0 \quad , \quad b_{2,n} = \frac{-2\alpha}{\pi lk\phi_n M\sqrt{-N}}, \\
&\lambda_1 = \frac{r}{2l}, \quad \lambda_{2,n} = \frac{1}{2}\sqrt{-N}, \\
&N = \frac{r^2}{l^2}-\frac{\phi_n^2\alpha^2k^2}{c_0 l}, \\
&M = \{J_0(\phi_n)\cdot Y_1(\phi_n k)-J_1(\phi_n k)\cdot Y_0(\phi_n)\} \\
&\qquad -k^{-1}\{J_0(\phi_n k)\cdot Y_1(\phi_n)-J_1(\phi_n)\cdot Y_0(\phi_n k)\},
\end{aligned}\right\} \tag{3.11}$$

and equation (3.9) will be reduced to the form

$$-2\sum_n b_{2,n}\,\varepsilon^{-\lambda_1 t}\sin\lambda_{2,n}t. \tag{3.12}$$

The oscillation periods T_2 and the time constant T_1 are

$$\left.\begin{aligned}
T_1 &= \frac{1}{\lambda_1} = \frac{2l}{r}, \\
T_{2,n} &= \frac{2\pi}{\lambda_{2,n}}.
\end{aligned}\right\} \tag{3.13}$$

On the contrary, for values of ϕ_n which satisfy

$$\frac{r^2}{l^2} \geqq \frac{\phi_n^2\alpha^2k^2}{c_0 l},$$

wave forms will become exponential, and equation (3.12) is verified to be

$$\sum_n b_{0,n}(\varepsilon^{p_{1,n}t}-\varepsilon^{p_{2,n}t}), \tag{3.14}$$

in which

$$\left.\begin{aligned}
b_{0,n} &= \frac{2\alpha}{\pi lk\phi_n M\sqrt{N}}, \\
p_{1,n} &= -\frac{r}{2l}+\frac{\sqrt{N}}{2}, \\
p_{2,n} &= -\frac{r}{2l}-\frac{\sqrt{N}}{2}.
\end{aligned}\right\} \tag{3.15}$$

For the value of $s_0 = -r/l$, a wave form will be represented by $(b_{0,0}\varepsilon^{s_0 t})$, where

$$b_{0,0} = -1/rh.$$

Equation (3.6) illustrates the oscillatory curve between the wave front and steady state, because the curves given by (3.12) become inharmonious with time passage. This fact can be understood by considering that the reflection coefficient takes a complicated form of when s is not large[9].

The oscillatory curves, which are given by (3. 12), and the exponential curves, which are represented by (3. 14), will become smaller and smaller with the passage of time, and they are died out at steady state. Thus the lightning current to the ground will reach to $(\pm A/rh)$ at steady state.

Conclusion

We have established the general equations for the traveling wave along the vertical conductor by consideration of the capacitance change of the vertical conductor which was ignored all this while.

And by solving these equations, we have clarified and explained many facts which could not be known with regard to distortion and attenuation of the lightning wave.

References

1) L. V. Bewley: G. E. Rev. Vol. 36 No. 12 (1933).
2) N. Mita: Researches of the Electrotechnical Laboratory No. 510 Jan. (1950).
3) L. V. Bewley: Traveling waves on Transmission Systems
4) J. W. Flowers: Phys. Rev. 64. 225 (1943).
5) W. W. Lewis: Protection of Transmission System 72.
6) Y. Moriwaki: Cal. of Transient Phenomena. 327–332
7) R. Rüdenberg: Wissenschaftliche Veröffentlichungen aus dem Siemens Konzern, Band IX, (1930)
8) Y. Kodaira: Phys. Math. Vol. 1. 351.
9) Lecture of I. E. E. J. No. 24, May, (1955).