The Shearing Stress in the Transition Region in the Flow along a Flat Plate

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Considering the correlation between the turbulent velocity in the main stream and the velocity fluctuation in the laminar boundary layer of a smooth flat plate which is set parallel to the main stream, we have obtained that the shearing stress in the transition region is exponentially depends on the intensity of the turbulence in the main stream and square-root of the distance froom the leading edge of the plate.

1. Introduction

For the sufficietly long smooth flat plate, the incompressible laminar flow in the boundary layer near the leading edge of a plate at zero incidence is followed by a transition to turbulence farther down stream. The flow does not become a fully developed turbulent flow immediately, it ceaces to follow the laws of laminar flow: there is a finite transition region. This transition starts by the effects of the various phenomena. In this paper, we discuss the case in which the transition depends on the turbulence in the main stream with the mean velocity U. The various experimental results concerning the transition report the following[1][2][3] evidences.

i) At a point in the neighbourhood of the surface of the plate in the boundary layer, the mean velocity of the laminar flow u decreases with the increase of the distance x from the leading edge of the plate and reaches a minimum value and in the next-down stream, suddenly increases to the maximum value, and in farther down stream, the mean velocity again decreases, but does not reach another minimum value. In general, the point at which the mean velocity u reaches the minimum value is called a transition point[1], and the part at which u increases in called a transition region, therefore the transition point is situated at the starting point of the transition region. The local turbulence arises in the transition region.

ii) The distribution line of the static pressure indicates the presure drop and the thickhess of the boundary layer δ increases suddenly in the transition region. The reading of the manomer oscillates strongly in the same region.

iii) The values of the Reynolds number $R_x=\frac{Ux}{\nu}$ (x is the distance from the leading edge of the plate, and ν is the kinematic viscosity) at the transition point vary in such a wide range as between 9×10^4 and 3×10^6, therefore it seems to be difficult to determine the critical Reynolds number theoretically. Dimensional considerations suggest that in any particular experiment the transition region starts when R_x reaches a certain value dependent on the turbulence in the main stream.

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iv) The shearing stress at the plate decreases with an increasing distance of the down-stream both in the laminar and in the fully developed turbulent region; in the transition region, however, the shearing stress at the upstream is considerably less than that at the downstream end.

In this paper, we discuss this shearing stress theoretically.

2. Equations concerning the shearing stress in the transition region

We take x_1-axis to the direction of the stream along the surface of the plate and x_2-axis perpendicular to the surface of the plate and x_3-axis along the surface and perpendicular to the direction of the stream, and u_1, u_2, u_3 are the turbulent velocity components in the main stream refer to the x_1, x_2, x_3 direction, and the pressur fluctuation is p, then the equation of motion for the turbulent velocity is

$$\frac{\partial u_i}{\partial t} = -\frac{\partial(u_i u_k)}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i, \tag{1}$$

where $$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

In the boundary layer, the components of the velocity fluctuation respecting to the direction of x_1 and x_2 are u_1' and u_2', and the equations of motion for these components are

$$\frac{\partial u_1'}{\partial t} = -U\frac{\partial u_1'}{\partial x_1} - u_2'\frac{dU}{dx_2} - \frac{1}{\rho}\frac{\partial p'}{\partial x_1} + \nu \nabla'^2 u_1', \tag{2}$$

$$\frac{\partial u_2'}{\partial t} = -U\frac{\partial u_2'}{\partial x_2} - \frac{1}{\rho}\frac{\partial p'}{\partial x_2} + \nu \nabla'^2 u_2', \tag{3}$$

where p' is the pressure fluctuation in the boundary layer, $\nabla'^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

We now assume that the turbulence in the main stream is homogeneous and isotropic, but in the present case, we consider the main stream and the boundary layer together, then this compound stream is non-homogeneous and non-isotropic, then we consider the correlation between u_2 and u_1'. The equation of motion for u_2 is derived from (1) and we have

$$\frac{\partial u_2}{\partial t} = -\frac{\partial(u_2 u_k)}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu \nabla^2 u_2. \tag{4}$$

On multiplying the equation (2) by u_2 and the equation (4) by u_1' and adding two, we obtain the time rate of change of $u_2 u_1'$ of which the average is

$$\begin{aligned}\frac{\partial(\overline{u_2 u_1'})}{\partial t} = &-\overline{u_1'\frac{\partial(u_2 u_k)}{\partial x_k}} - U\overline{u_2\frac{\partial u_1'}{\gamma x_1}} - \overline{u_2 u_2'}\frac{dU}{dx_2} \\ &-\frac{1}{\rho}\left(\overline{u_1'\frac{\partial p}{\partial x_2}} + \overline{u_2\frac{\partial p'}{\partial x_1}}\right) + \nu(\overline{u_1'\nabla^2 u_2 + u_2 \nabla'^2 u_1'}),\end{aligned} \tag{5}$$

Similarly, the equation for u_1 is

$$\frac{\partial u_1}{\partial t} = -\frac{\partial(u_1 u_k)}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu \nabla^2 u_1. \tag{6}$$

On multiplying equation (3) by u_1 and equation (4) by u_2' and adding two, we have

$$\frac{\partial(\overline{u_1 u_2'})}{\partial t} = -\overline{u_2'\frac{\partial(u_1 u_k)}{\partial x_k}} - U\overline{u_1\frac{\partial u_2'}{\partial x_2}} - \frac{1}{\rho}\left(\overline{u_2'\frac{\partial p}{\partial x_1}} + \overline{u_1\frac{\partial p'}{\partial x_2}}\right)$$
$$+\nu(\overline{u_2'\nabla^2 u_1} + \overline{u_1\nabla'^2 u_2'}) \,. \qquad (7)$$

In the above equations, u_1, u_2 are the function of $(x_1, x_2-\delta, x_3\,;\, t)$ and u_1', u_2' are restrict in $x_2 \leqq \delta$. According to the theory concerning the transition which is given by G. I. Taylor[(4)] and the results of the various experiments, it seems to be controlled by variations in the pressure gradient due to turbulence in the main stream, then the fluctuations in the pressure distribution will cause the transition in the boundary layer, and the velocity fluctuation is small as compared with the pressure gradient. $\frac{dU}{dx_2}$ is also small in the transition region except the immediate neighbourhood of the surface of the plate and we assume that the energy transfers among the fluid elements without dissipation which is due to the viscosity, then we obtain the following equations

$$\frac{\partial(\overline{u_2 u_1'})}{\partial t} = -\frac{1}{\rho}\left\{\frac{\partial}{\partial x_2}(\overline{u_1' p}) + \frac{\partial}{\partial x_1}(\overline{u_2 p'})\right\} + \frac{1}{\rho}\left(\overline{p\frac{\partial u_1'}{\partial x_2}} + \overline{p'\frac{\partial u_2}{\partial x_1}}\right), \qquad (8)$$

$$\frac{\partial(\overline{u_1 u_2'})}{\partial t} = -\frac{1}{\rho}\left\{\frac{\partial}{\partial x_1}(\overline{u_2' p}) + \frac{\partial}{\partial x_2}(\overline{u_1 p'})\right\} + \frac{1}{\rho}\left(\overline{p\frac{\partial u_2'}{\partial x_1}} + \overline{p'\frac{\partial u_1}{\partial x_2}}\right). \qquad (9)$$

The first term in the right hand-side of equation (8) concerns the diffusion of the energy, and similar in the equation (9). In general, the diffusion arises by the correlation between three components of the velocity fluctuation, the correlation between pressure and velocity fluctuation, and the effect of the viscosity. In the present case, however, these effects are small and we neglect the first term in the right hand-side of the equation (8) and (9), and put $p=p'$ approximately, then we have

$$-\frac{\partial(\overline{u_2 u_1'})}{\partial t} = \frac{1}{\rho}\overline{p\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)}, \qquad (10)$$

$$-\frac{\partial(\overline{u_1 u_2'})}{\partial t} = \frac{1}{\rho}\overline{p\left(\frac{\partial u_2'}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)}. \qquad (11)$$

J. Rotta (5) has calculated in the non-homogeneous turbulence

$$\frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} = -k\frac{\sqrt{E}}{L}(\overline{u_i u_j}), \quad i \neq j \qquad (12)$$

where $E=\frac{1}{2}\sum_{i=1}^{3} u_i^2$, k is numerical factor, L is diameter of the turbulent element. To determine the L, we take the momentum loss M in the boundary layer

$$M = \rho\int_0^h u(U-u)dx_2 = \rho U^2 \delta^{**}, \qquad (13)$$

where u is the stream velocity in the boundary layer, $x_2=0$ and $x_2=h$ include practically all those influenced by the friction, δ^{**} is the measure of the momentum change and equal to $\delta/6$ for the flat plate. In the transition region, the increase of the thickness of the boundary layer depends on the decrease of the static pressure, then we assume that the distance between the centre of the turbulent element and the centre of fluctuate element in the boundary layer equal to δ^{**} approximately when these elements impact each other and the energy transmites from the direction of the impact to another direction, then we can take δ^{**} instead of L and we have

$$-\frac{\partial(\overline{u_2 u_1'})}{\partial t} = \frac{k\sqrt{E}}{\delta^{**}}\overline{u_2 u_1'}. \tag{14}$$

In this equation we put $u_1 = u_1'$, $u_2 = u_2'$ nearly and integrate, we have

$$-\overline{u_2 u_1} = \frac{1}{f(r)} e^{\frac{k\sqrt{E}t}{\delta^{**}}}, \tag{15}$$

where $f(r)$ is the function of the position. We take

$$E = \frac{3}{2}u_1^2, \quad t = \frac{x}{U}, \quad \delta^{**} = 0.577\sqrt{\frac{\nu x}{U}}, \tag{16}$$

then

$$-\rho\overline{u_2 u_1} = \frac{\rho}{f(r)} e^{k_1\sqrt{\frac{Ux}{\nu}}\frac{u_1}{U}}, \tag{17}$$

where k_1 is a constant. We have same result from the equation (11).

These results indicate that the shearing stress along the plate is exponentially depends on $\sqrt{R_x}$ and the intensity of turbulence u_1/U in the main stream, this is also exponentially depends on $\sqrt{x}$. Therefore shearing stress along the plate at the up-stream and in the transition region is

$$\tau_{up} = \frac{d}{dx_1}\int_0^h \rho u(U-t)dx_2, \tag{18}$$

and at the down-stream end

$$\tau_{do} = \frac{d}{dx_1}\int_0^{h'} \rho u(U-u)dx_2 + \frac{2\rho}{f(r)} e^{k_1\sqrt{\frac{Ux}{\nu}}\frac{u_1}{U}}, \tag{19}$$

where h' is a length correspond to h.

3. Conclusion

In the equation (17), k_1 is an unknown constant and $f(r)$ is an unknown function, these are determined by the experiment. Moreover in the problems of the transition, we must determine the Reynolds number at the transition point as the functions of the turbulent intensity in the main stream, and the length of the transition region, which will be discussed in the successing papers.

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