Note on the Relationship between Imai's and Tomotika-Tamada's Formulae in the Theory of Two-dimensional Subsonic Flow of a Compressible Fluid

Note on the Relationship between Imai's and Tomotika-Tamada's Formulae in the Theory of Two-dimensional Subsonic Flow of a Compressible Fluid

By

Kinzo HIDA*

(Received February 28, 1954)

Abstract

Complex variables and conformal transformations are successfully employed in the theory of two-dimensional subsonic flow of a compressible fluid past an obstacle, and so far various formulae have been proposed for calculating the first-order effect of compressibility upon the complex velocity potential for the flow or similar other quantities.

In this paper a discussion is first made on the relationship between Imai's and Tomotika-Tamada's formulae, and it is shown that the latter formula can be derived from the former by a suitable choice of complementary functions contained in the former.

The relationship between Tomotika-Tamada's formula and another formula due to Kaplan is also discussed and it is shown that the latter formula can be derived from the former.

§1. Introduction

As is well known, complex variables and conformal transformations are employed with success for dealing with the two-dimensional subsonic flow of a compressible fluid past an obstacle. Thus, using the conjugate complex variables z and $\bar{z}$, Imai has given, in an earlier paper[1], an ingeneous formula useful for obtaining the first-order effect of compressibility upon the complex velocity potential for the two-dimensional subsonic flow past an obstacle. However, his method has a drawback that somewhat elabolate artificial devices are necessary to find complementary functions contained in his formula so as to satisfy the boundary conditions at the surface of the obstacle.** Later on, by modifying the well-known method due to Poggi, Tomotika and Tamada[4] have given another formula for calculating the first-order effect of compressibility upon the velocity potential for the two-dimensional subsonic flow past an obstacle. A great merit of their formula lies in the fact that as in the original Poggi's method and unlike in Imai's method, the boundary conditions at the surface of the obstacle concerned are automatically satisfied and that the difficult surface integrals in the original Poggi's method are replaced by contour integrals which can be evaluated comparatively easily by the use of the theorem of residues.

The main object of the present note is to show that Tomotika-Tamada's formula can

* Department of Applied Physics, College of Engineering

** It is to be noted however that in later papers,[2),3)] Imai himself has improved his method in such a way that such complementary functions can be determined more easily.

be derived from Imai's one by selecting the complementary functions adequately.

On the other hand, starting with the method due to Poggi as in the case of Tomotika and Tamada's paper, Kaplan[5] has also derived a formula useful for finding the first-order effect of compressibility upon the complex velocity for the two-dimensional subsonic flow past an obstacle. The difficult surface integrals in the original Poggi's method have been replaced, just as in Tomotika-Tamada's formula, by contour integrals which can be evaluated by the use of the theorem of residues.

It will be shown later however that Kaplan's formula can be readily derived from Tomotika-Tamada's formula by differentiation.

§2. Proposal of the problem

For reference we shall first give briefly the outlines of Imai's and Tomotika-Tamada's methods. Making, for the sake of convenience, all the quantities non-dimensional, we transform conformally the physical $\zeta(=\xi+i\eta)$-plane into the outside region of the unit circle $|z|=1$ in the $z\,(=x+iy)$-plane by the transformation:

$$\zeta = z+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots. \tag{2.1}$$

Let the density of the fluid at any point in the field of flow be denoted by ρ and also let the density of the fluid at infinity be denoted by ρ_∞. Then, in terms of the velocity potential Φ and the stream function Ψ, the velocity components u, v are given by

$$u=\frac{\partial\Phi}{\partial\xi}=\frac{\rho_\infty}{\rho}\frac{\partial\Psi}{\partial\eta}, \quad v=\frac{\partial\Phi}{\partial\eta}=-\frac{\rho_\infty}{\rho}\frac{\partial\Psi}{\partial\xi}.$$

We assume that Φ and Ψ as well as the complex velocity potential W defined as $W=\Phi+i\Psi$ can be respectively developed in power series of ascending M^2 as:

$$\begin{aligned}\Phi &= \Phi_0+M^2\Phi_1+M^4\Phi_2+\cdots,\\ \Psi &= \Psi_0+M^2\Psi_1+M^4\Psi_2+\cdots,\\ W &= W_0+M^2W_1+M^4W_2+\cdots,\end{aligned}$$

where $W_N=\Phi_N+i\Psi_N$ $(N=0,1,2,\cdots)$ and M is the Mach number corresponding to the state of fluid at infinity. The function $W_0=\Phi_0+i\Psi_0$ is evidently the complex velocity potential for incompressible fluid flow.

In the paper already cited[1], Imai has shown that the function W_1 can be obtained by a formula of the form:

$$\overline{W}_1(z,\bar z)=\frac{1}{4}\frac{d\overline{W}_0}{d\bar z}\frac{d\bar z}{d\bar\zeta}F(z)-\frac{1}{4}W_0(z)+\overline{G}_1(\bar z) \tag{2.2}$$

with

$$F(z)=\int\left(\frac{dW_0}{dz}\right)^2\left(\frac{dz}{d\zeta}\right)dz. \tag{2.2a}$$

Here, the bars denote the conjugate complex of the corresponding quantity, and $\overline{G}_1(\bar z)$ is an arbitrary complementary function of $\bar z$ alone which should be determined by the boundary conditions of the problem concerned.

When the undisturbed flow at infinity makes an angle δ with the positive direction of the x-axis and there exists a circulation $2\pi\kappa$ about the obstacle, W_0 is given by

$$W_0 = e^{-i\delta}\left(z+\frac{e^{2i\delta}}{z}\right)+i\kappa\log z\,. \tag{2.3}$$

In this case, the above Imai's formula (2.2) can be rewritten in the form:

$$\begin{aligned}\overline{W}_1(z,\bar{z}) = &\frac{1}{4}\frac{d\overline{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\zeta}\left\{F(z)+2i\kappa e^{-i\delta}\log\bar{z}\right\}\\ &-\frac{1}{4}\left\{W_0(z)+i\kappa\log\bar{z}\right\}+\overline{G}_1(\bar{z})\,,\end{aligned} \tag{2.4}$$

where the terms in the brackets are both one-valued.

On the other hand, by modifying the well-known method due to Poggi, Tomotika and Tamada have shown in the paper already cited[4] that the first-order effect of compressibility on the velocity potential can be obtained by a formula of the form:

$$\Phi_1 = \frac{1}{4}\Re\left[-e^{-i\delta}\left(z+\frac{1}{\bar{z}}\right)+\frac{1}{i\pi}\oint_{|t|=1}\bar{g}(t)\left\{F\left(\frac{1}{t}\right)-F(z)+2i\kappa e^{-i\delta}\log\frac{t}{\bar{z}}\right\}dt\right], \tag{2.5}$$

where

$$\bar{g}(t) = \frac{1}{2t}\left(\frac{d\overline{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\zeta}\right)_{\bar{z}=t}\frac{(z\bar{z}+1)t-2\bar{z}}{(zt-1)(t-\bar{z})}\,, \tag{2.5a}$$

and $F(z)$ is the same function as that given in (2.2a). Here, the integration is to be taken in the counter-clockwise sense round the unit circle with centre at the origin of the complex t-plane and $\Re$ means that the real part of complex quantities should be taken. A great merit of this formula lies in the fact that as in the case of the original Poggi's method, the boundary conditions are automatically satisfied, and that the difficult surface integrals in the original Poggi's method are replaced by contour integrals which can be evaluated comparatively easily by the use of the theorem of residues.

In the following lines we shall show that Imai's formula (2.2) can be transformed into Tomotika-Tamada's formula (2.5) by a suitable selection of the complementary function $\overline{G}_1(\bar{z})$.

§3. Equivalence of the two methods

In order to satisfy the condition $\Psi=0$ at the boundary of the obstacle, namely on the unit circle $|z|=1$ in the z-plane where $z\bar{z}=1$, the complementary function $\overline{G}_1(\bar{z})$ in (2.2) should be taken as:

$$\begin{aligned}\overline{G}_1(\bar{z}) = &-\frac{1}{4}\frac{d\overline{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\zeta}\left\{F\left(\frac{1}{\bar{z}}\right)+2i\kappa e^{-i\delta}\log\bar{z}\right\}\\ &+\frac{1}{4}\left\{W_0\left(\frac{1}{\bar{z}}\right)+i\kappa\log\bar{z}\right\}+h\left(\frac{1}{\bar{z}}\right)+\bar{h}(\bar{z})\,,\end{aligned} \tag{3.1}$$

where $h(z)$ is a one-valued function of z alone which is to be determined later on. The first two terms on the left-hand side are both one-valued in the outside region of the unit circle so that the sum of these two terms can be expressed in the form:

$$\sum_{k=1}^{n_H} H_k(\bar{z}) + \sum_{k=1}^{n_Z} Z_k(\bar{z}) + \sum_{p=1}^{\alpha} C_p \bar{z}^p \,, \tag{3.2}$$

where

$$\left.\begin{aligned} H_k(\bar{z}) &= \sum_{m=1}^{\alpha_k} \frac{C_{k,m}}{(\bar{z}-a_k)^m}, \quad |a_k|<1\,; \\ Z_k(\bar{z}) &= \sum_{m=1}^{\beta_k} \frac{d_{k,m}}{(\bar{z}-b_k)^m}, \quad |b_k|>1\,. \end{aligned}\right\} \tag{3.2a}$$

An arbitrary additive constant has been omitted here from the outset, since it has only trivial meaning in the complex velocity potential.

Further, from the fact that $\overline{W}_1(z, \bar{z})$ should be regular in the outside region of the unit circle $|z|=1$, it follows immediately that the function $\bar{h}(\bar{z})$ must be of the form:

$$-\sum_{k=1}^{n_Z} Z_k(\bar{z}) - \sum_{p=1}^{\alpha} C_p \bar{z}^p\,.$$

Thus, we have ultimately

$$\overline{G}_1(\bar{z}) = \sum_{k=1}^{n_H} H_k(\bar{z}) - \sum_{k=1}^{n_Z} \overline{Z}_k\left(\frac{1}{\bar{z}}\right) - \sum_{p=1}^{\alpha} \overline{C}_p \frac{1}{\bar{z}^p}\,. \tag{3.3}$$

Taking the closed contour as shown in Fig. 1, we have, by means of Cauchy's integral theorem,

$$\overline{G}_1(\bar{z}) = \frac{1}{2\pi i}\left\{\oint_{C_1} + \oint_{C_2}\right\}\left\{\sum_{k=1}^{n_H} H_k(\bar{z}) - \sum_{k=1}^{n_Z} \overline{Z}_k\left(\frac{1}{t}\right) - \sum_{p=1}^{\alpha} \overline{C}_p \frac{1}{t^p}\right\}\frac{dt}{t-\bar{z}}\,. \tag{3.4}$$

The integral taken round the outer circle C_1 of radius R tends, as $R \to \infty$, to become equal in magnitude to the residue at infinity of the integrand and can be easily evaluated from (3.2a) as:

$$\sum_{k=1}^{n_Z} \sum_{m=1}^{\beta_k} \frac{\bar{d}_{k,m}}{(-\bar{b}_k)^m}\,.$$

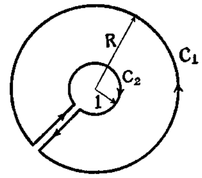

Fig. 1

Thus we have

$$\overline{G}_1(\bar{z}) = \sum_{k=1}^{n_Z} \sum_{m=1}^{\beta_k} \frac{\bar{d}_{k,m}}{(-\bar{b}_k)^m} - \frac{1}{2\pi i}\oint_{|t|=1} \sum_{k=1}^{n_H} H_k(t)\frac{dt}{t-\bar{z}}$$
$$+\frac{1}{2\pi i}\oint_{|t|=1} \sum_{k=1}^{n_Z} \overline{Z}_k\left(\frac{1}{t}\right)\frac{dt}{t-\bar{z}} + \frac{1}{2\pi i}\oint_{|t|=1} \sum_{p=1}^{\alpha} \overline{C}_p \frac{1}{t^p}\frac{dt}{t-\bar{z}}\,. \tag{3.5}$$

Further, after some calculations the first contour integral on the right-hand side can be transformed into the form:

$$-\frac{1}{2\pi i}\oint_{|t|=1} \sum_{k=1}^{n_H} H_k(t)\frac{dt}{t-\bar{z}}$$
$$= \frac{1}{2\pi i}\oint_{|t|=1} \frac{1}{4(t-\bar{z})}\left(\frac{d\overline{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\zeta}\right)_{\bar{z}=t}\left\{F\left(\frac{1}{t}\right) + 2i\kappa e^{-i\delta}\log t\right\}dt + \frac{1}{4}\frac{e^{-i\delta}}{\bar{z}}\,, \tag{3.6a}$$

while the second and third integrals can be evaluated as:

$$\frac{1}{2\pi i}\oint_{|t|=1}\sum_{k=1}^{n_Z}\bar{Z}_k\left(\frac{1}{t}\right)\frac{dt}{t-\bar{z}}=\sum_{k=1}^{n_Z}\bar{S}_k\left(\frac{1}{\bar{b}_k}\right), \tag{3.6b}$$

$$\frac{1}{2\pi i}\oint_{|t|=1}\sum_{p=1}^{\alpha}\bar{C}_p\frac{1}{t^p}\frac{dt}{t-\bar{z}}=-\sum_{p=1}^{\alpha}\frac{\bar{C}_p}{\bar{z}^p}, \tag{3.6c}$$

where $\bar{S}_k(1/\bar{b}_k)$ is the residue at $1/\bar{b}_k$ on the function $\bar{Z}_k(1/t)/(t-\bar{z})$.

Now, from (2.1) and (2.3) we have

$$\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}=\left\{e^{i\delta}\left(1-\frac{e^{-2i\delta}}{\bar{z}^2}\right)-\frac{i\kappa}{\bar{z}}\right\}\left\{1-\frac{\bar{a}_1}{\bar{z}^2}-2\frac{\bar{a}_2}{\bar{z}^3}-\cdots\right\}^{-1}, \tag{3.7}$$

which is also the regular function in the outside region of the unit circle $|z|=1$. Hence, proceeding in a similar manner to the case of deduction of (3.5) from (3.3), we have

$$\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}=e^{i\delta}-\frac{1}{2\pi i}\oint_{|t|=1}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)\frac{dt}{t-\bar{z}}. \tag{3.8}$$

Substitution of (3.5), (3.6) and (3.8) into (2.2) gives

$$\begin{aligned}\bar{W}_1(z,\bar{z})&=\frac{1}{4}e^{i\delta}\left\{F(z)+2i\kappa e^{-i\delta}\log\bar{z}\right\}-\frac{1}{4}\left\{W_0(z)+i\kappa\log\bar{z}\right\}\\&+\frac{1}{4}\frac{e^{-i\delta}}{\bar{z}}+\sum_{k=1}^{n_Z}\bar{S}_k\left(\frac{1}{\bar{b}_k}\right)-\sum_{p=1}^{\alpha}\frac{\bar{C}_p}{\bar{z}^p}+\sum_{k=1}^{n_Z}\sum_{m=1}^{\beta_k}\frac{\bar{d}_{k,m}}{(-\bar{b}_k)}\\&+\frac{1}{2\pi i}\oint_{|t|=1}\frac{1}{4(t-\bar{z})}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)_{\bar{z}=1}\left\{F\left(\frac{1}{t}\right)-F(z)+2i\kappa e^{-i\delta}\log\frac{t}{\bar{z}}\right\}dt.\end{aligned} \tag{3.9}$$

Rewriting $\bar{g}(t)$ as given by (2.5a) in the form:

$$\bar{g}(t)=\frac{1}{2}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)_{\bar{z}=t}\frac{1}{t-\bar{z}}+\frac{1}{2}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)_{\bar{z}=t}\frac{2-zt}{t(zt-1)},$$

and substituting $\frac{1}{2(t-\bar{z})}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)_{\bar{z}=t}$ from this relation, the integral in (3.9) is divided into three parts as:

$$\left.\begin{aligned}I_1&=\frac{1}{4\pi i}\oint_{|t|=1}\bar{g}(t)\left\{F\left(\frac{1}{t}\right)-F(z)+2i\kappa e^{-i\delta}\log\frac{t}{\bar{z}}\right\}dt,\\I_2&=\frac{1}{2\pi i}\oint_{|t|=1}\frac{1}{4}\frac{(2-zt)}{t(zt-1)}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)_{\bar{z}=t}\left\{F\left(\frac{1}{t}\right)+2i\kappa e^{-i\delta}\log t\right\}dt,\\I_3&=-\frac{1}{2\pi i}\oint_{|t|=1}\frac{1}{4}\frac{(2-zt)}{t(zt-1)}\left(\frac{d\bar{W}_0}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\right)_{\bar{z}=t}\left\{F(z)+2i\kappa e^{-i\delta}\log\bar{z}\right\}dt.\end{aligned}\right\} \tag{3.10}$$

Taking (3.2) and (3.7) into consideration and making use of the theorem of residues, the second and third integrals I_2 and I_3 can be evaluated and we have, after some reductions,

$$\left.\begin{aligned}I_2&=-\frac{1}{4z}e^{i\delta}+\sum_{p=1}^{\alpha}\frac{C_p}{z^p}-\sum_{k=1}^{n_Z}S_k\left(\frac{1}{b_k}\right)-\sum_{k=1}^{n_Z}\sum_{m=1}^{\beta_k}\frac{d_{k,m}}{(-b_k)^m},\\I_3&=-\frac{1}{4}e^{i\delta}\left\{F(z)+2i\kappa e^{-i\delta}\log\bar{z}\right\}.\end{aligned}\right\} \tag{3.11}$$

Thus, we have ultimately

$$\overline{W}_1(z,\bar{z}) = \frac{1}{4}\left\{-e^{-i\delta}\left(z+\frac{1}{\bar{z}}\right)+\frac{1}{i\pi}\oint_{|t|=1}\bar{g}(t)\left\{F\left(\frac{1}{t}\right)-F(z)+2i\kappa e^{-i\delta}\log\frac{t}{\bar{z}}\right\}dt\right\}$$
$$+\left[-\frac{1}{2}\left(\frac{e^{i\delta}}{z}-\frac{e^{-i\delta}}{\bar{z}}\right)-\frac{1}{4}i\kappa\log z\bar{z}-\sum_{k=1}^{n_Z}\left\{S_k\left(\frac{1}{b_k}\right)-\bar{S}_k\left(\frac{1}{\bar{b}_k}\right)\right\}\right.$$
$$\left.+\sum_{p=1}^{\alpha}\left(\frac{C_p}{z^p}-\frac{\bar{C}_p}{\bar{z}^p}\right)-\sum_{k=1}^{n_Z}\sum_{m=1}^{\beta_k}\left\{\frac{d_{k,m}}{(-b_k)^m}-\frac{\bar{d}_{k,m}}{(-\bar{b}_k)^m}\right\}\right]. \tag{3.12}$$

The terms in the square bracket are evidently all purely imaginary and therefore they contribute nothing to the velocity potential. Hence, we arrive at the formula:

$$\Phi_1 = \Re\left[\overline{W}_1(z,\bar{z})\right]$$
$$= \frac{1}{4}\Re\left[-e^{-i\delta}(z+1/\bar{z})+\frac{1}{i\pi}\oint_{|t|=1}\bar{g}(t)\left\{F(1/t)-F(z)+2i\kappa e^{-i\delta}\log(t/\bar{z})\right\}dt\right],$$

which is nothing but Tomotika-Tamada's formula (2.5).

§4. Relationship between Tomotika-Tamada's and Kaplan's formulae

Starting with the method due to Poggi as in Tomotika-Tamada's paper and using the complex variables z and $\bar{z}$, Kaplan has also derived another formula for calculating the first-order effect of compressibility upon the two-dimensional subsonic flow past an obstacle.

For simplicity we here assume that the fluid flows in the x–direction and there is no circulation about the obstacle. Let w denote the complex velocity in the physical ζ–plane and let it be developed in a power series of ascending M^2 as:

$$w = (w_0+M^2w_1+\cdots)(dz/d\zeta). \tag{4.1}$$

The function w_0 is evidently the complex velocity for incompressible flow past the body concerned, and remembering that $\delta=0$ and $\kappa=0$ in the present case, it is derived from W_0 given by (2.3) as:

$$w_0 = dW_0/dz = 1-1/z^2. \tag{4.2}$$

Kaplan has shown that the function w_1 expressing the first-order effect of compressibility upon the complex velocity is given by a formula of the form:

$$w_1 = \frac{1}{4}\left[S(z)-\frac{1}{z^2}\bar{S}\left(\frac{1}{z}\right)-\frac{1}{z^2}\bar{F}\left(\frac{1}{z},z\right)\right]+\frac{1}{4}F(z,\bar{z})-\frac{1}{4}\left(1-\frac{1}{z^2}\right), \tag{4.3}$$

where

$$F(z,\bar{z}) = w_0^2\bar{w}_0\frac{dz}{d\zeta}\frac{d\bar{z}}{d\bar{\zeta}}+\frac{d}{dz}\left(w_0\frac{dz}{d\zeta}\right)\int\bar{w}_0^2\frac{d\bar{z}}{d\bar{\zeta}}d\bar{z}, \tag{4.3a}$$

and $S(z)$ is the sum of residues of the function $1/(t-z)\cdot F(t,1/t)$ at the poles lying inside the unit circle $|t|=1$ in the t–plane. Also, $\bar{S}(1/z)$ and $\bar{F}(1/z,\bar{z})$ can be obtained respectively from $\bar{S}(\bar{z})$ and $\bar{F}(\bar{z},z)$ by replacing $\bar{z}$ by $1/z$.

In the following lines we shall show that the above Kaplan's formula can be derived from Tomotika-Tamada's formula (2.5), which, in the present case in which $\delta=0$ and $\kappa=0$, may be written in the form:

$$2\Phi_1 = \frac{1}{4}\Big[-\Big(z+\frac{1}{\bar{z}}\Big)+\frac{1}{i\pi}\oint_{|t|=1}\bar{g}(t)\Big\{F\Big(\frac{1}{t}\Big)-F(z)\Big\}dt$$

$$-\Big(\bar{z}+\frac{1}{z}\Big)-\frac{1}{i\pi}\oint_{|t|=1}g(\bar{t})\Big\{\bar{F}\Big(\frac{1}{\bar{t}}\Big)-\bar{F}(\bar{z})\Big\}d\bar{t}\Big]. \tag{4.4}$$

Also, the function $\bar{g}(t)$ can be resolved into partial fractions as:

$$\bar{g}(t) = \frac{1}{2}\Big(\bar{w}_0\frac{d\bar{z}}{d\bar{\zeta}}\Big)_{\bar{z}=t}\Big(\frac{1}{t-\bar{z}}-\frac{2}{t}+\frac{1}{t-1/z}\Big). \tag{4.5}$$

From the definition of w_1 in (4.1) it is easily seen that

$$w_1 = 2(\partial\Phi_1/\partial z). \tag{4.6}$$

Thus, retaining only those terms depending on z, we have

$$w_1 = \frac{1}{4}\Big[\Big(-1+\frac{1}{z^2}\Big)+\frac{\partial}{\partial z}(J_1+J_2+J_3)\Big], \tag{4.7}$$

where

$$J_1 = \frac{1}{2\pi i}\oint_{|t|=1}\Big(\bar{w}_0\frac{d\bar{z}}{d\bar{\zeta}}\Big)_{\bar{z}=t}\frac{F(1/t)}{t-1/z}dt, \tag{4.7a}$$

$$J_2 = -\frac{1}{2\pi i}\oint_{|t|=1}\Big(w_0\frac{dz}{d\zeta}\Big)_{z=\bar{t}}\frac{\bar{F}(1/\bar{t})}{\bar{t}-z}d\bar{t}, \tag{4.7b}$$

$$J_3 = -\frac{F(z)}{\pi i}\oint_{|t|=1}\bar{g}(t)dt+\text{conjugate complex}. \tag{4.7c}$$

Now, when $G(z)$ is a rational function having no singular points on the circumference of the unit circle $|z|=1$ in the z-plane, it can be proved that

$$\frac{d}{dz}\Big\{\frac{1}{2\pi i}\oint_{|t|=1}\frac{G(t)}{t-z}dt\Big\} = \frac{1}{2\pi i}\oint_{|t|=1}\frac{G'(t)}{t-z}dt.$$

Making use of this relation and taking (2.2a) and (4.2) into account, we have, after some reductions,

$$\frac{\partial J_1}{\partial z} = -\frac{1}{z^2}\frac{\partial J_1}{\partial(1/z)} = -\frac{1}{z^2}\frac{1}{2\pi i}\oint_{|t|=1}\frac{\bar{F}(t,1/t)}{t-1/z}dt. \tag{4.8}$$

The last integral can be evaluated by calculating the residues of the integrand at its poles inside the unit circle $|t|=1$ in the t-plane. Thus, it is easily found that

$$\frac{\partial J_1}{\partial z} = -\frac{1}{z^2}\Big\{\bar{S}\Big(\frac{1}{z}\Big)+\bar{F}\Big(\frac{1}{z},z\Big)\Big\}. \tag{4.9}$$

Next, $\partial J_2/\partial z$ can be evaluated in the same way as in the case of $\partial J_1/\partial z$. Since, however, $|z|>1$ for points in the region of fluid flow, it is found that

$$\frac{\partial J_2}{\partial z} = S(z). \tag{4.10}$$

Lastly, the value of J_3 can be evaluated conveniently by reversing the sense of integration along the contour $|t|=1$. In doing this, we make use of the fact that the function $\bar{g}(t)$ given by (4.5) has only one pole at $t=\bar{z}$ in the outside of the unit circle

$|t|=1$ and moreover, as is easily assured, its residue at infinity is zero. Then, differentiating the resulting value of J_3 with respect to z, we have

$$\frac{\partial J_3}{\partial z}=\left(w_0^2\frac{dz}{d\zeta}\right)\left(\overline{w}_0\frac{d\bar{z}}{d\bar{\zeta}}\right)+\frac{d}{dz}\left(w_0\frac{dz}{d\zeta}\right)\int\overline{w}_0^2\frac{d\bar{z}}{d\bar{\zeta}}d\bar{\zeta}=F(z,\bar{z})\,. \tag{4.11}$$

Thus, substituting (4.9), (4.10) and (4.11) into (4.7), we have ultimately

$$w_1=\frac{1}{4}\left[\left(-1+\frac{1}{z^2}\right)-\frac{1}{z^2}\left\{\overline{S}\left(\frac{1}{z}\right)+\overline{F}\left(\frac{1}{z},z\right)\right\}+S(z)+F(z,\bar{z})\right],$$

and this is just the formula due to Kaplan.

In conclusion the writer wishes to express his cordial thanks of Professor S. Tomotika for his continual guidance and his kind inspection of the manuscript. The writer's cordial thanks are also due to Dr. K. Tamada for suggesting the present problem, and to Professor T. Simasaki for his useful discussions.

References

1) I. Imai and T. Aihara: Report Aeron. Res. Inst., Tokyo Imp. Univ. **194** (1940).
2) I. Imai: Report Aero. Res. Inst., Tokyo Imp. Univ. **275** (1943), (in Japanese).
3) I. Imai: Report Aero. Res. Inst., Tokyo Imp. Univ. **290** (1944), (in Japanese with English summaries).
4) S. Tomotika and K. Tamada: Preliminary Report, Tomotika Lab. Univ. of Kyoto, No. **21** (1949), (in Japanese).
5) C. Kaplan: N. A. C. A. Rep. **728** (1942).