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On Critical Exponents of Dowling Matroids

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Abstract The *Critical Problem* in matroid theory is the problem to determine the critical exponent of a given representable matroid over a finite field. In this paper, we study the critical exponents of a class of representable matroids over finite fields, called Dowling matroids. Then the critical problem for a Dowling matroid is corresponding to the classical problem in coding theory to determine the maximum dimension k such that there exists an $[n, k, d]_q$ code for given n, d and q. We give a necessary and sufficient condition on the critical exponents of Dowling matroids by using a coding theoretical approach..

Keywords critical problem · critical exponent · matroid · linear code · Griesmer bound · Griesmer code

Mathematics Subject Classification (2000) 05B35 · 94B65

1 Introduction

Let $\mathbb{F}_q$ be a finite field of q elements. For any subset $S \subseteq \mathbb{F}_q^n$, define the *critical exponent* of S as follows:

$$c(S, q) := n - \max\{r \in \mathbb{Z}^+ \ : \ \text{there exists an } r\text{-dimensional subspace } D \text{ of } \mathbb{F}_q^n \text{ with } D \cap S = \emptyset\}.$$

This number was introduced in the context of matroid theory where it has attracted attention as the *critical exponent* $c(M, q)$ of an $\mathbb{F}_q$-representable matroid M. An $\mathbb{F}_q$-representable matroid is a matroid obtained by a set of vectors in $\mathbb{F}_q^n$. Thus we identify an $\mathbb{F}_q$-representable matroid M as a subset $S \subseteq \mathbb{F}_q^n$ in this paper.

The *Critical Problem* for matroids is the following problem introduced by Crapo and Rota in [6] to unify some problems in extremal combinatorial theory including such celebrated conjectures as the Tutte's 5-flow conjecture ([24]) and the Hadwiger conjecture ([9]) in graph theory. See [16] for further details.

Problem 1 (Crapo and Rota in [6]) *For given subset $S \subseteq \mathbb{F}_q^n$, find the critical exponent $c(S, q)$.*

For instance, see [2,4,5,7,14,15,16,17] for general results on the Critical Problem. As an application of this problem, Abbe, Alon and Bandeira consider the special case in which S is an annulus, motivated by the problem of correcting a black and white pixel image with respect to two possible corrected images, one light and one dark in [1].

For any vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, we define the *Hamming weight* of $\boldsymbol{x}$ as follows:

$$\mathrm{wt}(\boldsymbol{x}) := |\{i \ : \ x_i \neq 0\}|.$$

The *weight-t Dowling matroid* $\tilde{B}_{n,t}(q)$ is defined by

$$\tilde{B}_{n,t}(q) := \{\boldsymbol{x} \in \mathbb{F}_q^n \ : \ \mathrm{wt}(\boldsymbol{x}) \leq t\}.$$

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A Dowling matroid might contain parallel vectors. Hence we shall consider it as a subset of points in projective space $\mathrm{PG}(n-1,q)$ for convenience by regarding $\mathrm{PG}(n-1,q)$ as the set of all non-zero vectors in $\mathbb{F}_q^n$, modulo the equivalence relation $\boldsymbol{x} \sim \boldsymbol{y}$ whenever $\boldsymbol{y} = \alpha\boldsymbol{x}$ for some nonzero scalar α. Then we define a *Dowling geometry* $B_{n,t}(q)$ as a set of all points in $\mathrm{PG}(n-1,q)$ having weight at most t (cf. [16]). The Dowling geometry $B_{n,t}(q)$ can be viewed as a simplification of the Dowling matroid $\tilde{B}_{n,t}(q)$. In this paper, we mainly consider the Critical Problem for the special case $S = B_{n,t}(q)$.

The remainder of the paper is organized as follows. In Section 2, we show some known results on the critical exponents of Dowling geometries (matroids). In Section 3, we present some bounds on the critical exponents of Dowling geometries and we study a necessary and sufficient condition for the equality $c(B_{n,t}(q), q) = n - r$ to hold. In Section 4, we give more detailed necessary and sufficient conditions, given special values of q and r, for the equality $c(B_{n,t}(q), q) = n - r$ to hold.

2 Preliminary Results

An $[n, k, d]_q$ *code* C is a k-dimensional subspace of $\mathbb{F}_q^n$ with

$$d = \min\{\mathrm{wt}(\boldsymbol{x}) \ : \ \boldsymbol{0} \neq \boldsymbol{x} \in C\}.$$

One of the main problems in coding theory is to determine the minimum value of n for which there exists an $[n, k, d]_q$ code C. The following lower bound, known as the *Griesmer bound*, is essential in this paper (see, for instance, Theorem 2.7.4 in [12]):

$$n \geq g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer not less than x for any non-negative real number x. An $[n, k, d]_q$ code C is called a *Griesmer code* if C attains the Griesmer bound, that is, $n = g_q(k, d)$. The following is one of the existence theorems on Griesmer codes (Theorem 1.2 in [19]).

Theorem 2 ([19]) *There exists a $[g_q(k, d), k, d]_q$ Griesmer code for all d, $k = 1, 2$ and for $d \geq (k-2)q^{k-1} - (k-1)q^{k-2} + 1$, $k \geq 3$ for all q.*

The Critical Problem for $S = \tilde{B}_{n,t}(q)$ is equivalent to a classical problem in coding theory and it turns out that the Griesmer bound plays an important role in this problem.

Problem 3 *For given n, t, and q ($n, t \in \mathbb{Z}^+$, q : a prime power), determine the maximum dimension k such that there exists an $[n, k, t+1]_q$ code.*

In particular, $c(\tilde{B}_{n,t}(q), q) = n - k$ where k is the maximal dimension for which there exists an $[n, k, t+1]_q$ code.

Example 4 Consider

$$S = \tilde{B}_{4,2}(2) = \{\boldsymbol{x} \in \mathbb{F}_2^4 \ : \ \mathrm{wt}(\boldsymbol{x}) \leq 2\}.$$

We first note that there does not exist a $[4, 2, 3]_2$ code by the Griesmer bound. On the other hand, $D = \{(0, 0, 0, 0), (1, 1, 1, 0)\}$ is a $[4, 1, 3]_2$ code. Therefore it follows that

$$c(\tilde{B}_{4,2}(2), 2) = 4 - 1 = 3.$$

In [16], Kung gives a necessary and sufficient condition for the equality $c(B_{n,t}(q), q) = n - 1$ to hold.

Theorem 5 ([16]) *$B_{n,t}(q)$ has critical exponent $n - 1$ if and only if*

$$n - 1 \geq t \geq n - \left\lceil \frac{n}{q+1} \right\rceil.$$

In the proof of this theorem, he mainly argues about the Hamming weight of the sum of two vectors over $\mathbb{F}_q$. Now we provide another proof of this theorem by using the Griesmer bound for corresponding linear codes over $\mathbb{F}_q$.

(*Another Proof of Theorem* 5) There exists an $[n,1,t+1]_q$ code if and only if $n \geq t+1$. From Theorem 2, there exists an $[n,2,t+1]_q$ code if and only if $n \geq g_q(2,t+1) = t+1+\lceil (t+1)/q \rceil$, or, equivalently, $n \geq t+1+(t+1)/q$. Therefore, $B_{n,t}(q)$ has critical exponent $n-1$ if and only if

$$t+1 \leq n < t+1+\frac{t+1}{q} = (t+1)(1+1/q) = (t+1)(q+1)/q.$$

The proof follows easily. □

In this paper, we generalize the bound in Theorem 5 by using a similar argument as above for $c(B_{n,t}(q), q) = n-r$. In addition, Kung gives a sufficient condition for the equality $c(B_{n,t}(q), q) = n-2$ to hold.

Theorem 6 ([16]) *Let*

$$e = \left\lfloor \frac{1}{q+1+\frac{1}{q}} \left\lceil \frac{n}{q+1} \right\rceil \right\rfloor .$$

Suppose that $e \geq 1$ *and*

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 \geq t \geq n - \left\lceil \frac{n}{q+1} \right\rceil - e.$$

Then $B_{n,t}(q)$ *has critical exponent* $n-2$.

The converse of Theorem 6 is not true in general. For instance, there does not exist a $[23,3,13]_2$ code by the Griesmer bound. Here,

$$G = \begin{pmatrix} 1\,0\,1\,0\,1\,0 \cdots 1\,0\,1\,1\,1 \\ 0\,1\,0\,1\,0\,1 \cdots 0\,1\,1\,1\,1 \end{pmatrix}$$

is a generator matrix of a $[23,2,13]_2$ code; that is, a matrix whose rows span such a code. Therefore we have that

$$c(B_{23,12}(2), 2) = 23 - 2 = 21.$$

However it follows that

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 = 23 - 8 - 1 = 14,$$
$$n - \left\lceil \frac{n}{q+1} \right\rceil - e = 23 - 8 - 2 = 13.$$

One of the main purposes of this paper is to find necessary and sufficient conditions for when $c(B_{n,t}(q), q) = n-2$.

3 Conditions for $c(B_{n,t}(q), q) = n - r$

Let r be a positive integer with $1 \leq r \leq n-1$. Firstly, we shall prove some bounds on critical exponents of Dowling geometries to give a sufficient condition for when $c(B_{n,t}(q), q) = n-r$. The following lemma is essential.

Lemma 7 *If there exists an* $[n,k,d(\geq 2)]_q$ *code, then there also exists an* $[n,k,d-1]_q$ *code.*

Proof. Let C be an $[n,k,d]_q$ code with $d \geq 2$ and let $\boldsymbol{x}$ be a codeword in C with $\mathrm{wt}(\boldsymbol{x}) = d$. Suppose that i-th coordinate of $\boldsymbol{x}$ is nonzero. Consider the code

$$C' = \{(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_n) \,:\, \boldsymbol{y} = (y_1, \dots, y_n) \in C\}.$$

If there exist two distinct codewords $\boldsymbol{u} = (u_1, \dots, u_n), \boldsymbol{v} = (v_1, , \dots, v_n) \in C$ such that

$$(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n) = (v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_n),$$

then $\mathrm{wt}(\boldsymbol{u} - \boldsymbol{v}) = 1$, which contradicts that $d \geq 2$. Therefore, the code C' is an $[n,k,d-1]_q$ code. □

Set $\theta_i := q^i + q^{i-1} + \cdots + q + 1$, for any non-negative integer i, and set $\theta_{-1} := 0$.

Proposition 8 *Let r be a positive integer with $1 \le r \le n-1$. Suppose that*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil \le t.$$

Then $B_{n,t}(q)$ has critical exponent at least $n-r$.

Proof. Note that

$$\begin{aligned} n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil \le t &\Leftrightarrow n - \frac{n\theta_{r-1}}{\theta_r} < t+1 \\ &\Leftrightarrow \frac{q^r n}{\sum_{i=0}^{r} q^i} < t+1 \\ &\Leftrightarrow \frac{n}{\sum_{i=0}^{r} \frac{1}{q^i}} < t+1 \\ &\Leftrightarrow n < \sum_{i=0}^{r} \frac{t+1}{q^i} = (t+1)\sum_{i=0}^{r} \frac{1}{q^i}. \end{aligned}$$

Therefore, there does not exist an $[n, r+1, t+1]_q$ code. Hence we have $c(B_{n,t}(q), q) \ge n-r$. The proposition follows. □

We may modify Proposition 8 for some special cases of n. Let $s_0, s_1, \ldots, s_{r-1}$ be integers such that $0 \le s_i \le q$ for $i = 0, 1, \ldots, r-1$.

Lemma 9 *The parameter $s_0 = 0$ if and only if*

$$s_0\theta_{r-1} \le \sum_{i=1}^{r-1} s_i q^{i-1}.$$

Proof. If $s_0 = 0$, then $0 \le \sum_{i=1}^{r-1} s_i q^{i-1}$ holds from the definition.

Conversely, we assume that $s_0 \ge 1$. Then we have that

$$s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1} \ge \theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1} = \sum_{i=1}^{r-1} (q-s_i) q^{i-1} + 1 \ge 1 > 0.$$

□

Now we consider the following cases for given n and r.

Cases $\mathbf{A}_r$:

(1) $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ for some $m \ge 0$ and some $s_0, \ldots, s_{r-1}$ with $s_0 \ge s_1 \ge \cdots \ge s_{r-1}$ and $(s_0, s_1, \ldots, s_{r-1}) \ne (\alpha, \alpha, \ldots, \alpha), (\beta+1, \beta, \ldots, \beta)$ for any α, β such that $0 \le \alpha \le q$ and $0 \le \beta \le q-1$;
(2) $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ for some $m \ge 0$ and some $s_0, \ldots, s_{r-1}$ with $s_{r-1} \ne 0$ and $0 = s_0 \le s_1 \le \cdots \le s_{r-1}$, $s_{i+1} - s_i \le q-1$ for all $i = 0, 1, \ldots, r-2$.

Proposition 10 *For given n and $r (\ge 2)$, suppose that either of the cases $\mathbf{A}_r$ holds. If*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 \le t,$$

then $B_{n,t}(q)$ has critical exponent at least $n-r$.

Proof. Assume that there exists an $[n, r+1, d := n - \lceil n\theta_{r-1}/\theta_r \rceil]_q$ code. It follows from Lemma 9 that if the case $\mathbf{A}_r(1)$ holds, the the inequality $s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1} > 0$ holds, and if the case $\mathbf{A}_r(2)$ holds, the the inequality $s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1} \leq 0$ holds. Therefore, in the case $\mathbf{A}_r(j)$ for each $j = 1, 2$, we have that

$$
\begin{aligned}
d = n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil &= \theta_r m + \sum_{i=0}^{r-1} s_i q^i - \left\lceil \theta_{r-1} m + \sum_{i=1}^{r-1} s_i q^{i-1} + \frac{s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1}}{\theta_r} \right\rceil \\
&= q^r m + s_{r-1} q^{r-1} + \sum_{i=0}^{r-2} (s_i - s_{i+1}) q^i + (j-2), \\
\left\lceil \frac{d}{q^l} \right\rceil &= \left\lceil q^{r-l} m + s_{r-1} q^{r-l-1} + \sum_{i=l}^{r-2} (s_i - s_{i+1}) q^{i-l} + \frac{\sum_{i=0}^{l-1} (s_i - s_{i+1}) q^i + (j-2)}{q^l} \right\rceil, \ l = 1, 2, \ldots, r-1, \\
\left\lceil \frac{d}{q^r} \right\rceil &= \left\lceil m + \frac{s_{r-1} q^{r-1} + \sum_{i=0}^{r-2} (s_i - s_{i+1}) q^i + (j-2)}{q^r} \right\rceil.
\end{aligned}
$$

Suppose that the case $\mathbf{A}_r(1)$ holds. From the assumptions in this proposition, we have that $\sum_{i=0}^{l-1} (s_i - s_{i+1}) q^i - 1 \geq -1$ for all $l = 1, 2, \ldots, r-1$, and there exist some j such that $\sum_{i=0}^{j-1} (s_i - s_{i+1}) q^i - 1 > 0$. Then it follows that

$$
\begin{aligned}
\left\lceil \frac{d}{q^l} \right\rceil &\geq q^{r-l} m + s_{r-1} q^{r-l-1} + \sum_{i=l}^{r-2} (s_i - s_{i+1}) q^{i-l}, \ l = 1, 2, \ldots, r-1, \\
\left\lceil \frac{d}{q^r} \right\rceil &\geq m + 1.
\end{aligned}
$$

Moreover there exists some j such that

$$
\left\lceil \frac{d}{q^j} \right\rceil = q^{r-j} m + s_{r-1} q^{r-j-1} + \sum_{i=j}^{r-2} (s_i - s_{i+1}) q^{i-j} + 1.
$$

Thus we have that

$$
g_q(r+1, d) = \sum_{i=0}^{r} \left\lceil \frac{d}{q^i} \right\rceil \geq \theta_r m + \sum_{i=0}^{r-1} s_i q^i + 1 = n + 1 > n.
$$

Next we suppose that the case $\mathbf{A}_r(2)$ holds. Then we have that

$$
s_{r-1} q^{r-1} - \sum_{i=0}^{r-2} (s_{i+1} - s_i) q^i \geq s_{r-1} q^{r-1} - \sum_{i=0}^{r-2} (q-1) q^i = (s_{r-1} - 1) q^{r-1} + 1 > 0.
$$

Therefore it follows that

$$
\begin{aligned}
\left\lceil \frac{d}{q^l} \right\rceil &= q^{r-l} m + s_{r-1} q^{r-l-1} - \sum_{i=l}^{r-2} (s_{i+1} - s_i) q^{i-l}, \ l = 1, 2, \ldots, r-1, \\
\left\lceil \frac{d}{q^r} \right\rceil &= m + 1.
\end{aligned}
$$

Thus we have that

$$
g_q(r+1, d) = \sum_{i=0}^{r} \left\lceil \frac{d}{q^i} \right\rceil = \theta_r m + \sum_{i=0}^{r-1} s_i q^i + 1 = n + 1 > n.
$$

This argument shows that there does not exist the assumed code. Therefore the proposition follows. □

Now we study a necessary condition for the equality $c(B_{n,t}(q), q) = n - r$ to hold for any positive integer r with $1 \leq r \leq n-1$. From the definition of the critical exponent of $B_{n,t}(q)$ and Lemma 7, it is sufficient to show the existence of an $[n, r+1, d]_q$ code C such that $d \geq t+1$ when we prove that $c(B_{n,t}(q), q) \leq n - (r+1)$.

Lemma 11 *Suppose that $n = \theta_r m$ with $m \geq 1$. If*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 \geq t,$$

then $c(B_{n,t}(q), q) \leq n - (r+1)$.

Proof. In this case, set

$$d := n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil = \theta_r m - \theta_{r-1} m = q^r m.$$

Let S_r be a simplex $[\theta_r, r+1, q^r]_q$ code with a generator matrix G_r. Then we can construct a $[\theta_r m, r+1, q^r m]_q$ code by m concatenations of S_r, that is, a linear code having generator matrix $[G_r, G_r, \cdots, G_r]$. □

Lemma 12 *Suppose that $n = \theta_r m + \theta_r - \theta_l$ with $m \geq 0$ and $0 \leq l \leq r-1$. If*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 \geq t,$$

then $c(B_{n,t}(q), q) \leq n - (r+1)$.

Proof. Set

$$\begin{aligned} d := n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil &= (m+1)\theta_r - \theta_l - \left\lceil (m+1)\theta_{r-1} - \theta_{l-1} - \frac{\theta_{r-1} - \theta_{l-1}}{\theta_r} \right\rceil \\ &= q^r(m+1) - \theta_l + \theta_{l-1} = q^r(m+1) - q^l. \end{aligned}$$

It turns out that there exists an $[(m+1)\theta_r, r+1, q^r(m+1)]_q$ Griesmer code from Lemma 11. By Lemma 2.1 in [20], there exists an $[(m+1)\theta_r - \theta_l, r+1, d' \geq q^r(m+1) - q^l (= d)]_q$ code for $m \geq 0$ and $0 \leq l \leq r-1$. □

Consider $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ with $m \geq 0$ and $s_0 \leq s_1 \leq \cdots \leq s_{r-1}$. We assume that $s_0 = 0$. If $s_{r-1} = 0$, then $s_{r-2} = \cdots = s_1 = 0$ and so $n = \theta_r m$, which is corresponding to the case in Lemma 11. Thus we suppose that $s_{r-1} \neq 0$. If $s_l - s_{l-1} = q$ for some l, $1 \leq l \leq r-1$, then we have that $q = s_l (= s_{l+1} = \cdots = s_{r-1})$ and $0 = s_{l-1} (= s_{l-2} = \cdots = s_1)$, and so $n = \theta_r m + \theta_r - \theta_l$, which is corresponding to the case in Lemma 12. Moreover if $s_l - s_{l-1} \leq q - 1$ for all l, $1 \leq l \leq r-1$, then this case is corresponding to the case in Proposition 10 (2). Thus we may assume that $s_0 \neq 0$.

Lemma 13 *Set $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ with $m \geq r-1$. Suppose that $s_0 \leq s_1 \leq \cdots \leq s_{r-1}$ with $s_0 \neq 0$. If*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 \geq t,$$

then $c(B_{n,t}(q), q) \leq n - (r+1)$.

Proof. Set $d := n - \lceil n\theta_{r-1}/\theta_r \rceil$. From Lemma 9, we have that $s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1} > 0$. Thus it follows that

$$\begin{aligned} d &= \theta_r m + \sum_{i=0}^{r-1} s_i q^i - \left\lceil \theta_{r-1} m + \sum_{i=1}^{r-1} s_i q^{i-1} + \frac{s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1}}{\theta_r} \right\rceil \\ &= q^r m + s_{r-1} q^{r-1} + \sum_{i=0}^{r-2} (s_i - s_{i+1}) q^i - 1. \end{aligned}$$

It is straightforward that $s_{i+1} - s_i \neq q$ for all i, $1 \leq i \leq r-2$. So we have the following:

$$\begin{aligned} q^l - \sum_{i=0}^{l-1} (s_{i+1} - s_i) q^i &= \{q - (s_l - s_{l-1})\} q^{l-1} - (s_{l-1} - s_{l-2}) q^{l-2} - \cdots - (s_1 - s_0) \\ &\geq q^{l-1} - (q-1) q^{l-2} - \cdots - (q-1) q - (q-1) \\ &= 1 > 0, \quad l = 1, 2, \ldots, r-1. \end{aligned}$$

Therefore it follows that

$$\begin{aligned}\left\lceil \frac{d}{q^l} \right\rceil &= \left\lceil q^{r-l}m + s_{r-1}q^{r-l-1} - \sum_{i=l}^{r-2}(s_{i+1}-s_i)q^{i-l} - \frac{\sum_{i=0}^{l-1}(s_{i+1}-s_i)q^i + 1}{q^l} \right\rceil \\ &= q^{r-l}m + s_{r-1}q^{r-l-1} - \sum_{i=l}^{r-2}(s_{i+1}-s_i)q^{i-l},\ l = 1,2,\ldots,r-1, \\ \left\lceil \frac{d}{q^r} \right\rceil &= \left\lceil m + \frac{s_{r-1}q^{r-1} - \left(\sum_{i=0}^{r-2}(s_{i+1}-s_i)q^i + 1\right)}{q^r} \right\rceil = m + a,\end{aligned}$$

where a is equal to either 0 or -1. Thus we have that

$$g_q(k,d) = \sum_{i=0}^{r} \left\lceil \frac{d}{q^i} \right\rceil = \theta_r m + \sum_{i=0}^{r-1} s_i q^i + a = n + a.$$

From Lemma 2, there exist such Griesmer codes for $m \geq r-1$. □

Lemma 14 *Set $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ with $m \geq r-1$. Suppose that $(s_0, s_1, \ldots, s_{r-1}) = (\beta+1, \beta, \ldots, \beta)$ for some β such that $0 \leq \beta \leq q-1$. If*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 \geq t,$$

then $c(B_{n,t}(q), q) \leq n - (r+1)$.

Proof. Set

$$\begin{aligned} d := n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil &= q^r m + s_{r-1}q^{r-1} + \sum_{i=0}^{r-2}(s_i - s_{i+1})q^i - 1 = q^r m + \beta q^{r-1}, \\ \left\lceil \frac{d}{q^l} \right\rceil &= q^{r-l}m + \beta q^{r-l-1},\ l = 1,2,\ldots,r-1, \\ \left\lceil \frac{d}{q^r} \right\rceil &= \left\lceil m + \frac{\beta q^{r-1}}{q^r} \right\rceil = m + a, \end{aligned}$$

where a is equal to either 1 or 0. Thus we have that

$$g_q(r+1,d) = \theta_r m + \sum_{i=0}^{r-1} s_i q^i + b = n + b,$$

where b is equal to either 0 or -1. From Lemma 2, there exist such codes for $m \geq r-1$. □

In addition, we have the following lemma.

Lemma 15 *Suppose that $n = \theta_r m + \theta_r - \theta_l + 1$ with $m \geq 0$ and $0 \leq l \leq r-1$. If $n - \lceil n\theta_{r-1}/\theta_r \rceil - 1 \geq t$, then $c(B_{n,t}(q), q) \leq n - (r+1)$.*

Proof. Set

$$\begin{aligned} d := n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil &= (m+1)\theta_r - \theta_l + 1 - \left\lceil (m+1)\theta_{r-1} - \theta_{l-1} + \frac{\theta_{l-1}}{\theta_r} \right\rceil \\ &= (m+1)\theta_r - \theta_l + 1 - (m+1)\theta_{r-1} + \theta_{l-1} - 1 = q^r(m+1) - q^l. \end{aligned}$$

From Lemma 12, there exists an $[(m+1)\theta_r - \theta_l, r+1, d' \geq q^r(m+1) - q^l]_q$ code C. By adding 0 to each codewords in C as an extra coordinate, we can construct an $[(m+1)\theta_r - \theta_l + 1, r+1, d' \geq q^r(m+1) - q^l]_q$ code. □

Now we would label the cases that we consider in Lemmas 11–15 for given n and r as follows:

Cases $\mathbf{B}_r$:

(1) $n = \theta_r m$ for some $m \geq 1$;

(2) $n = \theta_r m + \theta_r - \theta_l$ for some $m \geq 0$ and some l with $0 \leq l \leq r-1$;
(3) $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ for some $m \geq r-1$ and some $s_0, \ldots, s_{r-1}$ with $1 \leq s_0 \leq s_1 \leq \cdots \leq s_{r-1}$;
(4) $n = \theta_r m + \beta\theta_{r-1} + 1$ for some $m \geq r-1$ and some β with $0 \leq \beta \leq q-1$;
(5) $n = \theta_r m + \theta_r - \theta_l + 1$ for some $m \geq 0$ and some l with $0 \leq l \leq r-1$.

Furthermore, we consider the following cases for given n and r:

Cases $\mathbf{C}_r$:

(1) $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ for some $m \geq r-1$ and some $s_0, \ldots, s_{r-1}$ with $s_0 = s_1 = \cdots = s_{r-2} > s_{r-1}$;
(2) $n = \theta_r m + \sum_{i=0}^{r-1} s_i q^i$ for some $m \geq r-1$ and some $s_0, \ldots, s_{r-1}$ with $s_{r-1} \neq 0$, $0 = s_0 \leq s_1 \leq \cdots \leq s_{r-1}$, $s_{i+1} - s_i \leq q-1$ for all $i = 0, 1, \ldots, r-2$.

Proposition 16 *For given n and $r (\geq 2)$, suppose that either of the cases $\mathbf{C}_r$ holds. If*

$$n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 2 \geq t,$$

then $c(B_{n,t}(q), q) \leq n - (r+1)$.

Proof. We prove that there exists an $[n, r+1, d := n - \lceil n\theta_{r-1}/\theta_r \rceil - 1]_q$ code.

We firstly suppose that the case $\mathbf{C}_r(1)$ holds. Then it follows that

$$\begin{aligned}
d = n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 &= q^r m + s_{r-1} q^{r-1} + \sum_{i=0}^{r-2} (s_i - s_{i+1}) q^i - 2 \\
&= q^r m + s_{r-1} q^{r-1} + (s_{r-2} - s_{r-1}) q^{r-2} - 2, \\
\left\lceil \frac{d}{q} \right\rceil &= \begin{cases} 2m + s_{r-1} & (q = 2,\ r = 2) \\ 2^{r-1} m + s_{r-1} 2^{r-2} + 2^{r-3} - 1 & (q = 2,\ r \geq 3) \\ q^{r-1} m + s_{r-1} q^{r-2} + (s_{r-2} - s_{r-1}) q^{r-3} & (q \neq 2) \end{cases}, \\
\left\lceil \frac{d}{q^l} \right\rceil &= q^{r-l} m + s_{r-1} q^{r-l-1} + (s_{r-2} - s_{r-1}) q^{r-l-2},\ l = 2, \ldots, r-2, \\
\left\lceil \frac{d}{q^{r-1}} \right\rceil &= \left\lceil qm + s_{r-1} + \frac{(s_{r-2} - s_{r-1}) q^{r-2} - 2}{q^{r-1}} \right\rceil = qm + s_{r-1} + 1 \text{ or } qm + s_{r-1}, \\
\left\lceil \frac{d}{q^r} \right\rceil &= \left\lceil m + \frac{s_{r-1} q^{r-1} + (s_{r-2} - s_{r-1}) q^{r-2} - 2}{q^r} \right\rceil = m + 1 \text{ or } m.
\end{aligned}$$

Thus we have that

$$g_q(r+1, d) \leq \theta_r m + \sum_{i=0}^{r-1} s_i q^i = n.$$

From Lemma 2, there exist such codes for $m \geq r-1$.

Next we suppose that the case $\mathbf{C}_r(2)$ holds. Then $s_{i+1} - s_i \leq q-1$ for all $i = 0, 1, \ldots, r-2$ implies that $\sum_{i=0}^{l-1} (s_{i+1} - s_i) q^i + 1 = q^l$ for any l if and only if $s_{i+1} - s_i = q-1$ for all $i = 0, 1, \ldots, l-1$. However this does not happen in the case $s_0 \le s_1 \le \cdots \le s_{r-1}$. Hence we have that

$$\begin{aligned}
d &= n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1 \\
&= \theta_r m + \sum_{i=0}^{r-1} s_i q^i - \left\lceil \theta_{r-1} m + \sum_{i=1}^{r-1} s_i q^{i-1} + \frac{s_0\theta_{r-1} - \sum_{i=1}^{r-1} s_i q^{i-1}}{\theta_r} \right\rceil - 1 \\
&= q^r m + s_{r-1} q^{r-1} + \sum_{i=0}^{r-2} (s_i - s_{i+1}) q^i - 1, \\
\left\lceil \frac{d}{q^l} \right\rceil &= \left\lceil q^{r-l} m + s_{r-1} q^{r-l-1} - \sum_{i=l}^{r-2} (s_{i+1} - s_i) q^{i-l} - \frac{\sum_{i=0}^{l-1} (s_{i+1} - s_i) q^i + 1}{q^l} \right\rceil \\
&= q^{r-l} m + s_{r-1} q^{r-l-1} - \sum_{i=l}^{r-2} (s_{i+1} - s_i) q^{i-l},\ l = 1, 2, \ldots, r-1, \\
\left\lceil \frac{d}{q^r} \right\rceil &= \left\lceil m + \frac{s_{r-1} q^{r-1} - \sum_{i=0}^{r-2} (s_{i+1} - s_i) q^i - 1}{q^r} \right\rceil = m + 1.
\end{aligned}$$

Thus we have that

$$g_q(r+1, d) = \sum_{i=0}^{r} \left\lceil \frac{d}{q^i} \right\rceil = \theta_r m + \sum_{i=0}^{r-1} s_i q^i = n.$$

From Lemma 2, there also exist such Griesmer codes for $m \ge r - 1$.

The proposition follows. □

Based on the previous argument, we prove a necessary and sufficient condition for the equality $B_{n,t}(q) = n - r$ to hold in each case on given n and r.

Proposition 8 holds without any condition. The following result is obtained by combining Proposition 8 and Lemmas 11–15.

Theorem 17 *For given n and r, suppose that both one of the cases $\mathbf{B}_r$ and one of the cases $\mathbf{B}_{r-1}$ hold. Then $B_{n,t}(q)$ has critical exponent $n - r$ if and only if*

$$n - \left\lceil \frac{n\theta_{r-2}}{\theta_{r-1}} \right\rceil - 1 \ge t \ge n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil.$$

Now we shall modify the above theorem for some special cases on n as discussed in Proposition 10.

By combining Proposition 10 and Proposition 16, we have the following results.

Theorem 18 *For given n and $r(\ge 3)$, suppose that all of one of the cases $\mathbf{A}_{r-1}$, one of the cases $\mathbf{B}_r$, and one of the cases $\mathbf{C}_{r-1}$ hold. Then $B_{n,t}(q)$ has critical exponent $n - r$ if and only if*

$$n - \left\lceil \frac{n\theta_{r-2}}{\theta_{r-1}} \right\rceil - 2 \ge t \ge n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil.$$

Theorem 19 *For given n and r, suppose that all of one of the cases $\mathbf{A}_r$, one of the cases $\mathbf{B}_{r-1}$, and one of the cases $\mathbf{C}_r$ hold. Then $B_{n,t}(q)$ has critical exponent $n - r$ if and only if*

$$n - \left\lceil \frac{n\theta_{r-2}}{\theta_{r-1}} \right\rceil - 1 \ge t \ge n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1.$$

Theorem 20 *For given n and $r(\ge 3)$, suppose that all of one of the cases $\mathbf{A}_r$, one of the cases $\mathbf{A}_{r-1}$, one of the cases $\mathbf{C}_r$, and one of the cases $\mathbf{C}_{r-1}$ hold. Then $B_{n,t}(q)$ has critical exponent $n - r$ if and only if*

$$n - \left\lceil \frac{n\theta_{r-2}}{\theta_{r-1}} \right\rceil - 2 \ge t \ge n - \left\lceil \frac{n\theta_{r-1}}{\theta_r} \right\rceil - 1.$$

4 Some Special Cases

4.1 Conditions for $c(B_{n,t}(q), q) = n - 2$, $2 \le q \le 9$

In this subsection, we study the conditions for when $c(B_{n,t}(q), q) = n - 2$. For a given q, any positive integer $n(\ge 3)$ is uniquely written as follows by using non-negative integers m, a, b, c:

$$n = (q^2 + q + 1)m + \begin{cases} aq + b & (0 \le a \le q-1 \text{ and } 0 \le b \le q-1;) \\ q^2 + c & (0 \le c \le q.) \end{cases}$$

Suppose that $n = (q^2+q+1)m+aq+b$ with $m \ge 1$. By Theorem 5 and Proposition 10, we have that in the cases that (1) $b > a$ with $b - a \ne 1$, and (2) $b = 0$ and $a \ne 0$, if $n - \lceil n/(q+1)\rceil - 1 \ge t \ge n - \lceil (q+1)n/(q^2+q+1)\rceil - 1$, then $c(B_{n,t}(q), q) = n - 2$. From Theorem 5 and Proposition 16, the converse is true for these cases.

By Theorem 5 and Proposition 8, it also finds that if $n - \lceil n/(q+1)\rceil - 1 \ge t \ge n - \lceil (q+1)n/(q^2+q+1)\rceil$, then $c(B_{n,t}(q), q) = n - 2$. By applying Lemma 14 for the case of $b - a = 1$, Lemma 13 for the case of $0 < b \le a$, and Lemma 11 for the case of $a = b = 0$, it turns out that the converse is also true for these cases.

In addition, if $n = (q^2 + q + 1)m + q^2 + c$ with $m \ge 1$, then the necessary and sufficient condition also holds from Lemmas 13 and 12.

Theorem 21 *Suppose that $n \ge q^2+q+1$. If $n = (q^2+q+1)m+aq+b$ for $m \ge 1$, $0 \le a \le q-1$, and $0 \le b \le q-1$ such that (1) $a < b$ with $b - a \ne 1$, or (2) $a > b = 0$ holds, then $B_{n,t}(q)$ has critical exponent $n-2$ if and only if*

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 \ge t \ge n - \left\lceil \frac{(q+1)n}{q^2+q+1} \right\rceil - 1.$$

Otherwise $B_{n,t}(q)$ has critical exponent $n-2$ if and only if

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 \ge t \ge n - \left\lceil \frac{(q+1)n}{q^2+q+1} \right\rceil.$$

In order to determine the condition for when $c(B_{n,t}(q), q) = n-2$ completely, it is sufficient to study the existences of some linear codes. From Theorem 2, it is sufficient to study the existences of $[n, 3, n - \lceil (q+1)n/(q^2+q+1)\rceil]_q$ codes or $[n, 3, n - \lceil (q+1)n/(q^2+q+1)\rceil - 1]_q$ codes for any $(3 \le)n = aq + b$, $0 \le a \le q-1$ and $0 \le b \le q-1$, or $m = q^2 + c$, $0 \le c \le q$. By using Lemmas 12 and 15, we find that there exist $[n, 3, n - \lceil (q+1)n/(q^2+q+1)\rceil]_q$ codes for $n = q^2 + q, q^2, q^2 + 1$. Therefore we study the existences of some $[n, 3, d]_q$ codes with the following types of parameters from Lemmas 13 and 14 ((I), (II), (III)), and Proposition 16 ((IV), (V)).

(I) $[aq + b, 3, d = (q-1)a + b - 1]_q$ codes with $1 \le b \le a(\le q-1)$;
(II) $[q^2 + c, 3, d = (q-1)q + c - 1]_q$ codes with $2 \le c \le q-1$;
(III) $[aq + a + 1, 3, d = aq]_q$ codes with $0 \le a \le q-2$;
(IV) $[aq + b, 3, d = (q-1)a + b - 2]_q$ codes with $(q - 1 \ge)b > a \ge 0$ and $b - a \ne 1$;
(V) $[aq, 3, d = (q-1)a - 1]_q$ codes with $1 \le a \le q-1$.

According to [3], [8] and [21], we can summarize the nonexistence of codes of types (I) and (III) for $q \le 9$ in Table 1. We remark that there always exists an $[n, 3, d-1]_q$ code for any parameters in Table 1.

On the codes of types (II) and (IV), we find that there exist such codes with these parameters for $q \le 9$ (see, [8]). The non-existing codes of type (V) for $q \le 9$ are the $[16, 3, 13]_8$ code and the $[18, 3, 15]_9$ code. In addition, we remark that there exist a $[16, 3, 12]_8$ code and the $[18, 3, 14]_9$ code (see, [3,8,21]).

Then we prove a necessary and sufficient condition for the equality $c(B_{n,t}(q), q) = n - 2$ to hold with $2 \le q \le 9$.

Corollary 22 *Let n be a positive integer with $n \ge 3$. Then $B_{n,t}(2)$ has critical exponent $n-2$ if and only if*

$$n - \left\lceil \frac{n}{3} \right\rceil - 1 \ge t \ge n - \left\lceil \frac{3n}{7} \right\rceil + a,$$

where

$$a = \begin{cases} -1 & (n \equiv 2 \pmod 7) \\ 0 & (\text{otherwise}). \end{cases}$$

q	3	4	4	5	5	5	5	5	5	7	7	7	7	7
(a,b)	$(1,2)$	$(2,2)$	$(2,3)$	$(2,2)$	$(3,2)$	$(3,3)$	$(1,2)$	$(2,3)$	$(3,4)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(3,3)$
n	5	10	11	12	17	18	7	13	19	16	23	30	37	24
d	3	7	8	9	13	14	5	10	15	13	19	25	31	20

q	7	7	7	7	7	7	7	7	7	7	8	8	8	8
(a,b)	$(4,3)$	$(5,3)$	$(4,4)$	$(5,4)$	$(5,5)$	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,6)$	$(2,1)$	$(2,2)$	$(4,2)$	$(6,2)$
n	31	38	32	39	40	9	17	25	33	41	17	18	34	50
d	26	32	27	33	34	7	14	21	28	35	14	15	29	43

q	8	8	8	8	8	8	8	8	8	8	8	8	8	9
(a,b)	$(4,3)$	$(5,3)$	$(6,3)$	$(4,4)$	$(5,4)$	$(6,4)$	$(5,5)$	$(6,5)$	$(6,6)$	$(2,3)$	$(4,5)$	$(5,6)$	$(6,7)$	$(2,1)$
n	35	43	51	36	44	52	45	53	54	19	37	46	55	19
d	30	37	44	31	38	45	39	46	47	16	32	40	48	16

q	9	9	9	9	9	9	9	9	9	9	9	9	9	9
(a,b)	$(2,2)$	$(3,2)$	$(3,3)$	$(4,2)$	$(4,3)$	$(4,4)$	$(5,4)$	$(5,5)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$	$(7,3)$
n	20	29	30	38	39	40	49	50	56	57	58	59	60	66
d	17	25	26	33	34	35	43	44	49	50	51	52	53	58

q	9	9	9	9	9	9	9	9	9	9	9
(a,b)	$(7,4)$	$(7,5)$	$(7,6)$	$(7,7)$	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,6)$	$(6,7)$	$(7,8)$
n	67	68	69	70	11	21	31	41	51	61	71
d	59	60	61	62	9	18	27	36	45	54	63

Table 1 Non-existing codes of types (I) and (Ⅲ)

Corollary 23 *Let n be a positive integer with $n \geq 3$ and let q be a prime power with $3 \leq q \leq 9$. If $(n,q) = (16,8)$, or $(18,9)$, then $B_{n,t}(q)$ has critical exponent $n-2$ if and only if*

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 \geq t \geq n - \left\lceil \frac{(q+1)n}{q^2+q+1} \right\rceil - 2.$$

If (n,q) is one of the pairs in Table 1, or $n = aq+b$ for any (a,b) such that $0 \leq a < b \leq q-1$ and $b-a \neq 1$, then $B_{n,t}(q)$ has critical exponent $n-2$ if and only if

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 \geq t \geq n - \left\lceil \frac{(q+1)n}{q^2+q+1} \right\rceil - 1.$$

Otherwise $B_{n,t}(q)$ has critical exponent $n-2$ if and only if

$$n - \left\lceil \frac{n}{q+1} \right\rceil - 1 \geq t \geq n - \left\lceil \frac{(q+1)n}{q^2+q+1} \right\rceil.$$

4.2 Conditions for $c(B_{n,t}(2),2) = n-r$, $r = 3,4$

Throughout of this subsection, we only consider the binary case.

According to Theorem 2 and [8], it is known that there always exists a Griesmer $[n,4,d]_2$ code for any positive integer d. From Theorem 17, if $n \equiv 0,1,7,8,9,11,12,13,14 \pmod{15}$, then there exists any $[n,4,n-\lceil 7n/15 \rceil]_2$ code and there does not exist any $[n,4,n-\lceil 7n/15\rceil+1]_2$ code. From Theorem 18, if $n \equiv 3,4,6,10 \pmod{15}$, then there exists any $[n,4,n-\lceil 7n/15\rceil-1]_2$ code and there does not exist any $[n,4,n-\lceil 7n/15\rceil]_2$ code. Moreover, by calculation, if $n \equiv 2 \pmod{15}$, then there exists any $[n,4,n-\lceil 7n/15\rceil-1]_2$ code and there does not exist any $[n,4,n-\lceil 7n/15\rceil]_2$ code, and if $n \equiv 5 \pmod{15}$, then there exists any $[n,4,n-\lceil 7n/15\rceil]_2$ code and there does not exist any $[n,4,n-\lceil 7n/15\rceil+1]_2$ code. On combining these facts, we immediately deduce the following.

Proposition 24 *Let n be a positive integer with $n \geq 4$. Set*

$$a := \begin{cases} -2 & (n \equiv 2 \pmod{7}) \\ -1 & (\text{otherwise}) \end{cases} \text{ and } b := \begin{cases} -1 & (n \equiv 2,3,4,6,10 \pmod{15}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then $B_{n,t}(2)$ has critical exponent $n-3$ if and only if

$$n - \left\lceil \frac{3n}{7} \right\rceil + a \geq t \geq n - \left\lceil \frac{7n}{15} \right\rceil + b.$$

According to Theorem 2 and [8], it is known that there exists any Griesmer $[n,5,d]_2$ code except for $[8,5,3]_2$, $[9,5,4]_2$, $[12,5,5]_2$, $[13,5,6]_2$ codes, and there exist $[8,5,2]_2$, $[9,5,3]_2$, $[12,5,4]_2$, $[13,5,5]_2$ codes. Let n be a positive integer with $n \geq 5$ and $n \neq 8,9,12,13$. From Theorem 17, if $n \equiv 0,1,15,16,23,24,27,28,29,30 \pmod{31}$, then there exists any $[n,5,n-\lceil 15n/31\rceil]_2$ code and there does not exist any $[n,5,n-\lceil 15n/31\rceil+1]_2$ code. From Theorem 18, if $n \equiv 7,8,12,14,20,22,26 \pmod{31}$, then there exists any $[n,5,n-\lceil 15n/31\rceil-1]_2$ code and there does not exist any $[n,5,n-\lceil 15n/31\rceil]_2$ code. Moreover, by calculation, if $n \equiv 2,3,4,5,6,10,11,18,19,21,25 \pmod{31}$, then there exists any $[n,5,n-\lceil 15n/31\rceil-1]_2$ code and there does not exist any $[n,5,n-\lceil 15n/31\rceil]_2$ code, and if $n \equiv 9,13,17 \pmod{31}$, then there exists any $[n,5,n-\lceil 15n/31\rceil]_2$ code and there does not exist any $[n,5,n-\lceil 15n/31\rceil+1]_2$ code. Therefore we have the following result.

Proposition 25 *Let n be a positive integer with $n \geq 5$. Set*

$$a := \begin{cases} -2 & (n \equiv 2,3,4,6,10 \pmod{15}) \\ -1 & (\text{otherwise}), \end{cases}$$

$$b := \begin{cases} -1 & (n \equiv 2,3,4,5,6,7,8,10,11,12,14,18,19,20,21,22,25,26 \pmod{31}) \\ 0 & (\text{otherwise}), \end{cases}$$

$$c := \begin{cases} -2 & (n = 8,12) \\ -1 & (n = 9,13). \end{cases}$$

Suppose that $n \neq 8,9,12,13$. Then $B_{n,t}(2)$ has critical exponent $n-4$ if and only if

$$n - \left\lceil \frac{7n}{15} \right\rceil + a \geq t \geq n - \left\lceil \frac{15n}{31} \right\rceil + b.$$

Suppose that $n = 8,9,12,$ or 13. Then $B_{n,t}(2)$ has critical exponent $n-4$ if and only if

$$n - \left\lceil \frac{7n}{15} \right\rceil - 2 \geq t \geq n - \left\lceil \frac{15n}{31} \right\rceil + c.$$

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