Nonexistence of some linear codes over the field of order four

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Abstract. We consider the problem of determining $n_4(5,d)$, the smallest possible length n for which an $[n,5,d]_4$ code of minimum distance d over the field of order 4 exists. We prove the nonexistence of $[g_4(5,d)+1,5,d]_4$ codes for $d = 31, 47, 48, 59, 60, 61, 62$ and the nonexistence of a $[g_4(5,d),5,d]_4$ code for $d = 138$ using the geometric method through projective geometries, where $g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$. This yields to determine the exact values of $n_4(5,d)$ for these values of d. We also give the updated table for $n_4(5,d)$ for all d except some known cases.

1 Introduction

We denote by $\mathbb{F}_q^n$ the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. An $[n,k,d]_q$ *code* $\mathcal{C}$ is a linear code of length n, dimension k and minimum Hamming weight d over $\mathbb{F}_q$. The *weight* of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. We only consider *non-degenerate* codes having no coordinate which is identically zero. A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists. This problem is sometimes called the optimal linear codes problem, see [5, 6]. A well-known lower bound on $n_q(k,d)$, called the Griesmer bound, says:

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil ,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k,d)$ are determined for all d only for some small values of q and k. The optimal linear codes problem for $q = 4$ is solved for $k \leq 4$ for all d, see [9, 15].

Theorem 1.1. *$n_4(4,d) = g_4(4,d)+1$ for $d = 3, 4, 7, 8, 13$-$16, 23$-$32, 37$-$44, 77$-80 and $n_4(4,d) = g_4(4,d)$ for any other d.*

As for the case $k = 5$, the value of $n_4(5,d)$ is unknown for 107 values of d, and the remaining cases look quite difficult because the only progress after the computer-aided research [1] was the nonexistence of Griesmer codes for $d = 287, 288$ [10], see also [15]. It is known that $n_4(5,d)$ is equal to $g_4(5,d)+1$ or $g_4(5,d)+2$ for $d = 31, 47, 48, 59, 60, 61, 62$ and that $n_4(5,d)$ is equal to $g_4(5,d)$ or $g_4(5,d)+1$ for $d = 138$. Our purpose is to prove the following theorems to determine $n_4(5,d)$ for these values of d.

Theorem 1.2. *There exists no $[g_4(5,d)+1,5,d]_4$ code for $d = 31, 47, 48, 59, 60, 61, 62$.*

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Theorem 1.3. *There exists no* $[g_4(5,d),5,d]_4$ *code for* $d = 138$.

We note that our proofs would heavily depend on the extension theorems which are valid only for linear codes over $\mathbb{F}_4$. So, generalizing the nonexistence results to $q \geq 5$ seems hopeless. The above theorems determine $n_4(5,d)$ for some d as follows.

Corollary 1.4. $n_4(5,d) = g_4(5,d) + 2$ *for* $d = 31, 47, 48, 59, 60, 61, 62$.

Corollary 1.5. $n_4(5,d) = g_4(5,d) + 1$ *for* $d = 138$.

For $k \geq 6$, we get the following by shortening since $g_q(k,d) = g_q(5,d) + k - 5$ for $k \geq 6$ if $d \leq q^5$.

Corollary 1.6. $n_4(k,d) \geq g_4(k,d) + 2$ *for* $d = 31, 47, 48, 59, 60, 61, 62$ *for* $k \geq 6$.

Corollary 1.7. $n_4(k,d) \geq g_4(k,d) + 1$ *for* $d = 138$ *for* $k \geq 6$.

We also give the updated table for $n_4(5,d)$ as Table 2. We give the values and bounds of $g = g_4(5,d)$ and $n = n_4(5,d)$ for all d except for $249 \leq d \leq 256$ and for $d \geq 369$ which are the cases satisfying $n_4(5,d) = g_4(5,d)$. Entries in boldface are given in this paper.

2 Preliminaries

In this section, we give the geometric method through $\mathrm{PG}(r,q)$, the projective geometry of dimension r over $\mathbb{F}_q$, and preliminary results to prove the main results. The 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats in $\mathrm{PG}(r,q)$ are called *points, lines, planes, solids, secundums* and *hyperplanes*, respectively.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$, denoted by $\mathcal{M}_\mathcal{C}$. An *i-point* is a point of Σ which has multiplicity i in $\mathcal{M}_\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_\mathcal{C}$ and let C_i be the set of i-points in Σ, $0 \leq i \leq \gamma_0$. For any subset S of Σ, *the multiplicity of* S *with respect to* $\mathcal{M}_\mathcal{C}$, denoted by $m_\mathcal{C}(S)$, is defined as $m_\mathcal{C}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|$, where $|T|$ denotes the number of elements in a set T. A line l with $t = m_\mathcal{C}(l)$ is called a *t-line*. A *t-plane* and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that $n = m_\mathcal{C}(\Sigma)$ and

$$n - d = \max\{m_\mathcal{C}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}, \tag{2.1}$$

where $\mathcal{F}_j$ denotes the set of j-flats of Σ. Conversely, such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n,k,d]_q$ code in the natural manner. For an m-flat Π in Σ, we define

$$\gamma_j(\Pi) = \max\{m_\mathcal{C}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\} \quad \text{for} \quad 0 \leq j \leq m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. Then $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$. For a Griesmer $[n,k,d]_q$ code, it is known (see [13]) that

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for} \quad 0 \leq j \leq k-1. \tag{2.2}$$

Let θ_j be the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$. An $[n,k,d]_q$ code, which is not necessarily Griesmer, satisfies the following:

$$\gamma_j \leq \gamma_{j+1} - \frac{n - \gamma_{j+1}}{\theta_{k-2-j} - 1}, \tag{2.3}$$

see [9]. We denote by λ_s the number of s-points in Σ. When $\gamma_0 = 2$, we have

$$\lambda_2 = \lambda_0 + n - \theta_{k-1}. \tag{2.4}$$

Denote by a_i the number of i-hyperplanes in Σ. The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following.

Lemma 2.1 ([8]). (a) $\sum_{i=0}^{n-d} a_i = \theta_{k-1}$. (b) $\sum_{i=1}^{n-d} i a_i = n\theta_{k-2}$.

(c) $\sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s$.

When $\gamma_0 \leq 2$, the above three equalities yield the following:

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \tag{2.5}$$

Lemma 2.2 ([16]). *Let Π be a w-hyperplane through a t-secundum δ. Then*

(a) $t \leq \gamma_{k-2} - (n-w)/q = (w + q\gamma_{k-2} - n)/q$.

(b) $a_w = 0$ *if an* $[w, k-1, d_0]_q$ *code with* $d_0 \geq w - \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

(c) $\gamma_{k-3}(\Pi) = \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor$ *if an* $[w, k-1, d_1]_q$ *code with* $d_1 \geq w - \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor + 1$ *does not exist.*

(d) *Let c_j be the number of j-hyperplanes through δ other than Π. Then* $\sum_j c_j = q$ *and*

$$\sum_j (\gamma_{k-2} - j)c_j = w + q\gamma_{k-2} - n - qt. \tag{2.6}$$

(e) *For a γ_{k-2}-hyperplane Π_0 with spectrum $(\tau_0, \ldots, \tau_{\gamma_{k-3}})$, $\tau_t > 0$ holds if $w + q\gamma_{k-2} - n - qt < q$.*

Lemma 2.3 ([7]). *Let Π be an i-hyperplane and let $\mathcal{C}_\Pi$ be an $[i, k-1, d_0]$ code generated by $\mathcal{M}_\mathcal{C}(\Pi)$. If any γ_{k-2}-hyperplane has no t-secundum with $t = \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$, then $d_0 \geq i - t + 1$.*

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1, k, d+1]_q$ code which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable*. It is known that the spectrum of a $[49, 4, 36]_4$ code is either

$$(a_1, a_9, a_{13}) = (1, 16, 68) \text{ or } (a_5, a_9, a_{13}) = (3, 13, 69)$$

and that every $[48, 4, 35]_4$ code is extendable [12]. The possible spectra of $[48, 4, 35]_4$ codes are given as follows.

Table 1: The spectra of some $[n, 4, d]_4$ codes.

parameters	possible spectra	reference
$[4, 4, 1]_4$	$(a_0, a_1, a_2, a_3) = (27, 36, 18, 4)$	[11]
$[5, 4, 2]_4$	$(a_0, a_1, a_2, a_3) = (20, 35, 20, 10)$	[11]
$[10, 4, 6]_4$	$(a_0, a_1, a_2, a_3, a_4) = (5, 20, 15, 20, 25)$	[3]
	$(a_0, a_1, a_2, a_3, a_4) = (6, 16, 21, 16, 26)$	
$[14, 4, 9]_4$	$(a_0, a_1, a_2, a_3, a_4, a_5) = (3, 14, 1, 12, 33, 22)$	[1]
$[15, 4, 10]_4$	$(a_0, a_1, a_3, a_4, a_5) = (2, 15, 5, 30, 33)$	[1]
$[16, 4, 11]_4$	$(a_0, a_1, a_4, a_5) = (1, 16, 20, 48)$	[1]
$[17, 4, 12]_4$	$(a_1, a_5) = (17, 68)$	[2]
$[23, 4, 16]_4$	$(a_3, a_7) = (28, 57)$	[1]
	$(a_1, a_3, a_5, a_7) = (6, 10, 18, 51)$	
	$(a_1, a_3, a_5, a_7) = (4, 12, 20, 49)$	
$[30, 4, 21]_4$	$(a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_9) = (1, 2, 4, 2, 11, 12, 22, 31)$	[1]
	$(a_2, a_3, a_5, a_6, a_7, a_8, a_9) = (4, 4, 4, 5, 12, 28, 28)$	
	$(a_1, a_2, a_3, a_6, a_7, a_8, a_9) = (1, 1, 7, 6, 23, 14, 33)$	
	$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = (1, 2, 4, 2, 2, 7, 12, 26, 29)$	
	$(a_1, a_3, a_6, a_7, a_8, a_9) = (1, 8, 9, 20, 12, 35)$	
	$(a_0, a_3, a_6, a_7, a_8, a_9) = (1, 8, 5, 24, 15, 32)$	
$[31, 4, 22]_4$	$(a_1, a_3, a_7, a_9) = (1, 8, 29, 47)$	[1]
$[48, 4, 35]_4$	$(a_0, a_9, a_{12}, a_{13}) = (1, 16, 20, 48)$	Lemma 2.4
	$(a_1, a_8, a_9, a_{12}, a_{13}) = (1, 3, 13, 18, 50)$	
	$(a_4, a_5, a_8, a_9, a_{12}, a_{13}) = (1, 2, 1, 12, 19, 50)$	
	$(a_5, a_8, a_9, a_{12}, a_{13}) = (3, 3, 10, 18, 51)$	

Lemma 2.4. *The spectrum of a* $[48, 4, 35]_4$ *code is one of the following:*
(a) $(a_0, a_9, a_{12}, a_{13}) = (1, 16, 20, 48)$
(b) $(a_1, a_8, a_9, a_{12}, a_{13}) = (1, 3, 13, 18, 50)$
(c) $(a_4, a_5, a_8, a_9, a_{12}, a_{13}) = (1, 2, 1, 12, 19, 50)$
(d) $(a_5, a_8, a_9, a_{12}, a_{13}) = (3, 3, 10, 18, 51)$.

Proof Let $\mathcal{C}$ be a Griesmer $[48, 4, 35]_4$ code. Since $\mathcal{C}$ is extendable, the spectrum of $\mathcal{C}$ satisfies $a_i = 0$ for $i \notin \{0, 1, 4, 5, 8, 9, 12, 13\}$ and (2.5) gives

$$39a_0 + 33a_1 + 18a_4 + 14a_5 + 5a_8 + 3a_9 = 87. \qquad (2.7)$$

Assume that $\mathcal{C}$ is extendable to a code with spectrum $(a_1, a_9, a_{13}) = (1, 16, 68)$. Then, the spectrum of $\mathcal{C}$ satisfies $a_0 + a_1 = 1$, $a_8 + a_9 = 16$, $a_{12} + a_{13} = 68$. Hence, the possible solutions for (2.6) are $(c_9, c_{13}) = (1, 3)$ and $c_{12} = 4$ for $(i, t) = (0, 0)$, and $a_0 > 0$ implies (a). Assume $a_1 > 0$. Since the possible solutions for (2.6) are $(c_8, c_{13}) = (1, 3)$ and $(c_9, c_{12}, c_{13}) = (1, 1, 2)$ for $(i, t) = (1, 0)$; $(c_{12}, c_{13}) = (1, 3)$ for $(i, t) = (1, 1)$, we have $a_1 = 1$, $a_9 = 16 - a_8$, $a_{12} = a_9 + 5 = 21 - a_8$, $a_{13} = 68 - a_{12} = 47 + a_8$. Hence we get (b) from (2.7). Assuming that $\mathcal{C}$ is extendable to a code with another spectrum, one can obtain (c) and (d) using (2.6) and (2.7) similarly. □

To prove Theorems 1.2 and 1.3, we employ the following results.

Theorem 2.5 ([14])**.** *Let* $\mathcal{C}$ *be an* $[n, k, d]_4$ *code with odd* d, $k \geq 3$. *Then* $\mathcal{C}$ *is extendable if* $A_i = 0$ *for all* $i \equiv 2 \pmod 4$ *or if* $i \equiv -d \pmod 4$.

Theorem 2.6 ([17])**.** *Let* $\mathcal{C}$ *be an* $[n, k, d]_4$ *code with* $k \geq 3$, $d \equiv 2 \pmod 4$ *such that* $A_i = 0$ *for all* $i \equiv 1 \pmod 4$. *Then* $\mathcal{C}$ *is extendable if there is a codeword* $c \in \mathcal{C}$ *with* $wt(c) \equiv 3 \pmod 4$.

3 Proof of Theorem 1.2

Lemma 3.1. *There exists no* $[84, 5, 61]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative $[84, 5, 61]_4$ code, where $84 = g_4(5, 61) + 1$. Then, we have $\gamma_0 \leq 2$, $\gamma_1 \leq 3$, $\gamma_2 \leq 7$, $\gamma_3 = 23$ from (2.3). Hence $\lambda_2 = 0$ or 1. Let Δ_{23} be a 23-solid. From Table 1, the spectrum of Δ_{23} is one of

(A) $(\tau_3, \tau_7) = (28, 57)$,
(B) $(\tau_1, \tau_3, \tau_5, \tau_7) = (6, 10, 18, 51)$,
(C) $(\tau_1, \tau_3, \tau_5, \tau_7) = (4, 12, 20, 49)$.

Hence, there is no 0-solid in $\Sigma = \mathrm{PG}(4, 4)$ since a j-plane on Δ_{23} satisfies $j \in \{1, 3, 5, 7\}$. If there exists a 2-solid, then one can find a 2-plane in the solid. Setting $(w, t) = (2, 2)$, any solution of (2.6) satisfies $c_{23} > 0$, which contradicts the fact that a 23-solid has no 2-plane. If there exists a 3-solid, then one can find a 3-plane there as well. But (2.6) has no solution for $(w, t) = (3, 3)$, a contradiction. If there exists a 7-solid, then it corresponds to a $[7, 4, 4]_4$ code by Lemma 2.2 (a), which does not exist by Theorem 1.1. If there exists a 19-solid, then it corresponds to a $[19, 4, 14]_4$ code by Lemma 2.3, which does not exist by the Griesmer bound. We can also prove $a_j = 0$ for $j = 6$, 8-11, 18. Thus, one can show $a_i = 0$ for all $i \notin \{1, 4, 5, 12\text{-}17, 20\text{-}23\}$ using Lemmas 2.2, 2.3, the Griesmer bound and Theorem 1.1 since an i-solid Δ_i can not meet Δ_{23} in a t-plane with $t \in \{0, 2, 4, 6\}$. We refer to this procedure as the **first sieve** in the proofs of the nonexistence results.

From (2.5), we get

$$\sum_{i=1}^{21} \binom{23-i}{2} a_i = 64\lambda_2 + 2399. \tag{3.1}$$

For any w-solid through a t-plane, (2.6) gives

$$\sum_j (23 - j)c_j = w + 8 - 4t \tag{3.2}$$

with $\sum_j c_j = 4$.

Suppose $a_1 > 0$ and let Δ_1 be a 1-solid. Then, the spectrum of Δ_1 is $(\tau_0, \tau_1) = (64, 21)$, and we have $a_1 = 1$ and $a_i = 0$ for $1 < i < 14$ from (3.2). Setting $w = 1$, the maximum possible contributions of c_j's to the LHS of (3.1) are $(c_{17}, c_{22}) = (1, 3)$ for $t = 0$ since $c_{23} = 0$; $(c_{20}, c_{21}, c_{23}) = (1, 1, 2)$ for $t = 1$. Estimating the LHS of (3.1) for the spectrum of Δ_1, we get

$$64\lambda_2 + 2399 \leq 15\tau_0 + (3 + 1)\tau_1 + 231 = 1275,$$

a contradiction. Hence $a_1 = 0$. One can similarly prove $a_4 = a_5 = 0$ using the spectra for a 4-plane and a 5-plane from Table 1.

Suppose that $a_{14} > 0$ and let Δ_{14} be a 14-solid. Then, the spectrum of Δ_{14} is $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = (3, 14, 1, 12, 33, 22)$ from Table 1. Setting $w = 14$, the maximum possible contributions of c_j's to the LHS of (3.1) are $(c_{12}, c_{14}, c_{22}) = (1, 1, 2)$ for $t = 0$; $(c_{12}, c_{16}, c_{23}) = (1, 1, 2)$ for $t = 1$; $(c_{12}, c_{22}) = (1, 3)$ for $t = 2$; $(c_{13}, c_{23}) = (1, 3)$ for $t = 3$; $(c_{20}, c_{22}) = (1, 3)$ for $t = 4$; $(c_{21}, c_{23}) = (1, 3)$ for $t = 5$ since we have $c_{23} = 0$ when t is even. Estimating the LHS of (3.1) for the spectrum of Δ_{14}, we get

$$64\lambda_2 + 2399 \leq (55 + 36)\tau_0 + (55 + 21)\tau_1 + 55\tau_2 + 45\tau_3 + 3\tau_4 + \tau_5 + 36 = 2089,$$

a contradiction. Hence $a_{14} = 0$.

Next, we suppose $a_{12} > 0$ and let Δ_{12} be a 12-solid. Then, Δ_{12} corresponds to a $[12, 4, 7]_4$ code. It is known from [1] that there are exactly 275 inequivalent such codes and 53 possible spectra $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ for Δ_{12}. We omit the list of them to save space in this paper, which is available from the author upon request. Setting $w = 12$, the maximum possible contributions of c_j's to the LHS of (3.1) are $(c_{12}, c_{16}, c_{22}) = (1, 1, 2)$ for

$t = 0$; $(c_{13}, c_{17}, c_{23}) = (1, 1, 2)$ for $t = 1$; $(c_{15}, c_{21}, c_{22}) = (1, 1, 2)$ for $t = 2$; $(c_{15}, c_{23}) = (1, 3)$ for $t = 3$; $c_{22} = 4$ for $t = 4$; $c_{23} = 4$ for $t = 5$ since we have $c_{23} = 0$ when t is even. Estimating the LHS of (3.1) for the possible spectra of Δ_{12}, we get

$$64\lambda_2 + 2399 \le (55 + 21)\tau_0 + (45 + 15)\tau_1 + (28 + 1)\tau_2 + 28\tau_3 + 0\tau_4 + 0\tau_5 + 55 \le 2351,$$

a contradiction. Hence $a_{12} = 0$.

Thus, $a_i > 0$ implies $i \in \{13, 15, 16, 17, 20, 21, 22, 23\}$. Setting $w = 23$, the maximum possible contributions of c_j's to the LHS of (3.1) are $(c_{13}, c_{16}, c_{23}) = (2, 1, 1)$ for $t = 1$; $(c_{13}, c_{15}, c_{22}, c_{23}) = (1, 1, 1, 1)$ for $t = 3$; $(c_{13}, c_{22}, c_{23}) = (1, 1, 2)$ for $t = 5$; $(c_{20}, c_{23}) = (1, 3)$ for $t = 7$. Estimating the LHS of (3.1) for the possible spectra (A)-(C) for Δ_{23}, we get

$$64\lambda_2 + 2399 \le (45 \cdot 2 + 21)\tau_1 + (45 + 28)\tau_3 + 45\tau_5 + 3\tau_7 \le 2367,$$

giving a contradiction. This completes the proof. □

Lemma 3.2. *There exists no* $[82, 5, 60]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative $[82, 5, 60]_4$ code, where $82 = g_4(5, 60) + 1$. Note that $\gamma_1 \le 3$ and $\gamma_0 \le 2$ from (2.3). Hence we have $\lambda_2 = 0$ or 1. By the first sieve, we have $a_i = 0$ for all $i \notin$ {0, 1, 2, 10, 14-18, 22}. For any w-solid through a t-plane, (2.6) gives

$$\sum_j (22 - j)c_j = w + 6 - 4t \tag{3.3}$$

with $\sum_j c_j = 4$. For $(i, t) = (1, 1)$, $(15, 5)$, $(16, 5)$, $(17, 5)$, the equation (3.3) has no solution. Thus, $a_1 = a_{15} = a_{16} = a_{17} = 0$. Then, for $(i, t) = (0, 0)$, the equation (3.3) has no solution again. Hence $a_0 = 0$. From the three equalities in Lemma 2.1, we get

$$190a_2 + 66a_{10} + 28a_{14} + 6a_{18} = 2142 + 64\lambda_2. \tag{3.4}$$

Suppose $a_2 > 0$ and let Δ_2 be a 2-solid. Then, the spectrum of Δ_2 is $(\tau_0, \tau_1, \tau_2) = (48, 32, 5)$, and we have $a_2 = 1$ and $a_{10} = 0$ from (3.3). Setting $w = 2$ in (3.3), the maximum possible contributions of c_j's to the LHS of (3.4) are $(c_{14}, c_{22}) = (1, 3)$ for $t = 0$; $(c_{18}, c_{22}) = (1, 3)$ for $t = 1$; $c_{22} = 4$ for $t = 2$. Estimating the LHS of (3.4) for the spectrum of Δ_2, we get

$$2142 + 64\lambda_2 \le 28\tau_0 + 6\tau_1 + 190 = 1726,$$

a contradiction. Hence $a_2 = 0$. One can prove $a_{10} = a_{14} = 0$ similarly using the spectra for a 10-plane and a 14-plane from Table 1. Note that we need to rule out a possible 14-plane before showing that $a_{10} = 0$. Now, we have $a_i = 0$ for all $i \notin \{18, 22\}$. Then, from the three equalities in Lemma 2.1, we get $a_{18} = 153$, $a_{22} = 231$, $\lambda_2 = 61905/64$, a contradiction. This completes the proof. □

Lemma 3.3. *There exists no* $[81, 5, 59]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative $[81, 5, 59]_4$ code, where $81 = g_4(5, 59) + 1$. We get $\gamma_1 \le 3$ and $\gamma_0 \le 2$ from (2.3), whence $\lambda_2 = 0$ or 1. By the first sieve, we have $a_i = 0$ for all $i \notin$ {0, 1, 2, 5, 9, 10, 13-18, 21, 22}. For any w-solid through a t-plane, (2.6) gives

$$\sum_j (22 - j)c_j = w + 7 - 4t \tag{3.5}$$

with $\sum_j c_j = 4$. From (2.5), we get

$$\sum_{i=0}^{18} \binom{22-i}{2} a_i = 2226 + 64\lambda_2. \tag{3.6}$$

Setting $w = t = 0$ in (3.5), the maximum possible contributions of c_j's to the LHS of (3.6) are $(c_{15}, c_{22}) = (1, 3)$. Estimating the LHS of (3.6) for the spectrum of a 0-solid, we get

$$2226 + 64\lambda_2 \le 21\tau_0 + 231 = 2016,$$

a contradiction. Hence $a_0 = 0$. We can show that $a_1 = 0$, similarly.

Next, suppose $a_{16} > 0$ and let Δ_{16} be a 16-solid. From Table 1, Δ_{16} has spectrum $(\tau_0, \tau_1, \tau_4, \tau_5) = (1, 16, 20, 48)$. We also suppose that $a_5 > 0$ and let Δ_5 be a 5-solid. Then, $a_5 = 1$ and $a_j = 0$ for other $j < 10$ from (3.5). Note that Δ_5 and Δ_{16} meets in a 0-plane or a 1-plane. Assume that $\Delta_5 \cap \Delta_{16}$ is a 1-plane. Setting $w = 16$ in (3.5), the maximum possible contributions of c_j's to the LHS of (3.6) are $(c_{10}, c_{13}, c_{21}) = (1, 1, 2)$ for $t = 0$; $(c_5, c_{21}, c_{22}) = (1, 2, 1)$ if $c_5 > 0$ for $t = 1$; $(c_{10}, c_{15}, c_{22}) = (1, 1, 2)$ if $c_5 = 0$ for $t = 1$; $(c_{15}, c_{22}) = (1, 3)$ for $t = 4$; $(c_{21}, c_{22}) = (3, 1)$ for $t = 5$. Estimating the LHS of (3.6) for the spectrum of Δ_{16}, we get

$$2226 + 64\lambda_2 \le 102\tau_0 + 136 + 87(\tau_1 - 1) + 21\tau_4 + 15 = 1978,$$

a contradiction. Assuming that $\Delta_5 \cap \Delta_{16}$ is a 0-plane, one can get a contradiction as well. Hence $a_{16} > 0$ implies $a_5 = 0$. Similarly, one can show that $a_{16} > 0$ implies $a_2 = 0$. Setting $w = 16$ in (3.5) again, the maximum possible contributions of c_j's to the LHS of (3.6) are $(c_9, c_{13}, c_{21}, c_{22}) = (1, 1, 1, 1)$ for $t = 0$; $(c_9, c_{16}, c_{22}) = (1, 1, 2)$ for $t = 1$; $(c_{15}, c_{22}) = (1, 3)$ for $t = 4$; $(c_{21}, c_{22}) = (3, 1)$ for $t = 5$. Estimating the LHS of (3.6) for the spectrum of Δ_{16}, we get

$$2226 + 64\lambda_2 \le 114\tau_0 + 93\tau_1 + 21\tau_4 + 15 = 2037,$$

a contradiction. Thus, $a_{16} = 0$.

Hence, $\mathcal{C}$ has no codeword whose weight is congruent to 1 mod 4. Then, $\mathcal{C}$ is extendable by Theorem 2.5, which contradicts Lemma 3.2. This completes the proof. □

It is known that there are exactly 20 inequivalent $[18, 4, 12]_4$ codes [1]. We need the following information from [1] about such codes.

Lemma 3.4. *Every* $[18, 4, 12]_4$ *code with* $a_0 = a_1 = 0$ *has spectrum* $(a_2, a_4, a_6) = (21, 24, 40)$.

Lemma 3.5. *There exists no* $[66, 5, 48]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative $[66, 5, 48]_4$ code, where $66 = g_4(5, 48) + 1$. Since $\gamma_1 \le 3$ and $\gamma_0 \le 2$ from (2.3), we have $\lambda_2 = 0$ or 1. We obtain $a_i = 0$ for all $i \notin \{0, 1, 2, 10, 14\text{-}18\}$ by the first sieve. For any w-solid through a t-plane, (2.6) gives

$$\sum_j (18 - j) c_j = w + 6 - 4t \tag{3.7}$$

with $\sum_j c_j = 4$. From (2.5), we get

$$\sum_{i=0}^{16} \binom{18-i}{2} a_i = 1848 + 64\lambda_2. \tag{3.8}$$

Suppose $a_0 > 0$ and let Δ_0 be a 0-solid. Then, the spectrum of Δ_0 is $\tau_0 = 85$, and we have $a_0 = 1$ and $a_j = 0$ for other $j < 14$ from (3.7). Setting $w = t = 0$ in (3.7), the maximum possible contributions of c_j's to the LHS of (3.8) are $(c_{14}, c_{16}, c_{18}) = (1, 1, 2)$ for $t = 0$. Estimating the LHS of (3.8) for the spectrum of Δ_0, we get

$$1848 + 64\lambda_2 \le 7\tau_0 + 153 = 748,$$

a contradiction. Hence $a_0 = 0$. One can prove $a_1 = a_2 = a_{10} = 0$ similarly. Now, we have $a_i = 0$ for all $i \notin \{14, 15, \ldots, 18\}$. Let Δ be a 18-solid. Setting $w = 18$, the equation (3.7) has no solution for $t = 0, 1$. Hence, Δ has spectrum $(\tau_2, \tau_4, \tau_6) = (21, 24, 40)$ by Lemma 3.4. Setting $w = 18$ in (3.7), the maximum possible contributions of c_j's to the LHS of (3.8) are $c_{14} = 4$ for $t = 2$; $(c_{14}, c_{18}) = (2, 2)$ for $t = 4$; $c_{18} = 4$ for $t = 6$. Estimating the LHS of (3.8) for the spectrum of Δ, we get

$$1848 + 64\lambda_2 \le 24\tau_2 + 12\tau_4 = 792,$$

a contradiction. This completes the proof. □

Lemma 3.6. *There exists no* $[65, 5, 47]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative $[65, 5, 47]_4$ code, where $65 = g_4(5, 47) + 1$. We get $\gamma_1 \le 3$ and $\gamma_0 \le 2$ from (2.3), whence $\lambda_2 = 0$ or 1. We have $a_i = 0$ for all $i \notin \{0, 1, 2, 5, 9, 10, 13\text{-}18\}$ by the first sieve. For any w-solid through a t-plane, (2.6) gives

$$\sum_j (22 - j)c_j = w + 7 - 4t \tag{3.9}$$

with $\sum_j c_j = 4$. From (2.5), we get

$$\sum_{i=0}^{16} \binom{18 - i}{2} a_i = 1928 + 64\lambda_2. \tag{3.10}$$

Setting $w = t = 0$ in (3.9), the maximum possible contributions of c_j's to the LHS of (3.10) are $(c_{13}, c_{16}, c_{18}) = (1, 1, 2)$. Estimating the LHS of (3.10) for the spectrum of a 0-solid, we get

$$1928 + 64\lambda_2 \le 11\tau_0 + 153 = 1088,$$

a contradiction. Hence $a_0 = 0$. We can show that $a_2 = 0$, similarly.

Next, suppose $a_{15} > 0$ and let Δ_{15} be a 15-solid. From Table 1, Δ_{15} has spectrum $(\tau_0, \tau_1, \tau_3, \tau_4, \tau_5) = (2, 15, 5, 30, 33)$. We also suppose that $a_1 > 0$ and let Δ_1 be a 1-solid. Then, $a_1 = 1$ and $a_j = 0$ for other $j < 10$ from (3.9). Note that Δ_1 and Δ_{15} meets in a 0-plane or a 1-plane. Assume that $\Delta_1 \cap \Delta_{15}$ is a 0-plane. Setting $w = 15$ in (3.9), the maximum possible contributions of c_j's to the LHS of (3.10) are $(c_1, c_{13}, c_{18}) = (1, 1, 2)$ if $c_1 > 0$ for $t = 0$; $(c_{10}, c_{13}, c_{17}) = (2, 1, 1)$ if $c_1 = 0$ for $t = 0$; $(c_{10}, c_{16}, c_{18}) = (2, 1, 1)$ for $t = 1$; $(c_{10}, c_{16}, c_{18}) = (1, 1, 2)$ for $t = 3$; $(c_{13}, c_{17}, c_{18}) = (1, 1, 2)$ for $t = 4$; $(c_{16}, c_{18}) = (1, 3)$ for $t = 5$. Estimating the LHS of (3.10) for the spectrum of Δ_{15}, we get

$$1928 + 64\lambda_2 \le 146 + 66(\tau_0 - 1) + 57\tau_1 + 29\tau_3 + 10\tau_4 + \tau_5 + 3 = 1548,$$

a contradiction. Assuming that $\Delta_1 \cap \Delta_{15}$ is a 1-plane, one can get a contradiction as well. Hence $a_{15} > 0$ implies $a_1 = 0$. Similarly, one can show that $a_{15} > 0$ implies $a_5 = 0$. Setting $w = 15$ in (3.9) again, the maximum possible contributions of c_j's to the LHS of (3.10) are $(c_9, c_{14}, c_{18}) = (2, 1, 1)$ for $t = 0$; $(c_9, c_{18}) = (2, 2)$ for $t = 1$; $(c_9, c_{17}, c_{18}) = (1, 1, 2)$ for

$t=3$; $(c_{13}, c_{17}, c_{18}) = (1, 1, 2)$ for $t=4$; $(c_{16}, c_{18}) = (1, 3)$ for $t=5$. Estimating the LHS of (3.10) for the spectrum of Δ_{15}, we get

$$1928 + 64\lambda_2 \le 78\tau_0 + 72\tau_1 + 36\tau_3 + 10\tau_4 + \tau_5 + 3 = 1752,$$

a contradiction. Thus, $a_{15} = 0$.

Hence, $\mathcal{C}$ has no codeword whose weight is congruent to 1 mod 4. Then, $\mathcal{C}$ is extendable by Theorem 2.5, which contradicts Lemma 3.5. This completes the proof. $\square$

The following result is known, see [4].

Lemma 3.7. *There exists no* $[45, 5, 32]_4$ *code.*

Lemma 3.8. *There exists no* $[44, 5, 31]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative $[44, 5, 31]_4$ code. It follows from Theorem 1.1 that $a_i = 0$ for all $i \notin \{0, 1, 2, 4, 5, 8, 9, 10, 12, 13\}$ by the first sieve. This implies that $\mathcal{C}$ has no codeword whose weight is congruent to 1 mod 4. Hence, $\mathcal{C}$ is extendable by Theorem 2.5, which contradicts Lemma 3.7. $\square$

Now, Theorem 1.2 follows from Lemmas 3.1-3.8.

4 Proof of Theorem 1.3

Theorem 4.1 ([12]). *There exists no* $[187, 5, 139]_4$ *code.*

Since $n_4(5, 138) = 186$, Theorem 1.3 follows from the following.

Lemma 4.2. *There exists no* $[186, 5, 138]_4$ *code.*

Proof Let $\mathcal{C}$ be a putative Griesmer $[186, 5, 138]_4$ code. We have $\gamma_0 = 1$, $\gamma_1 = 4$, $\gamma_2 = 13$, $\gamma_3 = 48$ from (2.2). Let Δ be a 48-solid. From Lemma 2.4, the spectrum of Δ is one of

(A) $(\tau_0, \tau_9, \tau_{12}, \tau_{13}) = (1, 16, 20, 48)$,
(B) $(\tau_1, \tau_8, \tau_9, \tau_{12}, \tau_{13}) = (1, 3, 13, 18, 50)$,
(C) $(\tau_4, \tau_5, \tau_8, \tau_9, \tau_{12}, \tau_{13}) = (1, 2, 1, 12, 19, 50)$,
(D) $(\tau_5, \tau_8, \tau_9, \tau_{12}, \tau_{13}) = (3, 3, 10, 18, 51)$.

Thus, a j-plane on Δ satisfies $j \in \{0, 1, 4, 5, 8, 9, 12, 13\}$. By the first sieve, we have $a_i = 0$ for all $i \notin \{$0, 1, 10, 14-17, 26-28, 30, 31, 42-44, 46-48$\}$. From (2.5), we get

$$\sum_{i=0}^{46} \binom{48-i}{2} a_i = 2883. \tag{4.1}$$

For any w-solid through a t-plane, (2.6) gives

$$\sum_j (48-j)c_j = w + 6 - 4t \tag{4.2}$$

with $\sum_j c_j = 4$.

Suppose $a_0 > 0$ and let Δ_0 be a 0-solid. Then, the spectrum of Δ_0 is $\tau_0 = 85$, and we have $a_0 = 1$ and $a_i = 0$ for $0 < i < 42$ from (4.2). Setting $w = 0$, the maximum possible contributions of c_j's to the LHS of (4.1) are $(c_{42}, c_{48}) = (1, 3)$ for $t = 0$. Estimating the LHS of (4.1) for the spectrum of Δ_0, we get

$$2883 \le 15\tau_0 + 1128 = 2403,$$

a contradiction. Hence $a_0 = 0$. One can prove $a_1 = a_{10} = a_{14} = 0$ similarly, see Table 1 for the possible spectra for a 10-solid and a 14-solid.

Suppose $a_{30} > 0$ and let Δ_{30} be a 30-solid. Then, the spectrum $(\tau_0, \tau_1, \dots, \tau_9)$ of Δ_{30} is one of the six spectra in Table 1 for $[30, 4, 21]_4$. Setting $w = 30$, the maximum possible contributions of c_j's to the LHS of (4.1) are $(c_{15}, c_{46}, c_{47}, c_{48}) = (1, 1, 1, 1)$ for $t = 0$; $(c_{16}, c_{48}) = (1, 3)$ for $t = 1$; $(c_{26}, c_{44}, c_{47}) = (1, 1, 2)$ for $t = 2$; $(c_{27}, c_{47}) = (1, 3)$ for $t = 3$; $(c_{28}, c_{48}) = (1, 3)$ for $t = 4$; $(c_{42}, c_{44}, c_{48}) = (2, 1, 1)$ for $t = 5$; $(c_{42}, c_{44}, c_{47}) = (1, 1, 2)$ for $t = 6$; $(c_{43}, c_{47}) = (1, 3)$ for $t = 7$; $(c_{44}, c_{48}) = (1, 3)$ for $t = 8$ and $c_{48} = 4$ for $t = 9$ since we have $c_{48} = 0$ when $t = 2, 3, 6, 7$. Estimating the LHS of (4.1) for each of the possible spectra of Δ_{30}, we get

$$2883 \le (528 + 1)\tau_0 + 496\tau_1 + (231 + 6)\tau_2 + 210\tau_3 + 190\tau_4 \\ +(30 + 6)\tau_5 + (15 + 6)\tau_6 + 10\tau_7 + 6\tau_8 + 153 \le 2838,$$

a contradiction. Hence $a_{30} = 0$.

Next, we suppose $a_{17} > 0$ and let Δ_{17} be a 17-solid. Then, the spectrum of Δ_{17} is $(\tau_1, \tau_5) = (17, 68)$ from Table 1. Setting $w = 17$, the maximum possible contributions of c_j's to the LHS of (4.1) are $(c_{31}, c_{46}, c_{48}) = (1, 1, 2)$ for $t = 1$; $(c_{46}, c_{47}, c_{48}) = (1, 1, 2)$ for $t = 5$. Estimating the LHS of (4.1) for the spectrum of Δ_{17}, we get

$$2883 \le (136 + 1)\tau_1 + \tau_5 + 465 = 2862,$$

a contradiction again. Hence $a_{17} = 0$.

Thus, $a_i > 0$ implies $i \in \{15, 16, 26, 27, 28, 31, 42, 43, 44, 46, 47, 48\}$, which implies that $A_i = 0$ for all $i \equiv 1 \pmod 4$. Since $\mathcal{C}$ is not extendable by Theorem 4.1, we have $A_i = 0$ for all $i \equiv 3 \pmod 4$ by Theorem 2.6, whence $a_i = 0$ for all odd i. Therefore, $a_i = 0$ for all $i \notin \{16, 26, 28, 42, 44, 46, 48\}$.

Suppose $a_{16} > 0$ and let Δ_{16} be a 16-solid. Then, the spectrum of Δ_{16} is $(\tau_0, \tau_1, \tau_4, \tau_5) = (1, 16, 20, 48)$ from Table 1. Setting $w = 16$, the maximum possible contributions of c_j's to the LHS of (4.1) are $(c_{26}, c_{48}) = (1, 3)$ for $t = 0$; $(c_{42}, c_{48}) = (3, 1)$ for $t = 1$; $(c_{42}, c_{48}) = (1, 3)$ for $t = 4$; $(c_{46}, c_{48}) = (1, 3)$ for $t = 5$. Estimating the LHS of (4.1) for the spectrum of Δ_{16}, we get

$$2883 \le 231\tau_0 + 45\tau_1 + 15\tau_4 + \tau_5 + 496 = 1795,$$

a contradiction. Hence $a_{16} = 0$.

Finally, setting $w = 48$, the maximum possible contributions of c_j's to the LHS of (4.1) are $(c_{26}, c_{42}, c_{44}) = (2, 1, 1)$ for $t = 0$; $(c_{26}, c_{42}, c_{48}) = (2, 1, 1)$ for $t = 1$; $(c_{28}, c_{42}) = (1, 3)$ for $t = 4$; $(c_{26}, c_{42}, c_{48}) = (1, 2, 1)$ for $t = 5$; $(c_{26}, c_{48}) = (1, 3)$ for $t = 8$; $(c_{42}, c_{48}) = (3, 1)$ for $t = 9$; $(c_{42}, c_{48}) = (1, 3)$ for $t = 12$; $(c_{46}, c_{48}) = (1, 3)$ for $t = 13$. Estimating the LHS of (4.1) for the possible spectra (A)-(D) for Δ_{48}, we get

$$2883 \le (462 + 15 + 6)\tau_0 + (462 + 15)\tau_1 + (190 + 45)\tau_4 + (231 + 30)\tau_5 \\ +231\tau_8 + 45\tau_9 + 15\tau_{12} + \tau_{13} \le 2247,$$

giving a contradiction. This completes the proof. □

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Table 2. Values and bounds for $n = n_4(5,d)$ with $g = g_4(5,d)$

d	g	n	d	g	n	d	g	n
1	5	5	41	57	58-59	81	111	111-112
2	6	6	42	58	59-60	82	112	112-113
3	7	8	43	59	60-61	83	113	113-114
4	8	9	44	60	61-62	84	114	114-115
5	10	10	45	62	63-64	85	116	117
6	11	11	46	63	64-65	86	117	118
7	12	13	47	64	**66**	87	118	119
8	13	14	48	65	**67**	88	119	120
9	15	16	49	68	69	89	121	122-123
10	16	17	50	69	70	90	122	123-124
11	17	19	51	70	71	91	123	124-125
12	18	20	52	71	72	92	124	125-126
13	20	21	53	73	74	93	126	127-128
14	21	22	54	74	75	94	127	128-129
15	22	23	55	75	76	95	128	129-130
16	23	24	56	76	77	96	129	130-131
17	26	27	57	78	79-80	97	132	133-134
18	27	28	58	79	80-81	98	133	134-135
19	28	29	59	80	**82**	99	134	135-136
20	29	30	60	81	**83**	100	135	136-137
21	31	32	61	83	**85**	101	137	138
22	32	33	62	84	**86**	102	138	139-140
23	33	34	63	85	87	103	139	140-141
24	34	35	64	86	88	104	140	141-142
25	36	37	65	90	90-91	105	142	143-144
26	37	38	66	91	91-92	106	143	144-145
27	38	39	67	92	92-93	107	144	145-146
28	39	40	68	93	93-94	108	145	146-147
29	41	42	69	95	95-96	109	147	148-149
30	42	43	70	96	96-97	110	148	149-150
31	43	**45**	71	97	98	111	149	150-151
32	44	46	72	98	99	112	150	151-152
33	47	48	73	100	101	113	153	154
34	48	49	74	101	102	114	154	155
35	49	50	75	102	103-104	115	155	156-157
36	50	51	76	103	104-105	116	156	157-158
37	52	53	77	105	106	117	158	159
38	53	54	78	106	107	118	159	160
39	54	55	79	107	108	119	160	161
40	55	56	80	108	109	120	161	162

Table 2 (continued)

d	g	n	d	g	n	d	g	n
121	163	164	161	217	218	201	270	270
122	164	165	162	218	219	202	271	271
123	165	166-167	163	219	220	203	272	272
124	166	167-168	164	220	221	204	273	273
125	168	169	165	222	223	205	275	276
126	169	170	166	223	224	206	276	277
127	170	171	167	224	225	207	277	278
128	171	172	168	225	226	208	278	279
129	175	175-176	169	227	228	209	281	281
130	176	176-177	170	228	229	210	282	282
131	177	177-178	171	229	230	211	283	283
132	178	178-179	172	230	231	212	284	284
133	180	180-181	173	232	233	213	286	286
134	181	181-182	174	233	234	214	287	287
135	182	182-183	175	234	235	215	288	289
136	183	183-184	176	235	236	216	289	290
137	185	185-186	177	238	239	217	291	292
138	186	**187**	178	239	240	218	292	293
139	187	188	179	240	241	219	293	294
140	188	189	180	241	242	220	294	295
141	190	191	181	243	244	221	296	297
142	191	192	182	244	245	222	297	298
143	192	193	183	245	246	223	298	299
144	193	194	184	246	247	224	299	300
145	196	197	185	248	249	225	302	302
146	197	198	186	249	250	226	303	303
147	198	199	187	250	251	227	304	304
148	199	200	188	251	252	228	305	305
149	201	202	189	253	253	229	307	307
150	202	203	190	254	254	230	308	308
151	203	204	191	255	255	231	309	309
152	204	205	192	256	256	232	310	310
153	206	207	193	260	260	233	312	312
154	207	208	194	261	261	234	313	313
155	208	209	195	262	262	235	314	314
156	209	210	196	263	263	236	315	315
157	211	212	197	265	265	237	317	317
158	212	213	198	266	266	238	318	318
159	213	214	199	267	267	239	319	319
160	214	215	200	268	268	240	320	320

Table 2 (continued)

d	g	n	d	g	n	d	g	n
241	323	323	289	388	388-389	329	441	441-442
242	324	324	290	389	389-390	330	442	442-443
243	325	325	291	390	390-391	331	443	443-444
244	326	326	292	391	391-392	332	444	444-445
245	328	328	293	393	393-394	333	446	446-447
246	329	329	294	394	394-395	334	447	447-448
247	330	330	295	395	395-396	335	448	449
248	331	331	296	396	396-397	336	449	450
257	346	346	297	398	399	337	452	452-453
258	347	347	298	399	400	338	453	453-454
259	348	348	299	400	401	339	454	454-455
260	349	349	300	401	402	340	455	455-456
261	351	351	301	403	404	341	457	457-458
262	352	352	302	404	405	342	458	458-459
263	353	353	303	405	406	343	459	459-460
264	354	354	304	406	407	344	460	460-461
265	356	356	305	409	410	345	462	462-463
266	357	357	306	410	411	346	463	463-464
267	358	358	307	411	412	347	464	465
268	359	359	308	412	413	348	465	466
269	361	361	309	414	415	349	467	468
270	362	362	310	415	416	350	468	469
271	363	363	311	416	417	351	469	470
272	364	364	312	417	418	352	470	471
273	367	367-368	313	419	420	353	473	473-474
274	368	368-369	314	420	421	354	474	474-475
275	369	369-370	315	421	422	355	475	476
276	370	370-371	316	422	423	356	476	477
277	372	372-373	317	424	425	357	478	479
278	373	373-374	318	425	426	358	479	480
279	374	374-375	319	426	427	359	480	481
280	375	375-376	320	427	428	360	481	482
281	377	377-378	321	431	431-432	361	483	484
282	378	378-379	322	432	432-433	362	484	485
283	379	379-380	323	433	433-434	363	485	486
284	380	380-381	324	434	434-435	364	486	487
285	382	382-383	325	436	436-437	365	488	489
286	383	383-384	326	437	437-438	366	489	490
287	384	385	327	438	438-439	367	490	491
288	385	386	328	439	439-440	368	491	492