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Design of time-delayed connection parameters for inducing amplitude death in high-dimensional oscillator networks

Yoshiki Sugitani, Keiji Konishi,[a] Luan Ba Le, and Naoyuki Hara
Department of Electrical and Information Systems, Osaka Prefecture University, 1-1 Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531, Japan

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The present paper studies time-delayed-connection induced amplitude death in high-dimensional oscillator networks. We provide two procedures for design of a coupling strength and a transmission delay: these procedures do not depend on the topology of oscillator networks (i.e., network structure and number of oscillators). A graphical procedure based on the Nyquist criterion is proposed and then is numerically confirmed for the case of five-dimensional oscillators, called generalized Rössler oscillators, which have two pairs of complex conjugate unstable roots. In addition, for the case of high-dimensional oscillators having two unstable roots, the procedure can be systematically carried out using only a simple algebraic calculation. This systematic procedure is numerically confirmed for the case of three-dimensional oscillators, called Moore-Spiegel oscillators, which have two positive real unstable roots. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4896318]

Time-delay-induced death has been actively investigated both experimentally and analytically, since it has great potential as a candidate phenomenon for stabilizing several unstable systems. Most of the previous studies on time-delay-induced death have shown their analytical results for the following cases: a two-dimensional normal form of supercritical Hopf bifurcation, which has a pair of complex conjugate unstable roots, is employed as an oscillator; the topology of the oscillator network is given in advance. However, in engineering systems, oscillators cannot always be described by such two-dimensional oscillators and the topology is generally unknown. The present paper applies a parametric approach well known in the field of robust control to time-delayed-induced amplitude death in high-dimensional oscillator networks. We provide two procedures for designing connection parameters. A graphical procedure based on the Nyquist criterion is proposed and then is numerically confirmed for the case of five-dimensional oscillators which have two pairs of complex conjugate unstable roots. In addition, for the case of high-dimensional oscillators having two unstable roots, the procedure can be systematically carried out using only a simple algebraic calculation. This systematic procedure is numerically confirmed for the case of three-dimensional oscillators which have two positive real unstable roots.

I. INTRODUCTION

The dynamical behavior of coupled oscillators has received considerable attention in nonlinear science.[1,2] In recent years, such dynamical behavior has been employed in the field of engineering. Examples include consensus and cooperation in multi-agent system networks[3] and dynamics of networks.[4–7] One of the most remarkable behaviors in coupled oscillators is synchronization: the oscillators follow the same pattern without a leader.[1,2,8] Among the several types of synchronization, chaotic synchronization is expected to be useful for secure communications.[9] On the other hand, it is well known that each oscillator can cease to oscillate due to its strong interaction. This phenomenon, called amplitude death, can be considered as a synchronization to an unstable steady state.[10–12]

Amplitude death in coupled oscillators has been studied intensively for almost a quarter-century.[13,14] This phenomenon can be observed in coupled non-identical oscillators, but, never in coupled identical oscillators.[14] It was reported that a transmission delay in connections, which naturally exists in real coupled oscillators, has the ability to induce this phenomenon even in coupled identical oscillators.[15] On the basis of this report, time-delay-induced death has been investigated both experimentally and analytically (see Refs. 11 and 12 and references therein). From an engineering point of view, amplitude death has great potential as a candidate phenomenon for stabilizing several unstable systems, since there is no need to employ controllers.[16] Therefore, amplitude death would be a useful phenomenon for stabilizing unstable oscillator networks without a leader (e.g., a centralized controller) or without decentralized controllers in various fields.

Most of the previous studies on time-delay-induced death[17–19] have shown their analytical results for the following cases: a two-dimensional normal form of supercritical Hopf bifurcation, called the Stuart-Landau oscillator, is employed as a oscillator for simplicity of analysis; the topology of the oscillator network is given in advance. Remark that the Stuart-Landau oscillator has a pair of complex conjugate unstable roots; thus, it is obvious that the analytical results for these cases cannot be generally used for engineering systems. The major reasons are as follows: oscillators in engineering systems cannot always be described by two-dimensional oscillators with unstable two roots in complex conjugate pairs; the topology of an oscillator network is generally unknown in engineering systems, such as sensor

[a]http://www.eis.osakafu-u.ac.jp/~ecs

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networks[20] and mobile agents;[21] and a system designer who wants to induce death needs not only a stability analysis but also a procedure for designing connection parameters.

In order to overcome the above major problems, the present paper applies a parametric approach well known in the field of robust control[22] to time-delayed-induced amplitude death in high-dimensional oscillator networks. We provide two procedures for designing connection parameters (i.e., coupling strength and transmission delay); these procedures are useful even when the topology of the oscillator network (i.e., network structure and number of oscillators) is unknown (see Fig. 1). We propose a graphical procedure based on the Nyquist criterion and apply it to five-dimensional oscillators, called generalized Rössler oscillators,[23,24] which have two pairs of complex conjugate unstable roots. Further, in the case that the high-dimensional oscillators have two unstable roots, we show that the procedure can be systematically carried out only using a simple algebraic calculation. This systematic procedure is applied to three-dimensional oscillators, called Moore-Spiegel oscillators,[25,26] which have two positive real unstable roots.

This paper employs the following notation. The principal argument of a complex number z is defined by $\mathrm{Arg}[z] \in [0, 2\pi)$. The largest integer $n \in \mathbb{Z}$ that is not greater than a real number x is defined by $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$. The symbol $:=$ denotes a definition.

II. OSCILLATOR NETWORKS

Consider N oscillators defined by

$$\dot{\boldsymbol{x}}_i(t) = \mathbf{F}[\boldsymbol{x}_i(t)] + \boldsymbol{b}u_i(t) \;\; (i = 1, 2, \ldots, N). \tag{1}$$

$\boldsymbol{x}_i(t) \in \mathbb{R}^m$ denotes the state of oscillator i, $u_i(t) \in \mathbb{R}$ is the coupling signal, and $\boldsymbol{b} \in \mathbb{R}^m$ is the coupling vector. The nonlinear function $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ is twice continuously differentiable, which has at least one unstable fixed point $\boldsymbol{x}^* : \mathbf{0} = \mathbf{F}(\boldsymbol{x}^*)$. Oscillator i is coupled by

$$u_i(t) = \boldsymbol{k}^T \left\{ \frac{1}{\eta_i} \left[\sum_{l=1}^{N} c_{il} \boldsymbol{x}_l(t-\tau) \right] - \boldsymbol{x}_i(t) \right\}, \tag{2}$$

where $\tau \geq 0$ and $\boldsymbol{k} \in \mathbb{R}^m$ are, respectively, the delay time and the coupling strength. The network topology depends on c_{il} as follows: $c_{il} = c_{li} = 1$ ($c_{il} = c_{li} = 0$) denotes that oscillator i is (not) connected to oscillator l. Here, $\eta_i := \sum_{l=1}^{N} c_{il}$ is the number of oscillators linked to oscillator i, where we assume $\eta_i > 0$, $\forall i$. Remark that η_i in connection (2) is introduced to make it diffusive.[64]

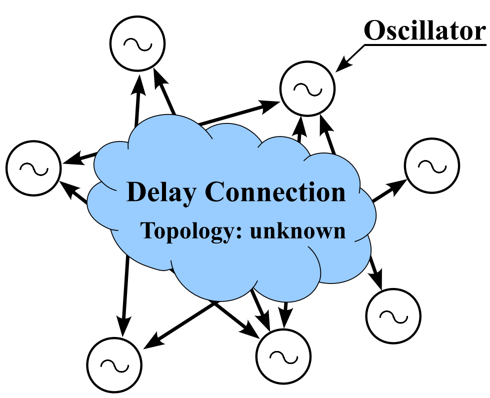

FIG. 1. Conceptual diagram of an oscillator network with unknown topology.

There exits at least one steady state

$$[\, \boldsymbol{x}_1^T \;\; \boldsymbol{x}_2^T \;\; \cdots \;\; \boldsymbol{x}_N^T \,]^T = [\, \boldsymbol{x}^{*T} \;\; \boldsymbol{x}^{*T} \;\; \cdots \;\; \boldsymbol{x}^{*T} \,]^T, \tag{3}$$

in oscillators (1) with delayed connection (2). It should be noted that the location of steady state (3) is fixed independent of $\boldsymbol{k}$, c_{il}, N, and τ owing to diffusive connection (2), but its stability depends on them.

The present paper tackles the following stabilization: an unstable steady state (3) in oscillators without connection (i.e., $\boldsymbol{k} \equiv \mathbf{0}$) is stabilized by connection (2) with suitable parameters $\boldsymbol{k}$ and τ. The main goal of this paper is to derive procedures for design of the connection parameters which can be used for any number of oscillators N and for any topology c_{il}.

III. LINEAR STABILITY ANALYSIS

Linearizing oscillators (1) coupled by connection (2) around steady state (3) yields

$$\begin{cases} \Delta\dot{\boldsymbol{x}}_i(t) = \mathbf{A}\Delta\boldsymbol{x}_i(t) + \boldsymbol{b}\Delta u_i(t), \\ \Delta u_i(t) = \boldsymbol{k}^T \left\{ \dfrac{1}{\eta_i} \left[\displaystyle\sum_{l=1}^{N} c_{il} \Delta\boldsymbol{x}_l(t-\tau) \right] - \Delta\boldsymbol{x}_i(t) \right\}, \end{cases} \tag{4}$$

where $\Delta\boldsymbol{x}_i(t) := \boldsymbol{x}_i(t) - \boldsymbol{x}^*$ is a small deviation from the fixed point $\boldsymbol{x}^*$ and the Jacobi matrix at $\boldsymbol{x}^*$ is described by $\mathbf{A} := \partial\mathbf{F}(\boldsymbol{x})/\partial\boldsymbol{x}|_{\boldsymbol{x}=\boldsymbol{x}^*}$. This paper assumes that $\mathbf{A}$ has at least one unstable eigenvalue. Further, the pair $(\mathbf{A}, \boldsymbol{b})$ is assumed to be controllable;[27,65] thus, a linear transformation $\Delta\boldsymbol{x}_i(t) = \boldsymbol{\Gamma}\boldsymbol{z}_i(t)$ allows us to obtain the controllability canonical form. Then, the dynamics of oscillator i at $\boldsymbol{x}^*$, that is system (4), are governed by

$$\dot{\boldsymbol{z}}_i(t) = \bar{\mathbf{A}}\boldsymbol{z}_i(t) + \bar{\boldsymbol{b}}\bar{\boldsymbol{k}}^T \left\{ \frac{1}{\eta_i} \left[\sum_{l=1}^{N} c_{il} \boldsymbol{z}_l(t-\tau) \right] - \boldsymbol{z}_i(t) \right\}, \tag{5}$$

where

$$\bar{\mathbf{A}} := \boldsymbol{\Gamma}^{-1}\mathbf{A}\boldsymbol{\Gamma} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_m & -a_{m-1} & \cdots & -a_2 & -a_1 \end{bmatrix},$$

$$\bar{\boldsymbol{b}} := \boldsymbol{\Gamma}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \bar{\boldsymbol{k}}^T := \boldsymbol{k}^T\boldsymbol{\Gamma} = \begin{bmatrix} \bar{k}_m \\ \bar{k}_{m-1} \\ \vdots \\ \bar{k}_2 \\ \bar{k}_1 \end{bmatrix}^T.$$

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The elements a_i in $\bar{\mathbf{A}}$ are the coefficients of the characteristic polynomial of $\mathbf{A}$

$$d(s) := \det[s\mathbf{I}_m - \mathbf{A}] = \det[s\mathbf{I}_m - \bar{\mathbf{A}}] \\ = s^m + a_1 s^{m-1} + \cdots + a_{m-1}s + a_m. \quad (6)$$

From Eq. (5), we notice that

$$\dot{\mathbf{Z}}(t) = (\mathbf{I}_N \otimes \bar{\mathbf{A}}^*)\mathbf{Z}(t) + (\mathbf{C} \otimes \bar{\boldsymbol{b}}\bar{\boldsymbol{k}}^T)\mathbf{Z}(t-\tau) \quad (7)$$

describes the dynamics of an oscillator network around a steady state, where $\mathbf{Z}(t) := [z_1(t)^T \cdots z_N(t)^T]^T$ and $\bar{\mathbf{A}}^* := \bar{\mathbf{A}} - \bar{\boldsymbol{b}}\bar{\boldsymbol{k}}^T$. The elements of $\mathbf{C}$ are as follows: $\{\mathbf{C}\}_{il} = c_{il}/\eta_i$ if $i \neq l$ and $\{\mathbf{C}\}_{il} = 0$ if $i = l$.

Now consider the stability of system (7). The characteristic equation associated with this system is written as

$$\det[s\mathbf{I}_{mN} - \mathbf{I}_N \otimes \bar{\mathbf{A}}^* - (\mathbf{C} \otimes \bar{\boldsymbol{b}}\bar{\boldsymbol{k}}^T)e^{-s\tau}] = 0. \quad (8)$$

It should be noted that $\mathbf{L} := \mathbf{I}_N - \mathbf{C}$ is similar to the real symmetric matrix $\bar{\mathbf{L}} := \mathbf{I}_N - \mathbf{D}^{-1/2}\mathbf{E}\mathbf{D}^{-1/2}$, where $\mathbf{D} := \mathrm{diag}\{\eta_1,\ldots,\eta_n\}$ and $\mathbf{E} := \mathbf{DC}$. As a result, $\mathbf{L}$ can be diagonalized with a diagonal transformation matrix $\mathbf{Q}$ as

$$\mathbf{Q}^{-1}\mathbf{L}\mathbf{Q} = \mathrm{diag}(\rho_1, \ldots, \rho_N),$$

where we have

$$0 = \rho_1 \le \rho_2 \le \cdots \le \rho_N \le 2, \quad (9)$$

for any network topology.[28–32] The relation between eigenvalues ρ_i $(i = 1,\ldots,N)$ of $\mathbf{L}$ and the network topology was revealed in Ref. 29. This diagonalization simplifies characteristic equation (8) as follows:

$$g(s) := \prod_{i=1}^{N} h(s, \rho_i), \quad (10)$$

where

$$h(s,\rho) := \det[s\mathbf{I}_m - \bar{\mathbf{A}}^* - e^{-s\tau}(1-\rho)\bar{\boldsymbol{b}}\bar{\boldsymbol{k}}^T]. \quad (11)$$

Let us recall our main purpose, which is to propose procedures for designing the connection parameters $\boldsymbol{k}^T$ and τ for stabilizing steady state (3) on the basis of uncertain information: the number of oscillators and the network topology. Thus, we have to deal with the stability of $h(s, \rho)$ for any $\rho \in [0, 2]$.

From Eq. (6) and

$$n(s) := \bar{\boldsymbol{k}}^T \mathrm{adj}(s\mathbf{I}_m - \bar{\mathbf{A}})\bar{\boldsymbol{b}} \\ = \bar{k}_1 s^{m-1} + \bar{k}_2 s^{m-2} + \cdots + \bar{k}_{m-1}s + \bar{k}_m, \quad (12)$$

the characteristic function $h(s, \rho)$ can be simplified as

$$h(s,\rho) = \det\begin{bmatrix} s\mathbf{I}_m - \bar{\mathbf{A}}^* & e^{-s\tau}(1-\rho)\bar{\boldsymbol{b}} \\ \bar{\boldsymbol{k}}^T & 1 \end{bmatrix} \\ = d(s) + n(s)\{1 - e^{-s\tau}(1-\rho)\}. \quad (13)$$

As a result, we see that our design problem can be reduced to a procedure for designing the gain $\bar{k}_i$ $(i = 1,\ldots, m)$, and the delay time τ such that $h(s, \rho)$ is stable for any $\rho \in [0, 2]$. Sections IV and V shall provide solutions to this problem.

IV. HIGH-DIMENSIONAL OSCILLATORS

A. Inherent limitations

To begin with, two inherent limitations of connection (2) are described below.

Lemma 1. Consider oscillators (1) coupled by connection (2). If connection (2) has no delay time ($\tau \equiv 0$), then steady state (3) cannot be stabilized for any $\boldsymbol{k}^T$ and $\mathbf{C}$.

Proof. See Subsection 1 of the Appendix. □

Lemma 2. Consider oscillators (1) coupled by connection (2). If $\mathbf{A}$ *has either a zero eigenvalue or an odd number of positive real eigenvalues, connection (2) never stabilizes steady state (3) for any τ, $\boldsymbol{k}^T$, and* $\mathbf{C}$.

Proof. See Subsection 2 of the Appendix. □

Lemma 2 guarantees that the well-known *odd-number property* for a single oscillator with delayed feedback control[33–35] or in two oscillators (i.e., $N = 2$)[36] remains in oscillator networks.

B. Design of connection

Our design problem is to determine the connection parameters such that we have stable $\Psi := \{h(s, \rho) : \rho \in [0, 2]\}$. Ψ can be described by

$$\Psi = \{(1-\mu)h(s,0) + \mu h(s,2) : \mu \in [0,1]\}, \quad (14)$$

because all of the coefficients of Eq. (13) are given by affine functions of ρ. Here, μ is the uncertain parameter corresponding to $\rho/2$. The parametric approach in robust control theory[32,37–39] gives us a useful stability condition.

Lemma 3. The family of quasi-polynomials Ψ, Eq. (14), is stable if all of

(a) *$h(s, 0)$* is stable;
(b) *$h(s, 2)$* is stable;
(c) for any $\omega \in [0, +\infty)$, $\phi(\omega) := \arg[h(j\omega, 0)] - \arg[h(j\omega, 2)] \neq \pm\pi$

are satisfied.

It is obviously difficult to check the conditions in Lemma 3. This is because they deal with the stability of quasi-polynomials (i.e., the transcendental equations). This difficulty motivates us to provide procedures for designing $\boldsymbol{k}^T$ and τ without using a direct way to numerically analyze the transcendental equations. From Lemma 3, we can provide a graphical procedure for designing $\boldsymbol{k}^T$ and τ.

Theorem 1. *For any* $\mathbf{C}$, *the steady state (3) is stable if $\boldsymbol{k}^T$ and τ are designed such that (i) $M(s) := n(s)/[d(s) + n(s)]$ is stable and (ii) the Nyquist plot $M(j\omega)e^{-j\omega\tau}$ never intersects the lines $\boldsymbol{l}^{(0)} \cup \boldsymbol{l}^{(2)}$ on the real axis, where $\boldsymbol{l}^{(0)} := \{r + jv : r \le -1, v = 0\}$ and $\boldsymbol{l}^{(2)} := \{r + jv : r \ge +1, v = 0\}$.*

Proof. See Subsection 3 of the Appendix. □

Theorem 1 provides us the following procedure for designing the connection parameters:

Step 0: Assume that $\mathbf{A}$ and $\boldsymbol{b}$ are known, but N and $\mathbf{C}$ are unknown.

Step 1: If $(\mathbf{A}, \boldsymbol{b})$ is controllable and $\mathbf{A}$ does not satisfy Lemma 2, then go to the next step, else quit.

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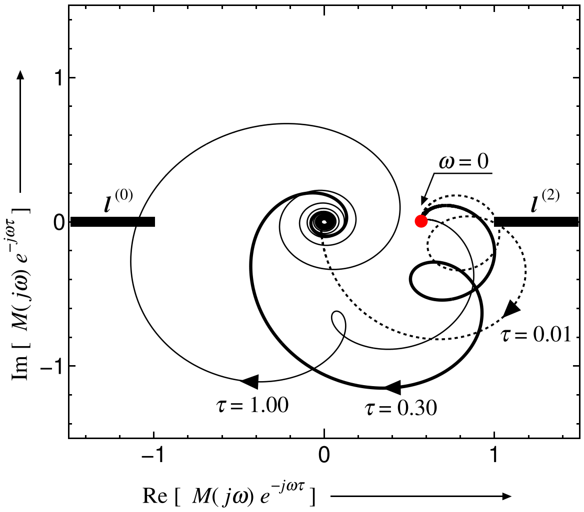

FIG. 2. Nyquist plots $M(j\omega)e^{-j\omega\tau}$ with $\tau = 0.01$ (dotted line), 0.30 (bold line), and 1.00 (thin line) for $\omega \in [0, +\infty)$.

Step 2: Obtain elements a_i $(i = 1,\ldots,m)$ from Eq. (6).

Step 3: Search for $\bar{\boldsymbol{k}}^T$ and τ which satisfy conditions (i) and (ii) in Theorem 1.

Step 4: Transform $\bar{\boldsymbol{k}}^T$ to $\boldsymbol{k}^T = \bar{\boldsymbol{k}}^T \boldsymbol{\Gamma}^{-1}$ after choosing $\boldsymbol{\Gamma}$ such that $\bar{\mathbf{A}} = \boldsymbol{\Gamma}^{-1}\mathbf{A}\boldsymbol{\Gamma}$ and $\bar{\boldsymbol{b}} = \boldsymbol{\Gamma}^{-1}\boldsymbol{b}$ take a controllable canonical form.

Remark that $\bar{\boldsymbol{k}}^T$ and τ estimated in step 3 are not unique: we may have several pairs of $\bar{\boldsymbol{k}}^T$ and τ numerically. This procedure is used in the following numerical examples.

C. Numerical examples

Consider a generalized Rössler oscillator (1) with

$$\mathbf{F}(\boldsymbol{x}_i) = \begin{bmatrix} a x_i^{(1)} - x_i^{(2)} \\ x_i^{(1)} - x_i^{(3)} \\ x_i^{(2)} - x_i^{(4)} \\ x_i^{(3)} - x_i^{(5)} \\ \varepsilon + b x_i^{(5)}(x_i^{(4)} - d) \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{15}$$

where $a = 0.04$, $b = 4.00$, $d = 2.00$, and $\varepsilon = 0.10$ are the parameters.[23,24] The oscillator has two fixed points $\boldsymbol{x}_-^* = [0.0125 \;\; 0.0005 \;\; 0.0125 \;\; 0.0005 \;\; 0.0125]^T$ and $\boldsymbol{x}_+^* = [49.9875 \;\; 1.9995 \;\; 49.9875 \;\; 1.9995 \;\; 49.9875]^T$: one which lies close to the origin and another which does not. We follow the procedure for designing the connection parameters proposed in subsection IV B. For step 0, the Jacobi matrix at each fixed point is calculated and $\boldsymbol{b}$ is given in Eq. (15). For step 1, we confirm that $\mathbf{A}$ at $\boldsymbol{x}_-^*$ does not satisfy the condition in Lemma 2, since the eigenvalues of $\mathbf{A}$ are -7.9918, $0.0047 \pm j1.6181$, and $0.0122 \pm j0.6181$. Further, we see that $(\mathbf{A}, \boldsymbol{b})$ is controllable. On the other hand, we do not have to consider the other fixed point, because its $\mathbf{A}$ has one real positive eigenvalue. For step 2, we obtain a_i $(i = 1,\ldots,5)$ from Eq. (6). For step 3, $\bar{\boldsymbol{k}}^T$ is set such that $M(s)$ has stable poles: -8.0, $-0.5 \pm j1.7$, and $-0.5 \pm j0.7$. These stable poles can be set arbitrarily.[66] The Nyquist plots $M(j\omega)e^{-j\omega\tau}$ with $\tau = 0.01$, 0.30, and 1.00 are drawn for $\omega \in [0, +\infty)$ as shown in Fig. 2. We see that $M(j\omega)e^{-j\omega\tau}$ with $\tau = 0.30$ does not cross the lines $\boldsymbol{l}^{(0)} \cup \boldsymbol{l}^{(2)}$ on the real axis. For step 4, the coupling strength is estimated: $\boldsymbol{k}^T = [0.5582 \;\; -0.0414 \;\; -1.9623 \;\; 2.0420 \;\; -1.2529]$. It must be emphasized that the delay time $\tau = 0.30$ and the coupling strength $\boldsymbol{k}^T$ designed by the above procedure are valid for any number of oscillators and for any topology.

Let us numerically confirm that the connection parameters designed above can be used for various networks as sketched in Fig. 3: the simplest network with $N = 2$, the ring-type network with $N = 10$, and the complete network with $N = 8$. The time series data of the three networks before and after coupling are shown in Fig. 4. The individual oscillators without connection behave oscillatory for $t < 100$. The connection with the designed parameters is applied to these oscillators at $t = 100$. It can be seen that each oscillator converges on its fixed point for $t > 100$. This result supports that the connection parameters designed by our procedure are valid independently of the number of oscillators and the network topology.

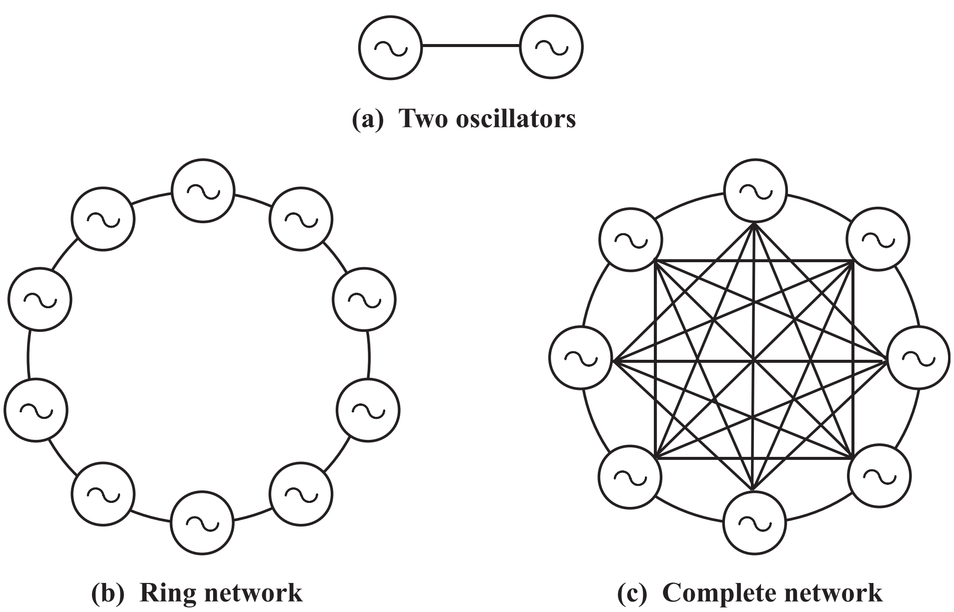

FIG. 3. Three networks for our numerical examples: (a) two oscillators, (b) ring-type network with $N = 10$, and (c) complete network with $N = 8$.

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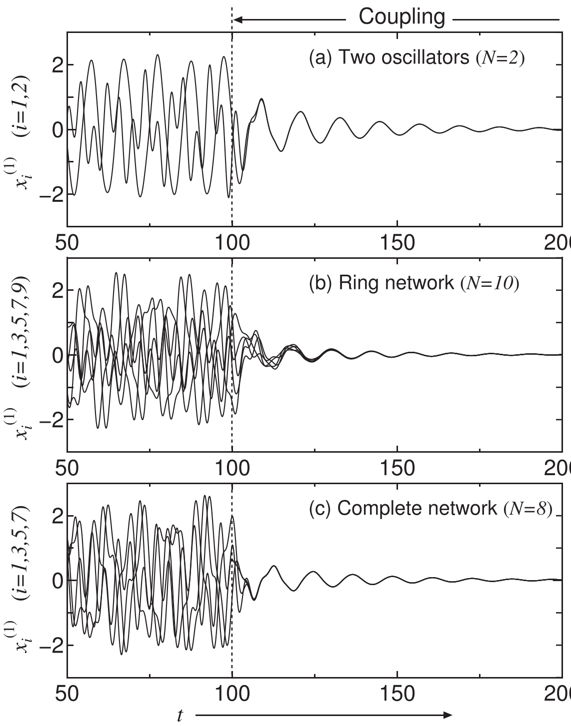

FIG. 4. Time series data of the three networks with generalized Rössler oscillators (15): (a) two oscillators, (b) ring type network with $N = 10$, and (c) complete network with $N = 8$. Individual oscillators without connection behave oscillatory for $t < 100$; they are coupled at $t = 100$.

Remark that step 3 is a troublesome task: although it is easy to design $\bar{\boldsymbol{k}}^T$ for condition (i) (i.e., for $M(s)$ to be stable), the design of $\bar{\boldsymbol{k}}^T$ and τ for condition (ii) requires a trial-and-error search in the $(m+1)$-dimensional parameter space. It is difficult to simplify the search in general; however, section V shall show that we can simplify it for oscillators (1) whose $\mathbf{A}$ has two non-zero unstable eigenvalues.

V. HIGH-DIMENSIONAL OSCILLATORS WITH TWO UNSTABLE ROOTS

A. Two-dimensional oscillators

This subsection deals with networks consisting of two-dimensional oscillators (i.e., $m = 2$) to obtain analytical results on stability. The stability of $h(s, \rho)$ given by Eq. (13) is investigated for the case that

$$d(s) = \det[s\mathbf{I}_2 - \mathbf{A}] = s^2 + a_1 s + a_2,\ n(s) = \bar{k}_1 s + \bar{k}_2. \quad (16)$$

For $a_2 \leq 0$, matrix $\mathbf{A}$ has one positive real eigenvalue or a zero eigenvalue. Therefore, $a_2 > 0$ is assumed in order to avoid the odd-number case in Lemma 2. Further, if $a_1 \geq 0$, matrix $\mathbf{A}$ is stable or has two eigenvalues on the imaginary axis. Thus, $a_1 < 0$ should be assumed in order to guarantee an unstable $\mathbf{A}$.

Our aim in this subsection is to design connection parameters $\bar{k}_1$, $\bar{k}_2$, and τ for $h(s, \rho)$ to be stable for any $\rho \in [0, 2]$. Our previous study[34] based on the direct method for stability analysis[40] allows us to deal with such a design problem.

Note that the equations $h(s, 0) = 0$ and $h(s, 2) = 0$ can be described by $e^{s\tau} = M(s)$ and $-e^{s\tau} = M(s)$, respectively. For $s = j\omega$, as we have $|e^{j\omega\tau}|^2 = 1$, the necessary condition for

$$\pm e^{j\omega\tau} = M(j\omega) \quad (17)$$

is $|M(j\omega)|^2 = 1$, which can be rewritten as

$$\begin{aligned} W(\omega^2) &:= |d(j\omega) + n(j\omega)|^2 - |n(j\omega)|^2 \\ &= \omega^4 + c_1\omega^2 + c_2 = 0, \end{aligned} \quad (18)$$

where

$$c_1 := a_1(a_1 + 2\bar{k}_1) - 2(a_2 + \bar{k}_2),\ c_2 := a_2(a_2 + 2\bar{k}_2). \quad (19)$$

Suppose that $W(\omega^2) = 0$ has two distinct positive roots, ω_α^2 and ω_β^2, where $\omega_\alpha^2 < \omega_\beta^2$. The direct method[40] guarantees that the characteristic root always moves from right to left through the points $\pm j\omega_\alpha$ in the complex plane if $W'(\omega_\alpha^2) := [\mathrm{d}W(\omega^2)/\mathrm{d}\omega^2]_{\omega=\omega_\alpha} < 0$, or always from left to right if $W'(\omega_\beta^2) > 0$. These points for which $W'(\omega_\alpha^2) < 0$ are stabilizing points and those for which $W'(\omega_\beta^2) > 0$ are destabilizing points.

Now we provide a way for designing the coupling strength $(\bar{k}_1, \bar{k}_2)$.

Lemma 4. If the coupling strength $(\bar{k}_1, \bar{k}_2)$ satisfies all of the inequalities

$$a_1 + \bar{k}_1 > 0, \quad a_2 + 2\bar{k}_2 > 0, \quad c_1^2 - 4c_2 > 0, \quad (20)$$

then we have the following three facts: (i) $W(\omega^2) = 0$ has the stabilizing points $\pm j\omega_\alpha$ and the destabilizing points $\pm j\omega_\beta$; (ii) the characteristic polynomial of $M(s)$, that is $d(s) + n(s)$, is stable; and (iii) characteristic quasi-polynomial $h(s, 2)$ with $\tau = 0$, that is $d(s) + 2n(s)$, is stable.

Proof. See Subsection 4 of the Appendix. □

Here, for the case of no delay (i.e., $\tau \equiv 0$), we now confirm the number of unstable roots of $h(s, 0) = 0$ and $h(s, 2) = 0$. According to our assumptions $a_1 < 0$ and $a_2 > 0$, we see that $h(s, 0) = 0$ with $\tau = 0$, that is $d(s) = 0$, has two unstable roots independently of the coupling strength. Since $h(s, 2) = 0$ with $\tau = 0$ is described by $d(s) + 2n(s)$, it has no unstable roots, due to fact (iii) in Lemma 4.

Remark that both $h(s, 0) = 0$ and $h(s, 2) = 0$ have the common points $\pm j\omega_{\alpha,\beta}$, and these points do not depend on τ. Further, it should be noted that the critical delays at which a pair of complex conjugate roots of $h(s, 0) = 0$ passes through the points $\pm j\omega_{\alpha,\beta}$ on the imaginary axis are derived from Eq. (17)

$$\tau_\alpha^{(0)}[l] := \frac{\psi_\alpha}{\omega_\alpha} + \frac{2\pi}{\omega_\alpha} l, \quad \tau_\beta^{(0)}[l] := \frac{\psi_\beta}{\omega_\beta} + \frac{2\pi}{\omega_\beta} l, \quad l = 0, 1, \ldots, \quad (21)$$

where

$$\psi_\alpha := \mathrm{Arg}[M(j\omega_\alpha)], \quad \psi_\beta := \mathrm{Arg}[M(j\omega_\beta)]. \quad (22)$$

The critical delays for $h(s, 2) = 0$ are described by

$$\tau_\alpha^{(2)}[l] := \frac{\psi_\alpha}{\omega_\alpha} - \frac{\pi}{\omega_\alpha} + \frac{2\pi}{\omega_\alpha} l, \quad \tau_\beta^{(2)}[l] := \frac{\psi_\beta}{\omega_\beta} - \frac{\pi}{\omega_\beta} + \frac{2\pi}{\omega_\beta} l,$$
$$l = 0, 1, \ldots. \quad (23)$$

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Figure 5 illustrates the relation of these critical delays.[67] Here, we shall consider the delay-dependent stability of $h(s, 0)$ and $h(s, 2)$ using the critical delays.

Lemma 5. $h(s, 0)=0$ with coupling strength $(\bar{k}_1, \bar{k}_2)$ satisfying (20) has no unstable roots if $\omega_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ satisfy

$$\frac{\psi_\alpha}{\omega_\alpha} < \frac{\psi_\beta}{\omega_\beta}, \tag{24}$$

and τ belongs to the intervals, $\Lambda^{(0)}$, where

$$\Lambda^{(0)} := \left\{ \tau \in \left(\tau_\alpha^{(0)}[l], \tau_\beta^{(0)}[l]\right), l = 0, 1, \ldots, \left\lfloor \frac{\omega_\alpha \psi_\beta - \omega_\beta \psi_\alpha}{2\pi(\omega_\beta - \omega_\alpha)} \right\rfloor \right\}. \tag{25}$$

Proof. See Subsection 5 of the Appendix. □

Lemma 6. $h(s, 2)=0$ with coupling strength $(\bar{k}_1, \bar{k}_2)$ satisfying (20) has no unstable roots if $\omega_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ satisfy

$$\frac{\psi_\alpha - \pi}{\omega_\alpha} < 0 < \frac{\psi_\beta - \pi}{\omega_\beta}, \tag{26}$$

and τ belongs to the intervals $\Lambda^{(2)}$ where

$$\Lambda^{(2)} := \left\{ \tau \in \left[0, \tau_\beta^{(2)}[0]\right), \tau \in \left(\tau_\alpha^{(2)}[l], \tau_\beta^{(2)}[l]\right), \right. \\ \left. l = 1, \ldots, \left\lfloor \frac{\omega_\alpha \psi_\beta - \omega_\beta \psi_\alpha + \pi(\omega_\beta - \omega_\alpha)}{2\pi(\omega_\beta - \omega_\alpha)} \right\rfloor \right\}. \tag{27}$$

Proof. See Subsection 6 of the Appendix. □

From Lemmas 4, 5, and 6, it is obvious that both $h(s, 0)$ and $h(s, 2)$ are stable if $(\bar{k}_1, \bar{k}_2)$ satisfies (20) and τ belongs to the intervals $\Lambda^{(0)} \cap \Lambda^{(2)}$. This fact corresponds to conditions (a) and (b) in Lemma 3; however, condition (c) still remains undiscussed. The following lemma shows that the designed $(\bar{k}_1, \bar{k}_2)$ and $\tau \in (\Lambda^{(0)} \cap \Lambda^{(2)})$ always guarantee condition (c).

Lemma 7. If coupling strength $(\bar{k}_1, \bar{k}_2)$ satisfies (20) and delay time τ belongs to the intervals $\Lambda^{(0)} \cap \Lambda^{(2)}$, condition (c) in Lemma 3 (i.e., Eq. (A1)) always holds.

Proof. See Subsection 7 of the Appendix. □

The above arguments are summarized by the following theorem.

Theorem 2. *Consider two-dimensional oscillators (1) $(m=2)$ with connection (2). Assume that the coefficients of $d(s)$ in Eq. (16) satisfy $a_1 < 0$ and $a_2 > 0$. The coupling strength $(\bar{k}_1, \bar{k}_2)$, given as the coefficients of $n(s)$ in Eq. (16), is supposed to satisfy*

$$a_1 + \bar{k}_1 > 0,\ a_2 + 2\bar{k}_2 > 0,\ c_1^2 - 4c_2 > 0, \tag{28}$$

$$\psi_\alpha/\omega_\alpha < \psi_\beta/\omega_\beta,\ (\psi_\alpha - \pi)/\omega_\alpha < 0 < (\psi_\beta - \pi)/\omega_\beta, \tag{29}$$

where

$$c_1 := a_1(a_1 + 2\bar{k}_1) - 2(a_2 + \bar{k}_2),\ c_2 := a_2(a_2 + 2\bar{k}_2), \tag{30}$$

$$\omega_\alpha := \sqrt{\frac{-c_1 - \sqrt{c_1^2 - 4c_2}}{2}},\ \omega_\beta := \sqrt{\frac{-c_1 + \sqrt{c_1^2 - 4c_2}}{2}}, \tag{31}$$

$$\psi_{\alpha,\beta} := \mathrm{Arg}\left(\frac{\bar{k}_2 + j\omega_{\alpha,\beta}\bar{k}_1}{a_2 + \bar{k}_2 - \omega_{\alpha,\beta}^2 + j\omega_{\alpha,\beta}(a_1 + \bar{k}_1)}\right). \tag{32}$$

If the delay time τ is in the intervals $\Lambda^{(0)} \cap \Lambda^{(2)}$, where

$$\Lambda^{(0)} := \left\{ \tau \in \left(\tau_\alpha^{(0)}[l], \tau_\beta^{(0)}[l]\right), l = 0, 1, \ldots, \left\lfloor \frac{\omega_\alpha \psi_\beta - \omega_\beta \psi_\alpha}{2\pi(\omega_\beta - \omega_\alpha)} \right\rfloor \right\}, \tag{33}$$

$$\Lambda^{(2)} := \left\{ \tau \in \left[0, \tau_\beta^{(2)}[0]\right), \tau \in \left(\tau_\alpha^{(2)}[l], \tau_\beta^{(2)}[l]\right), \right. \\ \left. l = 1, \ldots, \left\lfloor \frac{\omega_\alpha \psi_\beta - \omega_\beta \psi_\alpha + \pi(\omega_\beta - \omega_\alpha)}{2\pi(\omega_\beta - \omega_\alpha)} \right\rfloor \right\}, \tag{34}$$

$$\tau_\alpha^{(0)}[l] := \frac{\psi_\alpha}{\omega_\alpha} + \frac{2\pi}{\omega_\alpha} l,\ \tau_\beta^{(0)}[l] := \frac{\psi_\beta}{\omega_\beta} + \frac{2\pi}{\omega_\beta} l, \tag{35}$$

$$\tau_\alpha^{(2)}[l] := \frac{\psi_\alpha}{\omega_\alpha} - \frac{\pi}{\omega_\alpha} + \frac{2\pi}{\omega_\alpha} l,\ \tau_\beta^{(2)}[l] := \frac{\psi_\beta}{\omega_\beta} - \frac{\pi}{\omega_\beta} + \frac{2\pi}{\omega_\beta} l, \tag{36}$$

then steady state (3) is stable.

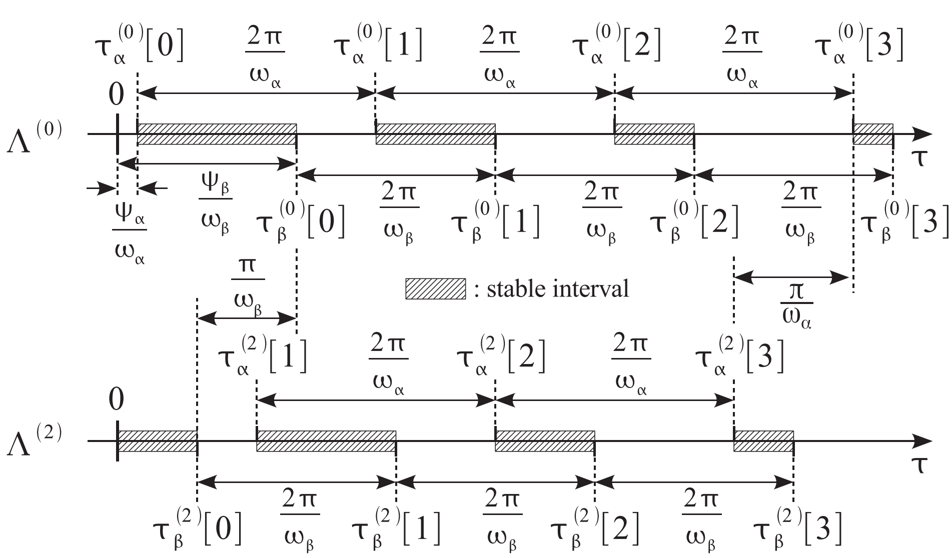

FIG. 5. Sketch of stable intervals $\Lambda^{(0)}$ (upper) and $\Lambda^{(2)}$ (lower).

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Proof. It is straightforward to prove this theorem from Lemmas 3, 4, 5, 6, and 7. Thus, we omit this proof. □

B. Extension to high-dimensional oscillators

In this subsection, we show that the procedure for two-dimensional oscillators mentioned in subsection V A can be easily extended to m-dimensional oscillators whose $\mathbf{A}$ has two non-zero unstable eigenvalues. To begin with, recall the characteristic quasi-polynomials $h(s,0) = d(s) + n(s)(1 - e^{-s\tau})$ and $h(s,2) = d(s) + n(s)(1 + e^{-s\tau})$. Here, $d(s)$ and $n(s)$ are given by Eqs. (6) and (12), respectively.

Now we assume that $d(s) = 0$ has two unstable roots (i.e., $\mathbf{A}$ has two unstable eigenvalues λ_1 and λ_2). Hence, we can decompose $d(s)$ as $d(s) = d_{\rm s}(s)d_{\rm u}(s)$, where $d_{\rm u}(s) := (s - \lambda_1)(s - \lambda_2) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2$ is the second-degree unstable polynomial and $d_{\rm s}(s)$ is the $(m - 2)$-degree stable polynomial. Since all the coefficients of $n(s)$, $\bar{k}_i$ $(i = 1,\ldots,m)$, can be chosen arbitrarily, we may take $n(s) = d_{\rm s}(s)n_{\rm u}(s)$, where $n_{\rm u}(s) := \sigma_1 s + \sigma_2$. As a result, we have

$$h(s,0) = d_{\rm s}(s)\{d_{\rm u}(s) + n_{\rm u}(s)(1 - e^{-s\tau})\},$$
$$h(s,2) = d_{\rm s}(s)\{d_{\rm u}(s) + n_{\rm u}(s)(1 + e^{-s\tau})\}. \qquad (37)$$

It should be noted that the quasi-polynomials enclosed by { } in Eq. (37) correspond to the $h(s, 0)$ and $h(s, 2)$ that subsection V A dealt with. The following corollary shows that Theorem 2 can be extended to a class of m-dimensional oscillators.

Corollary 1. Consider oscillators (1) with connection (2). Assume that $(\mathbf{A}, \boldsymbol{b})$ *is controllable;* $\mathbf{A}$ *has two unstable eigenvalues,* λ_1 *and* λ_2*; and the others are stable. Theorem 2 is valid when we use*

$$d_{\rm u}(s) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2,\ n_{\rm u}(s) = \sigma_1 s + \sigma_2, \qquad (38)$$

instead of d(s) and n(s) in Eq. (16).

From this corollary, we have a systematic design procedure for the connection parameters:

Step 0: Assume that $\mathbf{A}$ and $\boldsymbol{b}$ are known, but N and $\mathbf{C}$ are unknown.

Step 1: If $(\mathbf{A}, \boldsymbol{b})$ is controllable and $\mathbf{A}$ does not satisfy Lemma 2, then go to the next step, else quit.

Step 2: If $\mathbf{A}$ has two unstable eigenvalues λ_1 and λ_2 and the others are stable, then go to the next step, else go to Sec. IV.

Step 3: From $a_1 = -(\lambda_1 + \lambda_2) < 0$ and $a_2 = \lambda_1\lambda_2 > 0$, then we design $\bar{k}_1$, $\bar{k}_2$, and τ by Theorem 2.

Step 4: From the designed $\bar{k}_1$ and $\bar{k}_2$, obtain $\sigma_1 = \bar{k}_1$ and $\sigma_2 = \bar{k}_2$.

Step 5: From $d(s) = \det[s\mathbf{I}_m - \mathbf{A}]$ and $d_u(s) = (s - \lambda_1)(s - \lambda_2)$, obtain $d_{\rm s}(s) = d(s)/d_{\rm u}(s)$.

Step 6: Design all the elements of $\bar{\boldsymbol{k}}^T$, $\bar{k}_i$ $(i = 1,\ldots, m)$, such that $\bar{k}_1 s^{m-1} + \bar{k}_2 s^{m-2} + \cdots + \bar{k}_{m-1}s + \bar{k}_m = d_{\rm s}(s)(\sigma_1 s + \sigma_2)$ holds.

Step 7: Obtain coupling strength $\boldsymbol{k}^T$ from $\boldsymbol{k}^T = \bar{\boldsymbol{k}}^T\boldsymbol{\Gamma}^{-1}$, after choosing $\boldsymbol{\Gamma}$ such that $\bar{\mathbf{A}} = \boldsymbol{\Gamma}^{-1}\mathbf{A}\boldsymbol{\Gamma}$ and $\bar{\boldsymbol{b}} = \boldsymbol{\Gamma}^{-1}\boldsymbol{b}$ take a controllable canonical form.

This procedure is used in the following numerical examples.

C. Numerical examples

Let us consider three-dimensional oscillators (1) with

$$\mathbf{F}(\boldsymbol{x}_i) = \begin{bmatrix} x_i^{(2)} \\ x_i^{(3)} \\ -x_i^{(3)} - (a - b + b\{x_i^{(1)}\}^2)x_i^{(2)} - ax_i^{(1)} \end{bmatrix},$$
$$\boldsymbol{b} = [0 \quad 1 \quad 0]^T, \qquad (39)$$

where a and b denote the parameters. This oscillator is known as a Moore-Spiegel oscillator.[25,26] The oscillator has a fixed point at the origin $\boldsymbol{x}^* = \mathbf{0}$. Throughout this paper, the parameters are set to $a = 0.1$ and $b = 1.1$, where there exist the unstable fixed point $\boldsymbol{x}^* = \mathbf{0}$ with two positive real eigenvalues and a limit cycle in each isolated oscillator. It must be emphasized that the previous studies,[17–19,41] which treat only the fixed point with a pair of complex conjugate unstable eigenvalues, cannot be used for this oscillator.

Now we follow the procedure for designing the connection parameters proposed in subsection V B. For step 0, the Jacobi matrix $\mathbf{A}$ is calculated and $\boldsymbol{b}$ is given in Eq. (39). For step 1, we confirm that $\mathbf{A}$ does not satisfy the condition in Lemma 2, since the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0.5302$, $\lambda_2 = 0.1147$, and $\lambda_3 = -1.6449$. Further, we see that $(\mathbf{A}, \boldsymbol{b})$ is controllable. For step 2, we see that $\lambda_{1,2}$ are the two unstable roots and the other is stable. For step 3, we have $a_1 = -0.6449$ and $a_2 = 0.0608$. The critical delays $\tau^{(0)}_{\alpha,\beta}[l]$ and $\tau^{(2)}_{\alpha,\beta}[l]$ can be easily obtained from Theorem 2. Figures 6(a)–6(c) illustrate the relation between the critical delays and $\bar{k}_1$ for $\bar{k}_2 = 0.3, 0.7, 1.0$, respectively, where $\tau^{(0)}_{\alpha,\beta}[l]$ $(\tau^{(2)}_{\alpha,\beta}[l])$ are denoted by bold black curves (thin red curves). It can be seen that there is no stability region in Fig. 6(a) and the narrow strip region exits in Fig. 6(b). There is a wide region in Fig. 6(c). Here, we set $\bar{k}_1 = 2.0$ and $\bar{k}_2 = 0.7$. The delay time is chosen from the region: we set $\tau = 1.0$. For step 4, we have $\sigma_1 = 2.0$ and $\sigma_2 = 0.7$. For step 5, we obtain $d_{\rm s}(s) = s + 1.6449$. For step 6, we have $\bar{\boldsymbol{k}}^T = [1.1514 \quad 3.9898 \quad 2.0000]$. For step 7, the coupling strength is given by $\boldsymbol{k}^T = [1.2276 \quad 2.0000 \quad 0.7622]$.

The time series data of the three networks (see Fig. 3) before and after coupling are shown in Fig. 7. We see that each oscillator converges on its fixed point for $t > 100$. This result supports our analytical results.

VI. DISCUSSIONS

This section discusses the relations to the previous studies on amplitude death and to consensus problems which have been investigated in control theory.

Amplitude death is now a hot topic in nonlinear sciences.[11,12,42–46] Especially, time-delay-induced death[15] has been widely investigated for over fifteen years. It has been developed as follows: Reddy *et al.* discovered this phenomenon in globally coupled Stuart-Landau oscillators and analyzed its stability;[15,17] the stability analysis was extended to

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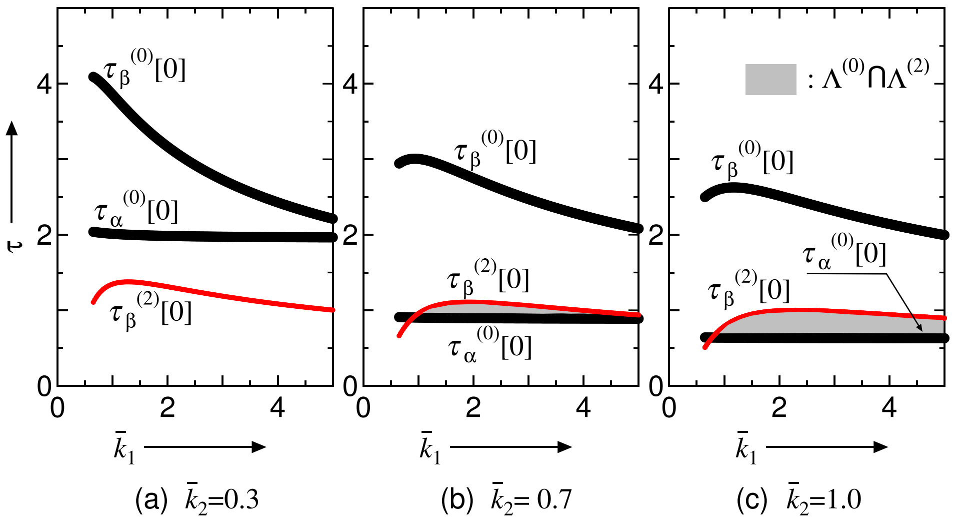

FIG. 6. Critical delays $\tau^{(0)}_{\alpha,\beta}[l]$ (bold black curves) and $\tau^{(2)}_{\alpha,\beta}[l]$ (thin red curves) in $\bar{k}_1$-τ plane for (a) $\bar{k}_2 = 0.3$, (b) $\bar{k}_2 = 0.7$, and (c) $\bar{k}_2 = 1.0$. Gray regions denote the stability region $\tau \in (\Lambda^{(0)} \cap \Lambda^{(2)})$.

a ring of Stuart-Landau oscillators;[18,19] Zou, Zheng, and Zhan analyzed the stability of time-delay-induced death on complex networks with Rössler oscillators.[41] Since these studies dealt with the oscillators which have a pair of complex conjugate unstable roots, their results cannot always be used for high-dimensional oscillators, for instance, the generalized Rössler oscillators with two pairs of complex conjugate unstable roots[23,24] (see Subsection IV C) and the Moore-Spiegel oscillators with two positive real unstable roots[25,26] (see Subsection V C). In contrast, our paper can deal with such high-dimensional oscillators.

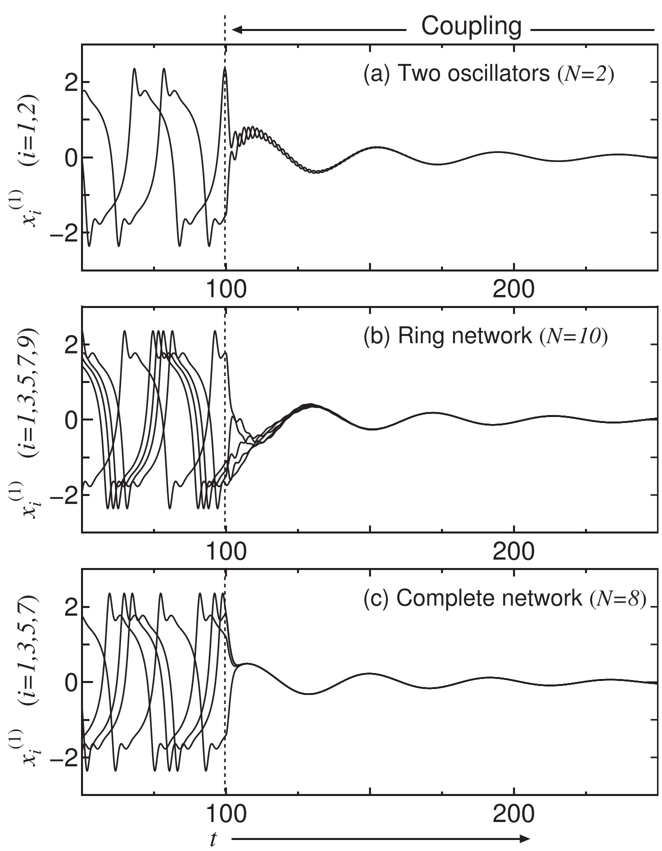

FIG. 7. Time series data of the three networks with Moore-Spiegel oscillators (39): (a) two oscillators, (b) ring type network with $N = 10$, and (c) complete network with $N = 8$. Individual oscillators without connection behave oscillatory for $t < 100$; they are coupled at $t = 100$.

Michiels and Nijmeijer provided the following two novel and valuable results:[10] the stability regions on the connection parameter space for m-dimensional oscillators coupled by directed connections can be directly computed in a computationally efficient way; for the case of Lorenz oscillators with sufficiently large coupling strength, there always exists the connection delay inducing death independently of the network topology. In contrast, for undirected connections, our paper provides the procedures for designing connection parameters for general high-dimensional oscillators without restriction on sufficiently large coupling strength.

Now let us briefly review our previous studies on the time-delay-induced death related to our present paper. Reference 47 dealt with high-dimensional oscillators coupled by time-delay connections on a *one-way ring* topology. As this previous study focused only on the specific network topology, its results cannot be used for oscillator networks with general topologies our present paper deals with. Reference 30 investigated amplitude death in Stuart-Landau oscillator networks with two connection delays: its results cannot be applied to the high-dimensional oscillators, which have two real positive unstable roots or more than two unstable roots. Reference 32 extended the results of Ref. 48, which investigated amplitude death in two scalar time-delay oscillators with a delayed connection, to scalar time-delay oscillator networks. Although Ref. 32 analyzed amplitude death using the concept of a robust stability,[37–39] which is also used in our present paper, the results of the previous study[32] are meant for only scalar time-delay oscillator networks.

It was reported that amplitude death can be induced by various time-delay connections: the distributed delay connection,[49–51] the partial delay connection,[52] the gradient delay connection,[53] the time-varying delay connection,[54–57] the digital delay connection,[58] the mismatched delay connection,[59] and the asymmetric delay connections.[60] It must be emphasized that most of them enlarge the parameter space where death occurs, as compared with the original delay connection.

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The consensus problems in networked multi-agent systems have been widely investigated in the field of control theory.[3] The problems under communication time-delays among agents[61] have attracted a growing interest in recent years.[62,63] Most of the studies on the consensus problems with communication time-delays investigate the dynamics of agents coupled by the delayed connections, in which each agent does not have unstable roots, and design the connection parameters such that the coupled agents are stable. One might conclude that the consensus problems with communication time-delays are similar to the design problems for time-delay-induced death, but they are totally opposite. The consensus problems always treat the time delay as a negative factor of instability; in contrast, our paper treats it as a positive factor of stability.

VII. CONCLUSION

The present paper provides procedures which allow us to obtain the coupling strength and connection delay for inducing stabilization of a steady state in oscillator networks. The procedures take into account the case in which the network structure and number of oscillators are unknown. In addition, we showed that if the oscillators have two unstable roots, the procedure can be systematically carried out using only a simple algebraic calculation. These analytical results are numerically confirmed for the cases of generalized Rössler oscillators and Moore-Spiegel oscillators.

ACKNOWLEDGMENTS

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APPENDIX: PROOFS OF LEMMAS AND THEOREMS

This appendix provides proofs of lemmas and theorems.

1. Proof of Lemma 1

From Eq. (9), we see that $\rho_1 = 0$ is fixed for any $\mathbf{C}$. Hence, if $h(s, \rho_1 = 0)$ is not stable, then $g(s)$ cannot be stable (see Eq. (10)). For $\tau = 0$, we have $h(s, 0) = d(s)$ independently of $\boldsymbol{k}^T$ and $\mathbf{C}$. As $\mathbf{A}$ has at least one unstable eigenvalue by assumption, $d(s) = 0$ has unstable roots. Therefore, for $\tau = 0$, steady state (3) cannot be stabilized by connection (2).

2. Proof of Lemma 2

Equation (9) suggests that $\rho_1 = 0$ is fixed for any $\mathbf{C}$. Thus, if $h(s, 0)$ is not stable, then $g(s)$ cannot be stable. Obviously if $\mathbf{A}$ has a zero eigenvalue (i.e., $d(0) = 0$), then $h(0, 0) = d(0) + n(0)\,\{1 - e^0\} = 0$ holds (i.e., $h(s, 0)$ is not stable due to the zero root) for any τ, $\boldsymbol{k}^T$, and $\mathbf{C}$. On the other hand, the function $h(s, 0)$ is continuous in s. For real positive s, we have that $\lim_{s\to+\infty} h(s,0) = +\infty$. We notice that $h(s, 0)$ with $s = 0$ is given by $h(0,0) = \det[-\mathbf{A}] = \prod_{i=1}^{m}(-\lambda_i)$. Here, λ_i $(i = 1,\ldots,m)$ are the eigenvalues of $\mathbf{A}$; thus, we have $h(0, 0) < 0$ when the number of real positive eigenvalues is odd. This fact suggests that $h(s, 0) = 0$ has at least one unstable root independently of τ, $\boldsymbol{k}^T$, and $\mathbf{C}$.

3. Proof of Theorem 1

This proof shows that the three conditions, (a), (b), and (c) in Lemma 3, hold if the conditions (i) and (ii) are satisfied.

(a) $h(s, 0) = d(s) + n(s)\,(1 - e^{-s\tau})$ is equivalent to the characteristic quasi-polynomial associated to a closed-loop system consisting of the feedforward part $M(s)$ and the feedback part $-e^{-s\tau}$. By the Nyquist stability criterion, it is obvious that if conditions (i) and (ii) are satisfied, then $h(s, 0)$ is stable.

(b) $h(s, 2) = d(s) + n(s)\,(1 + e^{-s\tau})$ corresponds to a closed-loop system consisting of $M(s)$ and $+e^{-s\tau}$. As is the case with $h(s, 0)$, the conditions (i) and (ii) also guarantee that $h(s, 2)$ is stable.

(c) Let us prove that

$$\phi(\omega) = \arg\left[\frac{h(j\omega,0)}{h(j\omega,2)}\right] = \arg\left[\frac{1 - M(j\omega)e^{-j\omega t}}{1 + M(j\omega)e^{-j\omega t}}\right] \neq \pm\pi, \forall\omega \in [0, +\infty). \tag{A1}$$

It is obvious that if condition (ii) is satisfied, then the two vectors $1 \pm M(j\omega)e^{-j\omega\tau}$ in the complex plane never have opposite directions for any $\omega \in [0, +\infty)$. Therefore, condition (A1) holds.

4. Proof of Lemma 4

This proof deals with each fact on the basis of the following relations: $a_1 + \bar{k}_1 > 0$ with the assumption $a_1 < 0$ is a sufficient condition for $a_1 + 2\bar{k}_1 > 0$; $a_2 + 2\bar{k}_2 > 0$ with the assumption $a_2 > 0$ is a sufficient condition for $a_2 + \bar{k}_2 > 0$.

(i) Note that $W(\omega^2) = 0$ obviously has two real positive roots, $\omega_\alpha^2 < \omega_\beta^2$, if the three inequalities $c_1 < 0$, $c_2 > 0$, and $c_1^2 - 4c_2 > 0$ are satisfied. It is easy to confirm that condition (20) with $a_1 < 0$ and $a_2 > 0$ is a sufficient condition for these inequalities to hold. Further, $W(0) = c_2 > 0$ shows that $\pm j\omega_\alpha$ and $\pm j\omega_\beta$ are stabilizing and destabilizing points, respectively.

(ii) The necessary and sufficient condition for $d(s) + n(s) = s^2 + (a_1 + \bar{k}_1)s + a_2 + \bar{k}_2$ to be stable is that $a_1 + \bar{k}_1 > 0$ and $a_2 + \bar{k}_2 > 0$. We easily see that condition (20) is a sufficient condition.

(iii) The necessary and sufficient condition for $d(s) + 2n(s) = s^2 + (a_1 + 2\bar{k}_1)s + a_2 + 2\bar{k}_2$ to be stable is that $a_1 + 2\bar{k}_1 > 0$ and $a_2 + 2\bar{k}_2 > 0$. We also easily see that condition (20) is a sufficient condition.

5. Proof of Lemma 5

Let us focus on the root movement of $h(s, 0) = 0$ in the case that τ is increased from 0. Recall that $h(s, 0) = 0$ with $\tau = 0$ has two unstable roots independently of the coupling strength. It is obvious that if $\tau_\alpha^{(0)}[0] < \tau_\beta^{(0)}[0]$, which is equivalent to condition (24), holds, then the following facts hold:

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the pair of complex conjugate unstable roots of $h(s, 0) = 0$ moves to the left through the stabilizing points $\pm j\omega_\alpha$ when τ exceeds $\tau_\alpha^{(0)}[0]$; the pair of complex conjugate stable roots of $h(s, 0) = 0$ moves to the right through the destabilizing points $\pm j\omega_\beta$ when τ exceeds $\tau_\beta^{(0)}[0]$. These stabilizing and destabilizing transitions are repeated (i.e., $l = 1,\ldots,$) until inequality $\tau_\alpha^{(0)}[l] < \tau_\beta^{(0)}[l]$ fails to hold. Therefore, $h(s, 0) = 0$ has no unstable roots if condition (24) holds and τ belongs to the intervals $\Lambda^{(0)}$.

6. Proof of Lemma 6

Recall that $h(s, 2) = 0$ both with $\tau = 0$ and $(\bar{k}_1, \bar{k}_2)$ satisfying (20) has no unstable roots due to fact (iii) in Lemma 4. We see that if $\tau_\alpha^{(2)}[0] < 0 < \tau_\beta^{(2)}[0]$, which is equivalent to condition (26), holds, then the following facts hold: the pair of complex conjugate stable roots of $h(s, 2) = 0$ moves to the right through $\pm j\omega_\beta$ when τ exceeds $\tau_\beta^{(2)}[0]$; the two unstable roots in complex conjugate pairs move to the left through $\pm j\omega_\alpha$ when τ exceeds $\tau_\alpha^{(2)}[0]$. These stabilizing and destabilizing transitions are repeated (i.e., $l = 1,\ldots,$) until inequality $\tau_\alpha^{(2)}[l] < \tau_\beta^{(2)}[l]$ fails to hold. Therefore, $h(s, 2) = 0$ has no unstable roots if condition (26) holds and τ belongs to the intervals $\Lambda^{(2)}$.

7. Proof of Lemma 7

Lemmas 4, 5, and 6 guarantee that if condition (20) and $\tau \in (\Lambda^{(0)} \cap \Lambda^{(2)})$ hold, then we have stable $M(s)$, $h(s, 0)$, and $h(s, 2)$. Remark that $h(s, 0) = 0$ and $h(s, 2) = 0$ can be rewritten as $1 - M(j\omega)e^{-j\omega\tau} = 0$ and $1 + M(j\omega)e^{-j\omega\tau} = 0$, respectively. We notice that on the assumption of a stable $M(s)$, stable functions $h(s, 0)$ and $h(s, 2)$ suggest that the Nyquist plot $M(j\omega)e^{-j\omega\tau}$ never intersects $\boldsymbol{l}^{(0)} \cup \boldsymbol{l}^{(2)}$. This implies that the two vectors $1 \pm M(j\omega)e^{-j\omega\tau}$ in the complex plane never have opposite directions for any $\omega \in [0, +\infty)$. Therefore, we notice that condition (c) in Lemma 3 holds if condition (20) is satisfied and τ belongs to $(\Lambda^{(0)} \cap \Lambda^{(2)})$.

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[64]In general, diffusive connections are often used for studies on coupled oscillators, since they describe physical diffusive connections and simplify stability analysis.

[65]The controllability can be checked by a simple procedure:[27] ($\mathbf{A}$, $\boldsymbol{b}$) is controllable if rank$[\boldsymbol{b}, \mathbf{A}\boldsymbol{b}, \mathbf{A}^2\boldsymbol{b}, \ldots, \mathbf{A}^{m-1}\boldsymbol{b}] = m$ holds.

[66]The local stability of steady state (3) is guaranteed even when these stable poles are set arbitrarily; however, we observe on numerical simulations that the oscillators with some other stable poles diverge when their initial states are not close to state (3). There is room for study on such divergence problem.

[67]It can be seen that the stable intervals periodically exist on the delay time τ. It is well known that this periodicity is a common characteristic on the stability of time delay systems.

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