A Conjecture on Optimal Ternary Linear Codes

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***Abstract*—We give a conjecture on the achievement of the Griesmer bound for ternary linear codes. We prove that our conjecture is valid for at most 7 dimensions.**

***Index Terms*—linear code, Griesmer bound, projective dual**

I. Introduction

Let $\mathbb{F}_q$ denote the finite field of order q. For a vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_q^n$, the *weight* of a, denoted by $wt(a)$, is the number of a_i's with $a_i \neq 0$. An $[n, k, d]_q$ code $\mathcal{C}$ is a subspace of $\mathbb{F}_q^n$ with dimension k and minimum weight d, which is the minimum non-zero weight of codewords in $\mathcal{C}$. Let A_i be the number of codewords of $\mathcal{C}$ with weight i. The list of non-zero A_i's is called the weight distribution of $\mathcal{C}$. The weight distribution "$A_0 = 1$, $A_d = w_d$, ..." is expressed as $0^1 d^{w_d} \cdots$ in this paper. A fundamental and classical problem in Coding Theory is to determine the exact values of $n_q(k, d)$ for all positive integer d for fixed q and k, where $n_q(k, d)$ is the smallest length n such that an $[n, k, d]_q$ code exists [8]. The Griesmer bound gives a lower bound on the function $n_q(k, d)$:

$$n_q(k, d) \geq g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil z \rceil$ stands for the minimum integer $\geq z$, see [8]. It is known for $k = 1, 2$ that the Griesmer bound is achieved for all d. So, we assume $k \geq 3$. For fixed q and k, it is also known that Griesmer codes with minimum weight d exist for all sufficiently large d [8]. A natural question is the following.

Problem 1. For fixed q and k, find the integer $D_{q,k}$ satisfying that $n_q(k, d) = g_q(k, d)$ for all $d > D_{q,k}$ and that $n_q(k, d) > g_q(k, d)$ for $d = D_{q,k}$. Then, determine $n_q(k, D_{q,k})$.

The above problem is partially solved as follows.

Theorem 1 ([11], [15]). *For $q \geq k$ with $k = 3, 4, 5$ and for $q \geq 2k - 3$ with $k \geq 6$, it holds that $D_{q,k} = (k-2)q^{k-1} - (k-1)q^{k-2}$ and that $n_q(k, D_{q,k}) = g_q(k, D_{q,k}) + 1$.*

We tackle Problem 1 for $q = 3$ (ternary linear codes) in this paper. We know $D_{3,3} = 3$ from Theorem 1. Hence, we assume $q = 3$ and $k \geq 4$ in what follows.

We define T_k as

- $T_k = 3^{k-2-l}(2(l-1) \cdot 3^{l+1} + 3^{k-l^2-l-1} + 2)$ for $l^2 + l + 2 \leq k \leq l^2 + 2l + 1$;
- $T_k = 3^{k-2-l}((2l-1) \cdot 3^{l+1} + 3^{k-l^2-2l-2} + 1)$ for $l^2 + 2l + 2 \leq k \leq l^2 + 3l + 2$;
- $T_k = 3^{k-2-l}(2l \cdot 3^{l+1} + 1)$ for $k = l^2 + 3l + 3$,

where l is a positive integer, see Table 1. Then, we can prove the following.

Theorem 2. *A $[g_3(k, d), k, d]_3$ code exists for all $d > T_k$.*

Our conjecture is the following.

Conjecture. $D_{3,k} = T_k$ and $n_3(k, T_k) = g_3(k, T_k) + 1$ for all $k \geq 4$.

To prove our conjecture, it suffices to show the following by Theorem 2:

(A) There exists no $[g_3(k, T_k), k, T_k]_3$ code;
(B) There exists a $[g_3(k, T_k) + 1, k, T_k]_3$ code.

It is known that $D_{3,4} = 15$, $D_{3,5} = 99$, $D_{3,6} = 351$ and that $n_3(k, d) = g_3(k, d) + 1$ for $(k, d) = (4, 15)$, $(5, 99)$, $(6, 351)$, see [18]. Hence, our conjecture is valid for $k \leq 6$. In this paper, we show that our conjecture is valid for $k = 7$ also.

Theorem 3. *$D_{3,7} = T_7$ and $n_3(7, T_7) = g_3(7, T_7) + 1$.*

We prove Theorem 2 in Section II. We give the geometric methods through projective geometry and preliminary results in Section III, which are needed to prove Theorem 3 in Section IV.

TABLE I

k	$g_3(k, T_k)$	T_k
4	23	15
5	149	99
6	527	351
7	2309	1539
8	7169	4779
9	23693	15795
10	90761	60507
11	278843	185895
12	895577	597051
13	3218171	2145447
14	9713561	6475707
15	29672123	19781415
16	93799337	62532891
17	324444731	216296487
18	978117161	652078107
19	2977398203	1984932135
20	9319615097	6213076731
21	31445629691	20963753127
22	94466029235	62977352823

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II. Proof of Theorem 2

We first recall the well-known conditions such that q-ary Griesmer codes with dimension k and minimum weight d exist. Let $\sigma = \lceil d/q^{k-1} \rceil$. Since Griesmer codes (called *σ-fold simplex codes*) with $d = \sigma q^{k-1}$ exist, we assume $d \not\equiv 0 \pmod{q^{k-1}}$. Then, d is uniquely expressed as

$$d = \sigma q^{k-1} - \sum_{j=1}^{r} q^{u_j - 1}, \tag{1}$$

where r and u_j's are integers satisfying

$$\begin{aligned} k-1 \geq u_1 \geq u_2 \geq \cdots \geq u_r \geq 1 \\ \text{and } u_j > u_{j+q-1} \text{ for } 1 \leq j \leq r-q+1. \end{aligned} \tag{2}$$

From the latter condition of (2), at most $q-1$ of $u_1, \ldots, u_r$ can take any given value. It is well-known that there exist Griesmer codes if $r \leq \sigma$. Assume $r \geq \sigma + 1$ and let $u = \sum_{i=1}^{\sigma+1} u_i$. The following theorem was originally found by Belov et al. [1] for binary linear codes, and by Hill [8] and Dodunekov [6] for q-ary linear codes.

Theorem 4 **([8]).** *A $[g_q(k,d), k, d]_q$ code exists if $u \leq \sigma k$.*

We prove Theorem 2 applying Theorem 4.

Lemma 5. *For $l^2 + l + 2 \leq k \leq (l+1)^2$ with positive integer l, a $[g_3(k,d), k, d]_3$ code exists if*

$$d > T_k = 3^{k-2-l}(2(l-1) \cdot 3^{l+1} + 3^{k-l^2-l-1} + 2).$$

Proof. When $d > T_k$, we get the following for σ in (1):

$$\sigma = \lceil d/3^{k-1} \rceil \geq \lceil T_k/3^{k-1} \rceil \geq 2l - 1.$$

We first assume that $\sigma \geq 2l$. Denote by u_{max} the maximum value of $u = \sum_{i=1}^{\sigma+1} u_i$. For even $\sigma = 2m$ $(m \geq l)$, we get

$$\begin{aligned} u_{max} &= 2\sum_{j=1}^{m}(k-j) + (k-m-1) \\ &= 2mk + k - (m+1)^2 \\ &\leq \sigma k + (l+1)^2 - (m+1)^2 \\ &\leq \sigma k. \end{aligned}$$

For odd $\sigma = 2m+1$ $(m \geq l)$, we have

$$\begin{aligned} u_{max} &= 2\sum_{j=1}^{m+1}(k-j) \\ &= (2m+1)k + k - (m+1)^2 - (m+1) \\ &\leq \sigma k + (l+1)^2 - (m+1)^2 - (m+1) \\ &< \sigma k. \end{aligned}$$

Hence, a $[g_3(k,d), k, d]_3$ code exists by Theorem 4.

Next, assume that $\sigma = 2l-1$. Let α be the integer satisfying $\alpha = l^2 + 2l + 2 - k$ $(1 \leq \alpha \leq l)$. Then, we obtain

$$T_k = (2l-1)3^{k-1} - \sum_{j=1}^{l} 3^{k-1-j} - \sum_{j=1, j\neq\alpha}^{l} 3^{k-1-j} - 3^{k-2-l}.$$

We denote by u_{T_k} the value of u for $d = T_k$. Then,

$$\begin{aligned} u_{T_k} &= \sum_{j=1}^{l}(k-j) + \sum_{j=1, j\neq\alpha}^{l}(k-j) + (k-1-l) \\ &= (2l-1)k + 1 \\ &= \sigma k + 1. \end{aligned}$$

It follows that the value u for $d > T_k$ satisfies $u < u_{T_k}$, i.e., $u \leq \sigma k$. Therefore, a $[g_3(k,d), k, d]_3$ code exists if $d > T_k$ when $\sigma = 2l-1$ by Theorem 4. □

The following two lemmas can be proved similarly.

Lemma 6. *For $(l+1)^2 + 1 \leq k \leq l^2 + 3l + 2$ with positive integer l, a $[g_3(k,d), k, d]_3$ code exists if*

$$d > T_k = 3^{k-2-l}((2l-1) \cdot 3^{l+1} + 3^{k-l^2-2l-2} + 1).$$

Lemma 7. *For $k = l^2 + 3l + 3$ with positive integer l, a $[g_3(k,d), k, d]_3$ code exists if*

$$d > T_k = 3^{k-2-l}(2l \cdot 3^{l+1} + 1).$$

Since $l^2 + 3l + 4 = (l+1)^2 + (l+1) + 2$, every dimension $k \geq 4$ satisfies either $l^2 + l + 2 \leq k \leq (l+1)^2$, $(l+1)^2 + 1 \leq k \leq l^2 + 3l + 2$ or $k = l^2 + 3l + 3$ with positive integer l. Hence Theorem 2 follows from Lemmas 5-7. □

III. Geometric methods

In this section, we give some geometric methods which are needed to prove Theorem 3. As usual, $\mathrm{PG}(r,q)$ denotes the projective geometry of dimension r over $\mathbb{F}_q$. The projective subspaces of dimension j in $\mathrm{PG}(r,q)$ are called *j-spaces*. The j-spaces for $j = 0, 1, 2, 3, r-2, r-1$ are usually called *points, lines, planes, solids, secundums, hyperplanes*, respectively [10].

Let $\mathcal{C}$ be an $[n,k,d]_q$ code with generator matrix G having no all-zero column. Then, the columns of G can be considered as a multiset of n points, denoted by $\mathcal{M}_{\mathcal{C}}$, in the projective space $\Sigma = \mathrm{PG}(k-1,q)$. We denote by $\mathcal{S}_j$ the set of j-spaces in Σ and by θ_j the number of points contained in a j-space, which can be calculated as $\theta_j = (q^{j+1}-1)/(q-1)$. An *$m$-point* is a point in Σ which appears exactly m times as columns of G. Let γ_0 be the maximum integer i such that an i-point of Σ exists. Let Λ_i be the set of i-points in Σ and let $\lambda_i = |\Lambda_i|$, $0 \leq i \leq \gamma_0$, where $|T|$ stands for the number of elements in a set T. For any set S in Σ, the multiplicity of S, denoted by $m_{\mathcal{M}_{\mathcal{C}}}(S)$, is naturally defined as $m_{\mathcal{M}_{\mathcal{C}}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \Lambda_i|$. This yields the partition $\bigcup_{i=0}^{\gamma_0} \Lambda_i$ of Σ such that $n = m_{\mathcal{M}_{\mathcal{C}}}(\Sigma)$ and that the maximum multiplicity of a hyperplane of Σ is equal to $n-d$ [8], and vice versa. A line ℓ is called an *m-line* if $m = m_{\mathcal{M}_{\mathcal{C}}}(\ell)$. An *$m$-plane* and so on are similarly defined. For a t-space T in Σ, let

$$\gamma_j(T) = \max\{m_{\mathcal{M}_{\mathcal{C}}}(\Delta) \mid \Delta \subset T,\ \Delta \in \mathcal{S}_j\},\ 0 \leq j \leq t.$$

Let $\lambda_m(T)$ be the number of m-points in T. We simply denote by γ_j for $\gamma_j(\Sigma)$ and by λ_s for $\lambda_s(\Sigma)$. We already know that $\gamma_{k-2} = n-d$ and $\gamma_{k-1} = n$.

Lemma 8 ([14]). *Let S be an $(s-1)$-space $(2 \le s \le k-1)$ in Σ with $m_{\mathcal{M}_\mathcal{C}}(S) = w$. Then an $(s-2)$-space δ in S satisfies*

$$m_{\mathcal{M}_\mathcal{C}}(\delta) \le \gamma_{s-1} - \frac{n-w}{\theta_{k-s}-1}.$$

And γ_t for $0 \le t \le k-3$ satisfies

$$\gamma_t \le \gamma_{t+1} - \frac{n-\gamma_{t+1}}{\theta_{k-2-t}-1}.$$

When $\mathcal{C}$ is a Griesmer code, the values $\gamma_0, \gamma_1, ..., \gamma_{k-3}$ are uniquely determined in [16] as follows:

$$\gamma_t = \sum_{v=0}^{t} \left\lceil \frac{d}{q^{k-1-v}} \right\rceil \quad \text{for } 0 \le t \le k-1. \tag{3}$$

When $\gamma_0 = 2$, we obtain

$$\lambda_2 = \lambda_0 + n - \theta_{k-1} \tag{4}$$

from $\lambda_0 + \lambda_1 + \lambda_2 = \theta_{k-1}$ and $\lambda_1 + 2\lambda_2 = n$. Let a_i be the number of i-hyperplanes in Σ. The list of non-zero a_i's is called the *spectrum* of $\mathcal{C}$. The spectrum can be obtained from the weight distribution as $a_{n-w} = A_w/(q-1)$ for $d \le w \le n$. To distinguish from the spectrum of $\mathcal{C}$, we use τ_i's for the spectrum of a hyperplane containing i-secundums. The following equalities for the spectrum are well-known [13]:

$$\sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1}, \tag{5}$$

$$\sum_{i=1}^{\gamma_{k-2}} i a_i = n\theta_{k-2}, \tag{6}$$

$$\sum_{i=2}^{\gamma_{k-2}} \binom{i}{2} a_i = \binom{n}{2}\theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} \binom{s}{2} \lambda_s. \tag{7}$$

When $\gamma_0 \le 2$, we get the following equality from (5)-(7):

$$\begin{aligned} \sum_{i=0}^{\gamma_{k-2}-2} \binom{\gamma_{k-2}-i}{2} a_i &= \binom{\gamma_{k-2}}{2}\theta_{k-1} \\ &-n(\gamma_{k-2}-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \end{aligned} \tag{8}$$

Lemma 9 ([12], [21]). *Put $\epsilon = q\gamma_{k-2} - n$ and $t_0 = \lfloor (w+\epsilon)/q \rfloor$, where $\lfloor x \rfloor$ stands for the largest integer $\le x$. Let H be a w-hyperplane containing a t-secundum T. Then $t \le (w+\epsilon)/q$ and the following hold.*

(i) *$a_w = 0$ if no $[w, k-1, d_0]_q$ code satisfying $d_0 \ge w - t_0$ exists.*

(ii) *$\gamma_{k-3}(H) = t_0$ if no $[w, k-1, d_1]_q$ code satisfying $d_1 \ge w - t_0 + 1$ exists.*

(iii) *Let c_m be the number of m-hyperplanes containing T except H. Then $\sum_m c_m = q$ and*

$$\sum_m (\gamma_{k-2} - m)c_m = w + \epsilon - qt. \tag{9}$$

(iv) *A γ_{k-2}-hyperplane with spectrum $(\tau_0, \ldots, \tau_{\gamma_{k-3}})$ satisfies $\tau_t > 0$ if $w + \epsilon - qt < q$.*

(v) *If any γ_{k-2}-hyperplane has no t_0-secundum, then $m_{\mathcal{M}_\mathcal{C}}(H) \le t_0 - 1$.*

An $[n, k, d]_q$ code $\mathcal{C}$ is called Γ-*divisible* (or Γ-div for short) if the weight of every codeword of $\mathcal{C}$ is divisible by an integer $\Gamma \ge 2$.

Lemma 10 ([21]). *For $q = p^h$ with p prime, let $\mathcal{C}$ be an m-div $[n, k, d]_q$ code with $m = p^r$ for some $1 \le r < h(k-2)$ satisfying $\lambda_0 > 0$ and*

$$\bigcap_{H \in \mathcal{S}_{k-2},\ m_{\mathcal{M}_\mathcal{C}}(H) < \gamma_{k-2}} H = \emptyset.$$

Then, there exists a t-div $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ such that

$$t = \frac{q^{k-2}}{m},\ n^* = ntq - \frac{d}{m}\theta_{k-1},\ d^* = (q\gamma_{k-2} - n)t$$

whose spectrum satisfies $a_{n^*-d^*-it} = \lambda_i$ for $0 \le i \le \gamma_0$.

A generator matrix of $\mathcal{C}^*$ can be obtained by taking the $(\gamma_{k-2} - im)$-hyperplanes for $\mathcal{C}$ as the i-points in the dual space of Σ for any non-negative integers i [21]. $\mathcal{C}^*$ is called a *projective dual* of $\mathcal{C}$, see [3] and [5].

Lemma 11 ([19]). *Let G be a generator matrix of an $[n, k, d]_q$ code $\mathcal{C}$ giving the multiset $\mathcal{M}_\mathcal{C}$ of n points in $\mathrm{PG}(k-1, q)$. If $\mathcal{M}_\mathcal{C}$ contains a u-space and if $d > q^u$, then an $[n - \theta_u, k, d']_q$ code $\mathcal{C}'$ with $d' \ge d - q^u$ exists.*

One can get a multiset $\mathcal{M}_{\mathcal{C}'}$ for the code $\mathcal{C}'$ in Lemma 11 from $\mathcal{M}_\mathcal{C}$ by removing the u-space U, that is, by deleting the θ_u columns of G corresponding to U. This construction method is called *geometric puncturing* [17].

Lemma 12 ([22]). *Let $\mathcal{C}$ be a Griesmer code with minimum weight d over the prime field $\mathbb{F}_p$. If p^ϵ divides d for some positive integer ϵ, then $\mathcal{C}$ is p^ϵ-divisible.*

Lemma 13 ([20]). *Let $\mathcal{C}$ be a p^r-divisible $[n, k, d]_q$ code with $q = p^h$, p prime, $r > h$. Then, any residual code of $\mathcal{C}$ corresponding to a hyperplane in Σ is p^{r-h}-divisible.*

Lemma 14. *If $\mathcal{C}$ satisfies $\gamma_0 = 1$ and $\gamma_1 \le 2$, it holds that*

$$\sum_{i=3}^{\gamma_{k-2}} \binom{i}{3} a_i = \binom{n}{3}\theta_{k-4}.$$

Proof. Counting the number of all possible $(\{P_1, P_2, P_3\}, H)$, where P_1, P_2, P_3 are distinct points in the n-set $\mathcal{M}_\mathcal{C}$ in Σ and H is a hyperplane containing the three points, one can get the required equality. $\square$

IV. Divisible codes of dimension $k \le 6$

In this section, we give the results on some divisible ternary linear codes, which we employ to prove Theorem 3.

Lemma 15. *No 3-div $[n, 6, 3]_3$ code exists for $8 \le n \le 11$.*

Proof. There exists no $[8, 6, 3]_3$ code since $n_3(6, 3) = 9$. We get the nonexistence of a 3-div $[9, 6, 3]_3$ code by the exhaustive computer search (we used the package Q-Extension [2]).

Suppose a 3-div $[10,6,3]_3$ code exists. Then, its projective dual is a 27-div $[446,6,297]_3$ code, which does not exist since $n_3(6,297) = 447$ or 448 [18]. Hence, there exists no 3-div $[10,6,3]_3$ code. One can prove the nonexistence of a 3-div $[11,6,3]_3$ code similarly. $\square$

We denote by $\mathcal{M}_1 + \mathcal{M}_2$ the multiset $\mathcal{M}$ consisting of the multisets $\mathcal{M}_1$ and $\mathcal{M}_2$ in Σ. In this case, we also write $\mathcal{M}_2 = \mathcal{M} - \mathcal{M}_1$. We write $2\mathcal{M}_1$ for $\mathcal{M}_1 + \mathcal{M}_2$ when $\mathcal{M}_1 = \mathcal{M}_2$.

Lemma 16. *The spectrum of a* 27*-div* $[770,6,513]_3$ *code is one of the following:*
(a) $(a_{176}, a_{230}, a_{257}) = (a, 14-3a, 350+2a)$, $a \in \{1,2\}$,
(b) $(a_{203}, a_{230}, a_{257}) = (b, 14-2b, 350+b)$, $b \in \{0,1,2\}$,
(c) $(a_{176}, a_{203}, a_{230}, a_{257}) = (1,1,9,353)$.

Proof. Let $\mathcal{C}$ be a 27-div $[770,6,513]_3$ code. We first assume that $\mathcal{C}$ has a 0-point in $\Sigma = \mathrm{PG}(5,3)$, i.e., $\lambda_0 > 0$. Then, A projective dual of $\mathcal{C}$ is a 3-div $[14,6,3]_3$ code $\mathcal{C}^*$. By the exhaustive computer search, we get 25 inequivalent 3-div $[14,6,3]_3$ codes. Let λ_w^* be the number of w-points for $\mathcal{C}^*$. Since $a_{257-27w} = \lambda_w^*$, we get the possible spectra for $\mathcal{C}$.

Next, assume that $\lambda_0 = 0$ in Σ. If $\lambda_1 = 0$, then the multiset $\mathcal{M}_\mathcal{C} - 2\Sigma$ gives a $[42,6,27]_3$ code, which does not exist [18], a contradiction. Hence $\lambda_1 > 0$, and the multiset $\mathcal{M}_\mathcal{C} - \Sigma$ gives a 27-div $[406(=770-\theta_5), 6, 270(=513-3^5)]_3$ code, say $\mathcal{D}$, by Lemma 11. A projective dual of $\mathcal{D}$ is a 3-div $[14,6,6]_3$ code. We get four inequivalent 3-div $[14,6,6]_3$ codes by the exhaustive computer search, and our assertion follows. $\square$

The following five lemmas can be proved similarly.

Lemma 17. *The spectrum of a* 27*-div* $[689,6,459]_3$ *code is* $(a_{203}, a_{230}) = (13, 351)$.

Lemma 18. *The spectrum of a* 27*-div* $[608,6,405]_3$ *code is* $(a_{176}, a_{203}) = (12, 352)$.

Lemma 19. *The spectrum of a* 9*-div* $[257,5,171]_3$ *code satisfies* $a_i = 0$ *for any* $i \notin \{32,41,50,59,68,77,86\}$.

Lemma 20. *The spectrum of a* 9*-div* $[230,5,153]_3$ *code satisfies* $a_i = 0$ *for any* $i \notin \{50,59,68,77\}$.

Lemma 21. *The spectrum of a* 9*-div* $[203,5,135]_3$ *code satisfies* $a_i = 0$ *for any* $i \notin \{41,50,59,68\}$.

Lemma 22. *Let* $\mathcal{C}$ *be a* 3*-div* $[86,4,57]_3$ *code with* $\lambda_0 = 4$. *Then, the four* 0*-points for* $\mathcal{C}$ *form a* 0*-line.*

Proof. A projective dual of $\mathcal{C}$, say $\mathcal{C}^*$, is a 3-div $[14,4,3]_3$ code. The value λ_0 for $\mathcal{C}$ is equal to the value a_{11}, the number of 11-planes for $\mathcal{C}^*$. By the exhaustive computer search, we confirmed that there exists only one 3-div $[14,4,3]_3$ code with $a_{11} = 4$ up to equivalence. For example, take a generator matrix of $\mathcal{C}^*$ as

$$G = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&2&2&2&2&2&2&2&2&0&1&0&0 \\ 0&0&1&1&1&1&2&2&2&2&0&0&1&0 \\ 1&2&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

From the above G, the 11-planes for $\mathcal{C}^*$ are $[1000]$, $[0001]$, $[1001]$, $[1002]$, which are collinear 0-points of $\mathcal{C}$ in the dual space of $\mathrm{PG}(3,3)$, where $[abcd]$ stands for the hyperplane $V(aX_0 + bX_1 + cX_2 + dX_3)$. $\square$

Lemma 23. *Every* 27*-div* $[689,6,459]_3$ *code with* $\lambda_0 > 0$ *satisfies* $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (1,75,250,38)$, $(3,69,256,36)$ *or* $(7,57,268,32)$.

Proof. Let $\mathcal{C}$ be a 27-div $[689,6,459]_3$ code with $\lambda_0 > 0$. A projective dual $\mathcal{C}^*$ of $\mathcal{C}$ is a 3-div $[13,6,3]_3$ code. By the exhaustive computer search, we get three inequivalent 3-div $[13,6,3]_3$ codes. Let a_w^* be the number of w-hyperplanes for $\mathcal{C}^*$. Since $\lambda_{(10-w)/3} = a_w^*$, we get the possible λ_i's for $\mathcal{C}$ as stated. $\square$

V. Proof of Theorem 3

Lemma 24. *There exists no* $[2309,7,1539]_3$ *code.*

Proof. Let $\mathcal{C}$ be a $[2309,7,1539]_3$ code. Then $\mathcal{C}$ is a 81-div code by Lemma 12. We first assume that $\lambda_0 = 0$. If also $\lambda_1 = 0$, then the multiset $\mathcal{M}_\mathcal{C} - 2\Sigma$ gives a $[123,7,81]_3$ code, which does not exist [7], a contradiction. Hence $\lambda_1 > 0$, and the multiset $\mathcal{M}_\mathcal{C} - \Sigma$ gives a Griesmer $[1216,7,810]_3$ code $\mathcal{D}$ with a 0-point. Since $\mathcal{D}$ is 81-div by Lemma 12, a projective dual of $\mathcal{D}$ is a 3-div $[14,7,6]_3$ code $\mathcal{D}^*$ from Lemma 10. Suppose that $\mathcal{D}^*$ contains a 2-point Q. Then, the projection of $\mathcal{M}_{\mathcal{D}^*}$ from Q to some hyperplane of Σ^* not containing Q gives a 3-div $[12,6,6]_3$ code, which is well known to be equivalent to the extended ternary Golay code. But it can be confirmed by the exhaustive computer search that it is impossible to construct a 3-div $[14,7,6]_3$ code from the extended Golay code. Hence, $\mathcal{D}^*$ has no 2-point and the shortening of $\mathcal{D}^*$ gives a 3-div $[13,6,6]_3$ code, which is also unique up to equivalence (cf. [4]). Constructing a 3-div $[14,7,6]_3$ code from the 3-div $[13,6,6]_3$ code is also impossible by the exhaustive computer search. Thus, a 3-div $[14,7,6]_3$ code does not exist.

Now, $\mathcal{C}$ has a 0-point, and a projective dual of $\mathcal{C}$ is a 3-div $[14,7,3]_3$ code $\mathcal{C}^*$ from Lemma 10. We denote by a_j^* for the spectrum of $\mathcal{C}^*$ and use λ_i^* and γ_s^* to stand for λ_i and γ_s for $\mathcal{C}^*$ to distinguish from a_j, λ_i and γ_s for $\mathcal{C}$. Then the spectrum of $\mathcal{C}^*$ satisfies $a_i^* = 0$ for any $i \notin \{2,5,8,11\}$, and $\gamma_0^* \le 6$ by Lemma 8. If $3 \le \gamma_0^* \le 6$, one can get a 3-div $[n,6,3]_3$ code with $8 \le n \le 11$ from $\mathcal{C}^*$ by shortening, which does not exist by Lemma 15. Hence $\gamma_0^* \le 2$. From the three equalities (5)-(7), we get

$$3a_2^* + a_5^* = 1471 + 27\lambda_2^*. \tag{10}$$

Since $\gamma_0^* \le 2$, the spectrum of $\mathcal{C}$ satisfies $a_i = 0$ for any $i \notin \{770, 689, 608\}$, and (9) in Lemma 9 gives

$$162c_{608} + 81c_{689} = w + 1 - 3t. \tag{11}$$

Setting $(w,t) = (608,176)$ from Lemma 18, the solution of (11) is $(c_{608}, c_{689}, c_{770}) = (0,1,2)$, which contradicts that a 689-hyperplane has no 176-secundum by Lemma 17. Hence $a_{608} = \lambda_2^* = 0$, and $\gamma_0^* = 1$.

Setting $(w, t) = (689, 203)$ from Lemma 17, the solution of (11) is

$$(c_{689}, c_{770}) = (1, 2). \tag{12}$$

Setting $(w, t) = (689, 230)$, the solution of (11) is

$$(c_{689}, c_{770}) = (0, 3). \tag{13}$$

Since a 689-hyperplane for $\mathcal{C}$ is considered as a 1-point for $\mathcal{C}^*$, we have $m_{\mathcal{M}_{\mathcal{C}^*}}(\ell) \leq 2$ for all line ℓ from (12) and (13). Hence it follows from Lemma 14 and (10) with $\lambda_2^* = 0$ that

$$(a_2^*, a_5^*, a_8^*, a_{11}^*) = (251, 718, 120, 4). \tag{14}$$

Then $\mathcal{C}$ satisfies $\lambda_0 = a_{11}^* = 4$ from (14). Let Δ be an x-solid containing all the 0-points of $\mathcal{C}$. Since a 689-hyperplane has at most three 0-points by Lemma 23, Δ is not contained in a 689-hyperplane. Hence Δ is contained in a 257-secundum in some 770-hyperplane Π by Lemmas 16 and 17. Since Π corresponds to a 27-div $[770, 6, 513]_3$ code by Lemma 13, (9) in Lemma 9 gives

$$54c_{203} + 27c_{230} = w' + 1 - 3t' \tag{15}$$

for a w'-secundum through a t'-solid in Π from Lemma 16. Setting $w' = 257$, $c_{203} = c_{230} = 0$ and $t' = x$ in (15), one can deduce that $x = 86$. Then, Δ gives a 3-div $[86, 4, 57]_3$ code by Lemmas 13 and 19. And the four 0-points in Δ form a 0-line, say ℓ_0, by Lemma 22. Let δ be a y-plane through ℓ_0 in Δ. Since the number of solids through δ is θ_3, we have

$$y + (86 - y)\theta_3 = 2309,$$

i.e., $y = 29$. Hence

$$m_{\mathcal{M}_{\mathcal{C}}}(\Delta) = 4 \cdot 29 = 116,$$

a contradiction. Thus, a $[2309, 7, 1539]_3$ code does not exist. □

Lemma 25. *There exists a* $[2310, 7, 1539]_3$ *code.*

Proof. Let $\mathcal{C}$ be a $[23, 7, 9]_3$ code with generator matrix

$$\begin{pmatrix} 11100000001221111122 1100 \\ 00001000002120001202011 \\ 00000100000222000220211 \\ 00000010000012220021011 \\ 00000001020001202001111 \\ 00000000121000110100111 \\ 00010000001110001101 0011 \end{pmatrix}.$$

The weight distribution of $\mathcal{C}$ is $0^1 9^{64} 12^{438} 15^{954} 18^{646} 21^{84}$ and is 3-divisible. Hence, we get a 81-div $[2310, 7, 1539]_3$ code as a projective dual of $\mathcal{C}$. □

There exists no $[g_3(7, T_7), 7, T_7]_3$ code by Lemma 24 and there exists a $[g_3(7, T_7)+1, 7, T_7]_3$ code by Lemma 25. Hence Theorem 3 follows. □

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