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Yuto Inoue[a], Tatsuya Maruta[a,1,*]

[a] *Department of Mathematical Sciences, Osaka Prefecture University, Sakai, Osaka 599-8531, Japan*

Abstract

We introduce a new concept "geometric extending" for linear codes over finite fields and consider the extendability of divisible codes. As an application, we construct new Griesmer $[n, 5, d]_q$ codes for $3q^4 - 5q^3 + 1 \le d \le 3q^4 - 5q^3 + q^2$ with $q \ge 3$, combining the known geometric methods such as projective dual, geometric extending and geometric puncturing.

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1. Introduction

An $[n, k, d]_q$ code $\mathcal{C}$ is a k dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming weight $d = \min\{wt(\boldsymbol{c}) > 0 \mid \boldsymbol{c} \in \mathcal{C}\}$ over $\mathbb{F}_q$, the field of q elements. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists for given q, k, d [5, 6]. The Griesmer bound is a best known lower bound on the length n:

$$n \ge g_q(k, d) = \sum_{i=0}^{k-1} \left\lceil d/q^i \right\rceil ,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Linear codes of length $n = g_q(k, d)$ are called *Griesmer codes*. The values of $n_q(k, d)$ are determined for all d only for some small values of q and k [4, 13]. For the case $k = 5$, it is well-known that $[g_q(5, d), 5, d]_q$ codes exist for $q^4 - 2q^2 + 1 \le d \le q^4$, $2q^4 - 2q^3 - q^2 + 1 \le d \le 2q^4 + q^2 - q$ and $d \ge 3q^4 - 4q^3 + 1$ for all q [10, 11], see also [3]. Recently, we have proved the following.

*Corresponding author

Email addresses: `braveclimb9510@gmail.com` (Yuto Inoue), `maruta@mi.s.osakafu-u.ac.jp` (Tatsuya Maruta)

Theorem 1.1 ([9]). *There exist $[g_q(5,d),5,d]_q$ codes for $2q^4-3q^3+1 \le d \le 2q^4-3q^3+q^2$, $3q^4-5q^3+q^2+1 \le d \le 3q^4-5q^3+2q^2$ for all q.*

The Griesmer codes of dimension 5 in Theorem 1.1 whose minimum weights are divisible by q are constructed by the following algorithm:

Step 1.	Construct a q-divisible code $\mathcal{C}$ with smaller length.
Step 2.	Construct a q^2-divisible Griesmer code $\mathcal{C}^*$ as a projective dual of $\mathcal{C}$.
Step 3.	Find skew lines in the multiset consisting of the columns of a generator matrix of $\mathcal{C}^*$ and apply geometric puncturing.

Our aim of this paper is to construct new Griesmer codes of dimension 5, by adding the following new step between Steps 1 and 2.

$\diamond$ Construct a q-divisible code $\bar{\mathcal{C}}$ from $\mathcal{C}$ by geometric extending.

Our main theorem is the following:

Theorem 1.2. *There exist $[g_q(5,d),5,d]_q$ codes for $3q^4-5q^3+1 \le d \le 3q^4-5q^3+q^2$ for all q.*

Corollary 1.3. *$n_q(5,d) = g_q(5,d)$ for $3q^4-5q^3+1 \le d \le 3q^4-5q^3+2q^2$ for all q.*

We give the geometric methods such as projective dual, geometric puncturing and geometric extending in Section 2. We prove Theorem 1.2 in Section 3.

Remark 1. We apologize that there are some typos in [9]:

- in Abstract, "$3q^4+5q^3+1 \le d \le 3q^4+5q^3+q^2$" must be "$3q^4-5q^3+q^2+1 \le d \le 3q^4-5q^3+2q^2$",
- in Theorem 1.2 and Corollary 1.4, the same as in Abstract,
- in Lemma 3.10, "$3q^4+5q^3+2q^2-sq$" must be "$3q^4-5q^3+2q^2-sq$".

2. Geometric methods to construct optimal linear codes

We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A t-flat is a t dimensional projective subspace of $\mathrm{PG}(r,q)$. The 0-flats, 1-flats, 2-flats, 3-flats and $(r-1)$-flats are called *points, lines, planes, solids* and *hyperplanes*, respectively. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r,q)$ and by θ_j the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$. A hyperplane of $\mathrm{PG}(r,q)$ is defined as the set of points with homogeneous coordinates $(x_0,x_1,\ldots,x_r)$ satisfying a linear equation $a_0x_0+a_1x_1+\cdots+a_rx_r=0$ with non-zero vector $(a_0,a_1,\ldots,a_r)$ over $\mathbb{F}_q$. We denote such a hyperplane by $[a_0,a_1,\ldots,a_r]$.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. Then, the columns of a generator matrix G of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted by $\mathcal{M}_{\mathcal{C}}$. An *s-point* is a point of Σ which has multiplicity s in $\mathcal{M}_{\mathcal{C}}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_{\mathcal{C}}$. Let Λ_s be the set of s-points in Σ, $0 \leq s \leq \gamma_0$, and let $\lambda_s = |\Lambda_s|$, where $|\Lambda_s|$ denotes the number of elements in a set Λ_s. For any subset S of Σ, *the multiplicity of* S, denoted by $m_{\mathcal{C}}(S)$, is defined as $m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \Lambda_i|$. For $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{F}_q^k \setminus \{(0, 0, \ldots, 0)\}$, take the hyperplane $H = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ in Σ and let $\boldsymbol{c} = \lambda_1 \boldsymbol{g}_1 + \lambda_2 \boldsymbol{g}_2 + \cdots + \lambda_k \boldsymbol{g}_k \in \mathcal{C}$, where $\boldsymbol{g}_j$ is the j-th row of G. Then, it is well known (see [1, 6]) that

$$n - wt(\boldsymbol{c}) = m_{\mathcal{C}}(H). \tag{2.1}$$

Hence, we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} \Lambda_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and the maximum multiplicity of hyperplanes is exactly $n - d$, i.e.,

$$n - d = \max\{m_{\mathcal{C}}(H) \mid H \in \mathcal{F}_{k-2}\}, \tag{2.2}$$

see Theorem 2.3 in [6]. Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} \Lambda_i$ as above gives an $[n, k, d]_q$ code in the natural manner. A hyperplane H with $t = m_{\mathcal{C}}(H)$ is called a *t-hyperplane*. A *t-line*, a *t-plane* and *t-solid* are defined similarly. Denote by a_i the number of i-hyperplanes in Σ. The list of the values a_i is called the *spectrum* of $\mathcal{C}$. It follows from (2.1) that the spectrum of $\mathcal{C}$ can be calculated from the weight distribution by $a_i = A_{n-i}/(q-1)$ for $0 \leq i \leq n - d$.

The method for constructing new codes from a given $[n, k, d]_q$ code by deleting the coordinates corresponding to some geometric object in $\mathrm{PG}(k-1, q)$ is called *geometric puncturing* [12]. An instant geometric puncturing is to remove some flat from the multiset $\mathcal{M}_{\mathcal{C}}$ if possible.

Lemma 2.1 ([12, 14])**.** *Let $\mathcal{C}$ be an $[n, k, d]_q$ code. If the multiset $\mathcal{M}_{\mathcal{C}}$ contains a t-flat Δ and if $d > q^t$, then the multiset $\mathcal{M}_{\mathcal{C}} - \Delta$ gives an $[n - \theta_t, k, d']_q$ code $\mathcal{C}'$ with $d' \geq d - q^t$.*

We note that the equality $d' = d - q^t$ in Lemma 2.1 holds if an $(n-d)$-hyperplane not containing Δ exists. We consider the converse of the above lemma, namely *geometric extending*. We get the following from (2.2).

Lemma 2.2. *Let $\mathcal{C}$ be an $[n, k, d]_q$ code. If there exists a t-flat Δ such that every hyperplane of Σ through Δ has multiplicity at most $n - d - q^t$, then the multiset $\mathcal{M}_{\mathcal{C}} + \Delta$ gives an $[n + \theta_t, k, \bar{d}]_q$ code with $\bar{d} \geq d + q^t$.*

When can we find a t-flat Δ to construct an $[n + \theta_t, k, d + q^t]_q$ code $\bar{\mathcal{C}}$ with multiset $\mathcal{M}_{\mathcal{C}} + \Delta$ for a given $[n, k, d]_q$ code $\mathcal{C}$? In general, it is not easy to find a t-flat Δ satisfying the condition in Lemma 2.2 even when $t = 1$. We consider the possibility of geometric extending for divisible codes. An $[n, k, d]_q$ code is called *m-divisible* if all codewords have weights divisible by an integer $m > 1$.

Lemma 2.3. *Let $\mathcal{C}$ be a q^s-divisible $[n, k, d]_q$ code with positive integer s. If there exists a t-flat Δ which is not contained in an $(n-d)$-hyperplane of Σ with $t \leq s$, then the multiset $\mathcal{M}_{\mathcal{C}} + \Delta$ gives a q^t-divisible $[n + \theta_t, k, d + q^t]_q$ code.*

Proof. Let $\bar{\mathcal{C}}$ be an $[n+\theta_t, k, \bar{d}]_q$ code given by the multiset $\mathcal{M}_{\mathcal{C}} + \Delta$. For a hyperplane H in Σ, it follows from (2.1) that $m_{\mathcal{C}}(H) = n - d - jq^s$ with some non-negative integer j since $\mathcal{C}$ is q^s-divisible. For $j = 0$, $m_{\bar{\mathcal{C}}}(H) = n - d + \theta_{t-1}$ since H meets Δ in a $(t-1)$-flat. For $j \geq 1$, we have

$$m_{\bar{\mathcal{C}}}(H) = n - d - jq^s + \theta_t \leq n - d + \theta_{t-1}$$

if H contains Δ and $m_{\bar{\mathcal{C}}}(H) = n - d - jq^s + \theta_{t-1} < n - d + \theta_{t-1}$ otherwise. From the condition $t \leq s$, $\bar{\mathcal{C}}$ is q^t-divisible from (2.1) and the maximum multiplicity of hyperplanes for $\bar{\mathcal{C}}$ is $n - d + \theta_{t-1}$. Hence, from (2.2), we obtain

$$\bar{d} = n + \theta_t - (n - d + \theta_{t-1}) = d + q^t.$$

□

Finally, we recall the projective dual for divisible codes.

Lemma 2.4 ([15]). *Let $\mathcal{C}$ be an m-divisible $[n, k, d]_q$ code with $q = p^h$, p prime, whose spectrum is*

$$(a_{n-d-(w-1)m}, a_{n-d-(w-2)m}, \cdots, a_{n-d-m}, a_{n-d}) = (\alpha_{w-1}, \alpha_{w-2}, \cdots, \alpha_1, \alpha_0),$$

where $m = p^r$ for some $1 \leq r < h(k-2)$ satisfying $\lambda_0 > 0$ and

$$\bigcap_{H \in \mathcal{F}_{k-2},\ m_{\mathcal{C}}(H) < n-d} H = \emptyset.$$

Then there exists a t-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $t = q^{k-2}/m$, $n^* = \sum_{j=0}^{w-1} j\alpha_j = ntq - \frac{d}{m}\theta_{k-1}$, $d^* = ((n-d)q - n)t$ whose spectrum is

$$(a_{n^*-d^*-\gamma_0 t}, a_{n^*-d^*-(\gamma_0-1)t}, \cdots, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_{\gamma_0}, \lambda_{\gamma_0-1}, \cdots, \lambda_1, \lambda_0).$$

Note that a generator matrix for $\mathcal{C}^*$ is given by considering $(n-d-jm)$-hyperplanes as j-points in the dual space Σ^* of Σ for $0 \leq j \leq w-1$ [15]. $\mathcal{C}^*$ is called a *projective dual* of $\mathcal{C}$, see also [2] and [6].

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 for $q \geq 5$ using projective dual, geometric extending and geometric puncturing. Note that Theorem 1.2 is known for $q \leq 5$, see [13]. A set S of s points in $\mathrm{PG}(r, q)$, $r \geq 2$, is called an *s-arc* if no $r+1$ points are on the same hyperplane, see [7] and [8] for arcs. When $q \geq r$, one can take a normal rational curve as a $(q+1)$-arc, see Theorem 27.5.1 in [8].

In $\mathrm{PG}(4, q)$, $q \geq 5$, take a normal rational curve $K = \{P_0, P_1, \ldots, P_q\}$ in the hyperplane $H = [0, 0, 0, 0, 1]$ as

$$P_0(1, 0, 0, 0, 0),\ P_i(1, \alpha^i, \alpha^{2i}, \alpha^{3i}, 0),\ P_q(0, 0, 0, 1, 0),$$

Table 1: $(4q-2)$-solids for $\mathcal{C}$.

type	number
A1	$\binom{q}{2}$
A2	q
A3	$q-1$
A4	q
A5	q

where α is a primitive element of $\mathbb{F}_q$. Take a line $l_0 = \{P_0, Q_1, \dots, Q_q\}$ as

$$Q_i(1,0,0,0,\alpha^i) \text{ with } 1 \le i \le q-1,\ Q_q(0,0,0,0,1),$$

and a point Q, lines $l_1, \dots, l_q$ and a plane δ as

$$Q(0,1,0,0,1),\ l_i = \langle P_i, Q_i \rangle \text{ with } 1 \le i \le q,\ \delta = \langle l_0, Q \rangle,$$

where $\langle \chi_1, \chi_2, \cdots \rangle$ denotes the smallest flat containing $\chi_1, \chi_2, \cdots$. Setting the set of s-points Λ_s as

$$\Lambda_1 = (\bigcup_{i=1}^{q-1} l_i) \setminus l_0,\ \Lambda_{q-1} = \{P_0, Q_q\},\ \Lambda_q = \{P_q, Q\},\ \Lambda_0 = \Sigma \setminus (\Lambda_1 \cup \Lambda_{q-1} \cup \Lambda_q),$$

we get the following q-divisible code, which was called $\mathcal{C}_2$ in [9].

Lemma 3.1 ([9]). *There exists a q-divisible $[q^2+3q-2, 5, q^2-q]_q$ code $\mathcal{C}$ with spectrum*

$$(a_{q-2}, a_{2q-2}, a_{3q-2}, a_{4q-2}) = (q^4 - 4q^3 + 6q^2 - 4q + 1,$$
$$5q^3 - 12q^2 + 10q - 3 - \binom{q}{2}, 7q^2 - 9q + 4, \binom{q}{2} + 4q - 1)$$

which has five types of $(4q-2)$-solids:

(A1) *$H_{ij} = \langle l_i, l_j \rangle$ for $1 \le i < j \le q$;*
(A2) *$\langle \delta, l_i \rangle$ for $1 \le i \le q$;*
(A3) *$\langle Q, P_q, l_i \rangle$ for $1 \le i \le q-1$;*
(A4) *the solids through the plane $\langle Q, l_q \rangle$ not containing l_0;*
(A5) *the solids through the plane $\langle Q, P_0, P_q \rangle$ not containing l_0*

and nine types of $(3q-2)$-solids:

(B1) *the solids containing l_0 and only one of $l_1, \dots, l_q$ and not containing Q;*
(B2) *the solid through δ containing none of $l_1, \dots, l_q$;*
(B3) *the solids through the line $\langle P_0, P_q \rangle$ not containing Q, l_0;*
(B4) *the solids through the line $\langle P_0, Q \rangle$ not containing P_q, l_0;*

Table 2: $(3q-2)$-solids for $\mathcal{C}$.

type	number
B1	q
B2	1
B3	q^2-q
B4	q^2-q
B5	q^2-q
B6	q^2-q
B7	q^2-2q+1
B8	q^2-2q+1
B9	q^2-2q+1

(B5) *the solids through* l_q *not containing* Q, l_0*;*

(B6) *the solids through the line* $\langle Q, Q_q\rangle$ *not containing* P_q, l_0*;*

(B7) *the solids through the plane* $\langle l_i, P_q\rangle$ *with* $1 \le i \le q-1$ *not containing* Q, l_0*;*

(B8) *the solids through the plane* $\langle l_i, Q\rangle$ *with* $1 \le i \le q-1$ *not containing* P_q, l_0*;*

(B9) *the solids through the plane* $\langle Q, P_q, Q_i\rangle$ *with* $1 \le i \le q-1$ *not containing* l_i, l_0*,*

see Tables 1 and 2 for the number of solids of each type.

Now, we shall construct a q-divisible $[q^2+4q-1, 5, q^2]_q$ code $\bar{\mathcal{C}}$ from $\mathcal{C}$ by geometric extending. Take two points R_1, R_2 and the line L as follows:

$$R_1(1,1,0,0,0),\ R_2(0,0,1,0,1),\ L = \langle R_1, R_2\rangle.$$

Let $\bar{\mathcal{C}}$ be the code with $\mathcal{M}_{\bar{\mathcal{C}}} = \mathcal{M}_{\mathcal{C}} + L$.

Lemma 3.2. *$\bar{\mathcal{C}}$ is a q-divisible $[q^2+4q-1, 5, q^2]_q$ code.*

Proof. By Lemma 2.3, it suffices to prove that L is not contained in a $(4q-2)$-solid for $\mathcal{C}$. There are five types of $(4q-2)$-solids for $\mathcal{C}$ as in Lemma 3.1. The solid $H_{ij} = \langle l_i, l_j\rangle$ of type A1 does not contain L since $H_{ij} = [0, \alpha^i\alpha^j, -\alpha^i-\alpha^j, 1, 0]$ with $1 \le i < j \le q-1$ and $H_{iq} = [0, -\alpha^i, 1, 0, 0]$ with $1 \le i \le q-1$ do not contain the point R_1. The solid $\langle \delta, l_i\rangle$ with $1 \le i \le q$ of type A2 does not contain L since $\langle \delta, l_i\rangle = [0, 0, -\alpha^i, 1, 0]$ with $1 \le i \le q-1$ and $\langle \delta, l_q\rangle = [0,0,1,0,0]$ do not contain the point R_2. Assume that the solid $H_1^{(i)} = \langle Q, P_q, l_i\rangle$ with $1 \le i \le q-1$ of type A3 contains R_1. Then $\alpha^i = -1$, and $H_1^{(i)} = [1,-1,-2,0,1]$ does not contain R_2. Hence $H_1^{(i)}$ does not contain L. Suppose that a solid H_2 of type A4 containing Q, $l_q = \langle P_q, Q_q\rangle$ and not containing l_0 contains the point R_1. Then the solid $\langle Q, l_q, R_1\rangle = [0,0,1,0,0]$ contains P_0, and H_2 contains l_0, giving a contradiction. Hence H_2 does not contain L. Finally, suppose a solid H_3 of type A5 containing Q, P_0, P_q and not containing l_0 contains R_1. Then the solid $\langle Q, P_0, P_q, R_1\rangle = [0, 0, 1, 0, 0]$ contains l_0, a contradiction. Hence H_3 does not contain L. Thus, no $(4q-2)$-solid for $\mathcal{C}$ contains the line L, and our assertion follows from Lemma 2.3. $\square$

As a projective dual of $\bar{\mathcal{C}}$, we get the following.

Lemma 3.3. *There exists a* q^2*-divisible* $[3q^4 - 2q^3 - q^2 - q, 5, 3q^4 - 5q^3 + q^2]_q$ *code* $\bar{\mathcal{C}}^*$.

Lemma 3.4. *The multiset* $\mathcal{M}_{\bar{\mathcal{C}}^*}$ *contains* $q-1$ *skew lines.*

Proof. Note that the 0-points for $\bar{\mathcal{C}}^*$ are the $(4q-1)$-solids for $\bar{\mathcal{C}}$. There are nine types of $(3q-2)$-solids for $\mathcal{C}$ as in Lemma 3.1. Note that the solids of type B1 or B2 contain the line l_0 and the others do not.

Suppose the solid H_1' of type B1 containing l_0 and l_i with $1 \le i \le q$ and not containing Q contains L. Then, H_1' contains the plane $\langle l_0, R_1\rangle$ containing Q, a contradiction. Hence H_1' does not contain L. The solid H_2' of type B2 through δ containing none of $l_1, \dots, l_q$ contains L since the solid $\langle \delta, L\rangle = [0, 0, 0, 1, 0]$ does not contain $l_1, \dots, l_q$. There is only one solid of type B3 containing the lines L and $\langle P_0, P_q\rangle$ and not containing Q since the solid $\langle P_0, P_q, L\rangle = [0, 0, 1, 0, -1]$ does not contain Q. Suppose a solid H_4' of type B4 containing the line $\langle P_0, Q\rangle$ and not containing P_q contains L. Then, H_4' contains δ, and H_4' contains l_0, a contradiction. Hence H_4' does not contain L. Since the solid $\langle l_q, L\rangle = [1, -1, 0, 0, 0]$ does not contain Q, there is only one solid of type B5 containing the lines L, l_q and not containing Q.

Suppose a solid H_6' of type B6 containing the line $\langle Q, Q_q\rangle$ and not containing P_q contains L. Then, H_6' contains δ, and H_6' contains l_0, a contradiction. Hence H_6' does not contain L. A solid of type B7 containing the plane $\langle l_i, P_q\rangle$ and not containing Q does not contain L since the solid $\langle L, l_i\rangle = [-\alpha^{3i}, \alpha^{3i}, -\alpha^{2i}, 1, \alpha^{2i}]$ does not contain P_q. Let H_8' be a solid of type B8 containing the plane $\langle l_i, Q\rangle$ and not containing P_q. Note that $\langle L, l_i\rangle$ contains Q if and only if $\alpha^i = -1$. Hence $\langle L, l_i\rangle = [1, -1, -1, 1, 1]$ does not contain P_q, and there is only one solid of type B8 containing L. Finally, let H_9' be a solid of type B9 containing the plane $\langle Q, P_q, Q_i\rangle$ and not containing l_i. Since the solid $\langle Q, P_q, L\rangle = [1, -1, -1, 0, 1]$ contains Q_i if and only if $\alpha^i = -1$, $\langle Q, P_q, L\rangle$ does not contain P_i, and there is only one solid of this type containing L.

Thus there are five $(3q-2)$-solids for $\mathcal{C}$ containing L. Then the $(4q-1)$-solids for $\bar{\mathcal{C}}$ consist of the $(4q-2)$-solids for $\mathcal{C}$ and the above five $(3q-2)$-solids for $\mathcal{C}$ containing L. Recall that the 0-points for $\mathcal{C}^*$ are the $(4q-2)$-solids for $\mathcal{C}$. Since l_0 is contained in H_{ij} and δ in Σ, the plane l_0^* contains exactly $\binom{q}{2} + q + 1$ 0-points in Σ^* corresponding to the solids H_{ij} $(1 \le i < j \le q)$, $\langle \delta, l_i\rangle$ $(1 \le i \le q)$ and $\langle \delta, L\rangle$. The number of s-points with $s \ge 1$ on l_0^* is $\theta_2 - \binom{q}{2} - q - 1 \ge q - 1$. Recall that the plane l_0^* is contained in the solids $Q_1^*, \dots, Q_q^*$ in Σ^*. We set $\alpha^\rho = -1$, i.e., $\rho = (q-1)/2$ if q is odd and $\rho = q-1$ if q is even. Then the solid Q_ρ^* contains the plane l_0^*, the points $\langle Q, P_q, l_\rho\rangle^*$, $\langle L, l_\rho, Q\rangle^*$ and $\langle Q, P_q, L\rangle^*$. The 0-points contained in Q_ρ^* out of the plane l_0^* are only these three points. Hence, the number of s-points in Q_ρ^* out of the plane l_0^* with $s \geq 1$ is $\theta_3 - (\theta_2 + 1 + 1 + 1) = q^2 - 3$, and we can take $q - 1$ lines in the solid Q_ρ^* containing no 0-point in Σ^*, meeting l_0 in a point. □

Applying Lemma 2.1 (geometric puncturing) by deleting j skew lines contained in the multiset $\mathcal{M}_{\bar{\mathcal{C}}^*}$, we get an $[n_j = 3q^4 - 2q^3 - q^2 - q - j(q+1), 5, d_j]_q$ code with $d_j \geq 3q^4 - 5q^3 + q^2 - jq$ for $1 \leq j \leq q-1$. Since $g_q(5, 3q^4 - 5q^3 + q^2 - jq) = n_j$ for $q \geq 5$, we obtain the following.

Lemma 3.5. *There exist* $[3q^4 - 2q^3 - q^2 - q - j(q+1), 5, 3q^4 - 5q^3 + q^2 - jq]_q$ *codes for* $1 \leq j \leq q - 1$.

Proof of Theorem 1.2. Lemma 3.5 provides the codes needed in Theorem 1.2, when d is divisible by q. The rest of the codes required for the theorem can be obtained by the normal puncturing of these divisible codes. □

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