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Abstract. We prove the nonexistence of some ternary linear codes of dimension 6, which implies that $n_3(6,d) = g_3(6,d)+2$ for $d = 48, 49, 66, 67, 149, 150$, where $g_3(k,d) = \sum_{i=0}^{k-1} \lceil d/3^i \rceil$ and $n_q(k,d)$ denotes the minimum length n for which an $[n,k,d]_q$ code exists. To prove the nonexistence of a putative code through projective geometry, we introduce some proof techniques such as i-Max and i-Max-NS to rule out some possible weights of codewords.

1 Introduction

We denote by $\mathbb{F}_q^n$ the vector space of n-tuples over $\mathbb{F}_q$, the field of order q. The *weight* of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero entries in $\boldsymbol{x}$. An $[n,k,d]_q$ code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with $d = \min\{wt(\boldsymbol{c}) > 0 \mid \boldsymbol{c} \in \mathcal{C}\}$, which is also called a linear code of length n, dimension k and minimum weight d over $\mathbb{F}_q$. We only consider linear codes having no coordinate which is identically zero. A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists for all k, d, q. An $[n,k,d]_q$ code is called *optimal* if $n = n_q(k,d)$. See [18] for the updated tables of $n_q(k,d)$ for some small q and k. See also [3] for optimal linear codes for small q. The Griesmer bound ([6, 23]) gives a lower bound on $n_q(k,d)$:

$$n_q(k,d) \geq g_q(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. An $[n,k,d]_q$ code $\mathcal{C}$ is called *Griesmer* if it attains the Griesmer bound, i.e. $n = g_q(k,d)$. For ternary linear codes, $n_3(k,d)$ is known for $k \leq 5$ for all d ([14]), but the value of $n_3(6,d)$ is still unknown for 76 integers d.

Theorem 1.1. *$n_3(4,d) = g_3(4,d) + 1$ for $d = 3, 7, 8, 9, 13, 14, 15$ and $n_3(4,d) = g_3(4,d)$ for all the other d.*

Theorem 1.2. (1) *$n_3(5,d) = g_3(5,d)$ for $d = 1, 2, 4, 5, 6, 10, 11, 12, 28$-$31, 34, 35, 36, 52$-$60, 64$-$93$ and for $d \geq 100$.*

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(2) $n_3(5,d) = g_3(5,d) + 1$ *for* $d = 3, 7, 8, 9, 13\text{-}24, 32, 33, 37\text{-}51,\ 61, 62, 63, 94\text{-}99$.

(3) $n_3(5,d) = g_3(5,d) + 2$ *for* $d = 25, 26, 27$.

It is known that $n_3(6,d) = g_3(6,d)+1$ or $g_3(6,d)+2$ for $d = 48, 49, 66, 67, 149, 150$, see [7, 12, 19, 20, 21, 24]. Our purpose is to prove the following.

Theorem 1.3. *There exist no* $[g_3(6,d)+1, 6, d]_3$ *codes for* $d = 48, 49, 66, 67, 149, 150$.

Corollary 1.4. $n_3(6,d) = g_3(6,d) + 2$ *for* $d = 48, 49, 66, 67, 149, 150$.

We recall the geometric method through projective geometry and preliminary results in Section 2. We give some results on ternary linear codes of dimension 5 in Section 3, which are needed to prove Theorem 1.3 in Section 4. To prove the nonexistence of putative codes through projective geometry, we introduce some proof techniques such as "i-Max" and "i-Max-NS" to rule out the hyperplanes of some possible multiplicities. The updated $n_3(6,d)$ table for $d \le 360$ is given as Table 3. In the table, "s-t" stands for $g_3(6,d) + s \le n_3(6,d) \le g_3(6,d) + t$. Entries in boldface are given in this paper. Note that $n_3(6,d) = g_3(6,d)$ for all $d \ge 352$ by Theorem 2.12 of [9].

2 Preliminary results

We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A t-flat is a t dimensional projective subspace of $\mathrm{PG}(r,q)$. The 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes*, respectively. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r,q)$ and by θ_j the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. Then, the columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted by $\mathcal{M}_\mathcal{C}$. An *i-point* is a point of Σ which has multiplicity $m_{\mathcal{M}_\mathcal{C}}(P) = i$ in $\mathcal{M}_\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_\mathcal{C}$. Let Λ_i be the set of i-points in Σ, $0 \le i \le \gamma_0$, and let $\lambda_i = |\Lambda_i|$, where $|\Lambda_i|$ denotes the number of elements in a set Λ_i. For any subset S of Σ, *the multiplicity of* S, denoted by $m_{\mathcal{M}_\mathcal{C}}(S)$, is defined as $m_{\mathcal{M}_\mathcal{C}}(S) = \sum_{P\in S} m_{\mathcal{M}_\mathcal{C}}(P) = \sum_{i=1}^{\gamma_0} i\cdot|S\cap\Lambda_i|$. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0}\Lambda_i$ such that $n = m_{\mathcal{M}_\mathcal{C}}(\Sigma)$ and

$$n - d = \max\{m_{\mathcal{M}_\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \tag{2.1}$$

Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0}\Lambda_i$ as above gives an $[n,k,d]_q$ code. A hyperplane H with $t = m_{\mathcal{M}_\mathcal{C}}(H)$ is called a *t-hyperplane*. A *t-line*, a *t-plane* and so on are defined similarly. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{M}_\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m;$$

$$\lambda_s(\Pi) = |\Pi \cap \Lambda_s|,\ 0 \le s \le \gamma_0.$$

We denote simply by γ_j and λ_s instead of $\gamma_j(\Sigma)$ and $\lambda_s(\Sigma)$, respectively. It holds that $\gamma_{k-2} = n - d,\ \gamma_{k-1} = n$.

Denote by a_i the number of i-hyperplanes in Σ. The list of a_i's is called the *spectrum* of $\mathcal{C}$. The spectrum can be calculated from the weight distribution of $\mathcal{C}$ by $a_i = A_{n-i}/(q-1)$ for $0 \le i \le n-d$, where A_w is the number of codewords of $\mathcal{C}$ with weight w. Let τ_j be the number of j-secundums in a fixed hyperplane Π of Σ. The list of τ_j's is called the *spectrum* of Π. Simple counting arguments yield the following [15].

$$\sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1}, \tag{2.2}$$

$$\sum_{i=1}^{\gamma_{k-2}} ia_i = n\theta_{k-2}, \tag{2.3}$$

$$\sum_{i=2}^{\gamma_{k-2}} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s. \tag{2.4}$$

When $\gamma_0 \le 2$, we get the following equality from (2.2)-(2.4).

$$\sum_{i=0}^{\gamma_{k-2}-2} \binom{\gamma_{k-2}-i}{2} a_i = \binom{\gamma_{k-2}}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \tag{2.5}$$

Note that λ_2 can be calculated from the spectrum and (2.5) when $\gamma_0 = 2$.

Lemma 2.1 ([16])**.** *For $0 \le j \le k-3$, $\gamma_j \le \gamma_{j+1} - (n-\gamma_{j+1})/(\theta_{k-2-j}-1)$.*

Lemma 2.2 ([13, 24])**.** *Put $\epsilon = q\gamma_{k-2} - n$ and $t_0 = \lfloor (w+\epsilon)/q \rfloor$, where $\lfloor x \rfloor$ stands for the largest integer $\le x$. Let H be a w-hyperplane containing a t-secundum T. Then $t \le (w+\epsilon)/q$ and the following hold.*

(1) *$a_w = 0$ if no $[w, k-1, d_0]_q$ code satisfying $d_0 \ge w - t_0$ exists.*

(2) *$\gamma_{k-3}(H) = t_0$ if no $[w, k-1, d_1]_q$ code satisfying $d_1 \ge w - t_0 + 1$ exists.*

(3) *Let c_j be the number of j-hyperplanes through T except H. Then $\sum_j c_j = q$ and*

$$\sum_j (\gamma_{k-2} - j)c_j = w + \epsilon - qt. \tag{2.6}$$

(4) *A γ_{k-2}-hyperplane with spectrum $(\tau_0, \ldots, \tau_{\gamma_{k-3}})$ satisfies $\tau_t > 0$ if $w+\epsilon-qt < q$.*

(5) *If any γ_{k-2}-hyperplane has no t_0-secundum, then $m_{\mathcal{M}_\mathcal{C}}(H) \le t_0 - 1$.*

It follows from (2.1) that an i-hyperplane Π corresponds to an $[i, k-1, d_0]_q$ code with $d_0 = i - \gamma_{k-3}(\Pi)$. A linear code $\mathcal{C}$ is called *projective* if $\gamma_0 = 1$.

Lemma 2.3. *Let $\mathcal{C}$ be an $[n,k,d]_q$ code. If every $(n-d)$-hyperplane H satisfies $\gamma_0(H)=1$, then $\mathcal{C}$ is projective.*

Proof. Suppose an s-point P with $s>1$ exists. Take a hyperplane π not containing P and let $\mathcal{M}_{\bar{\mathcal{C}}}$ be the multiset for a code $\bar{\mathcal{C}}$ obtained by the projection of $\mathcal{M}_{\mathcal{C}}$ from P onto π. Then, $\bar{\mathcal{C}}$ is an $[n-s,k-1,d]_q$ code and there is an $(n-d-s)$-secundum for $\bar{\mathcal{C}}$ in π, say Δ. Then, $H=\langle P,\Delta\rangle$ is an $(n-d)$-hyperplane in Σ with $\gamma_0(H)>1$, a contradiction. Hence Σ has no s-point with $s>1$, i.e., $\gamma_0=1$. □

3 Spectra of some $[n,k,d]_3$ codes with $k=4,5$

In this section, we give some results on ternary linear codes of dimensions 4 or 5, which are needed to investigate ternary linear codes of dimension 6 in Section 4. Tables 1 and 2 can be obtained from the known results. There are exactly 9 $[12,5,6]_3$ codes, 11 $[27,5,16]_3$ codes and 444 $[24,5,14]_3$ codes up to equivalence [2]. From the classifications, we get the following lemmas.

Lemma 3.1. *Let $\mathcal{C}$ be a $[12,5,6]_3$ code with $a_5=\lambda_2=0$. Then, the spectrum of $\mathcal{C}$ is $(a_0,a_3,a_6)=(3,76,42)$.*

Lemma 3.2. *The spectrum of a $[24,5,14]_3$ code is one of the following:*
(a) $(a_0,a_4,a_6,a_7,a_9,a_{10})=(1,12,13,30,26,39)$,
(b) $(a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_{10})=(2,4,1,16,4,12,18,32,32)$,
(c) $(a_1,a_3,a_4,a_6,a_7,a_9,a_{10})=(b,a,15-a-3b,16-2a,24+2a+3b,24+a,42-a-b)$ *with* $a\in\{0,1,2,3,4\}$, $b=0,1$.

Lemma 3.3. *Let $\mathcal{C}$ be a $[27,5,16]_3$ code. If $\lambda_2=0$, then the spectrum of $\mathcal{C}$ is one of the following:*
(a) $(a_0,a_5,a_8,a_9,a_{11})=(1,18,18,39,45)$,
(b) $(a_1,a_4,a_5,a_7,a_8,a_9,a_{10},a_{11})=(1,4,14,4,14,13,45,26)$,
(c) $(a_1,a_4,a_5,a_7,a_8,a_9,a_{10},a_{11})=(1,2,16,8,10,13,43,28)$.

The following two lemmas can be obtained by the exhaustive computer search (e.g. using the package Q-Extension [1]).

Lemma 3.4. *The spectrum of a $[27,4,17]_3$ code satisfies $a_2=a_5=0$.*

Lemma 3.5. *The spectrum of a $[36,5,22]_3$ code is one of the following:*
(a) $(a_0,a_8,a_{10},a_{12},a_{13},a_{14})=(1,18,18,12,36,36)$,
(b) $(a_1,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14})=(1,4,14,4,6,8,9,43,32)$,
(c) $(a_1,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14})=(1,4,14,1,12,5,12,37,35)$,
(d) $(a_2,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14})=(1,1,6,11,2,6,10,10,42,32)$,
(e) $(a_2,a_6,a_7,a_8,a_{10},a_{11},a_{12},a_{13},a_{14})=(1,2,4,12,10,8,11,40,33)$,
(f) $(a_2,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14})=(1,8,10,1,8,9,12,38,34)$,
(g) $(a_3,a_6,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14})=(1,6,8,4,12,2,20,33,35)$.

Table 1: The spectra of some ternary linear codes of dimension 4 [5].

parameters	possible spectra
$[5,4,2]_3$	$(a_0,a_1,a_2,a_3)=(5,15,10,10)$
$[8,4,4]_3$	$(a_0,a_1,a_2,a_3,a_4)=(3,4,10,12,11)$
	$(a_0,a_1,a_2,a_3,a_4)=(2,8,4,16,10)$
	$(a_0,a_2,a_3,a_4)=(4,16,8,12)$
$[14,4,8]_3$	$(a_1,a_2,a_3,a_4,a_5,a_6)=(1,4,4,8,9,14)$
	$(a_1,a_2,a_4,a_5,a_6)=(2,4,10,12,12)$
	$(a_1,a_2,a_3,a_4,a_5,a_6)=(2,2,5,7,11,13)$
	$(a_1,a_2,a_3,a_4,a_5,a_6)=(3,1,2,12,9,13)$
	$(a_1,a_2,a_3,a_4,a_5,a_6)=(3,3,3,6,10,15)$
	$(a_0,a_2,a_3,a_4,a_5,a_6)=(1,3,4,9,10,13)$
	$(a_0,a_2,a_3,a_5,a_6)=(1,3,10,10,16)$
	$(a_2,a_3,a_5,a_6)=(3,12,10,15)$
	$(a_0,a_2,a_4,a_5,a_6)=(1,4,15,6,14)$
	$(a_0,a_3,a_5,a_6)=(1,13,13,13)$

Lemma 3.6. *The spectrum of a projective* $[17,5,9]_3$ *code satisfies* $a_1 \le 12$.

Proof. Let $\mathcal{C}$ be a $[17,5,9]_3$ code with $\gamma_0 = 1$ and $a_1 > 0$. Let Δ be a 1-solid in $\mathrm{PG}(4,3)$ and let c_j be the number of j-solids through a fixed t-plane in Δ. From (2.5) and (2.6) with $w = 1$, we have

$$28a_0 + 21a_1 + 15a_2 + 10a_3 + 6a_4 + 3a_5 + a_6 = 396 \tag{3.1}$$

and

$$\sum_j (8-j)c_j = 8 - 3t \tag{3.2}$$

with $\sum_j c_j = 3$. Suppose $a_1 \ge 13$. Note that the spectrum of Δ is $(\tau_0, \tau_1) = (27, 13)$. Let $L = 28c_0 + 21c_1 + 15c_2 + 10c_3 + 6c_4 + 3c_5 + c_6$ from (3.1). Since the solution of c_j's in (3.2) with $c_1 > 0$ is $(c_1, c_7, c_8) = (1,1,1)$ for $t = 0$ giving $L = 21$ and since the minimum possible contributions of c_j's in (3.2) to L are $(c_5, c_6) = (2,1)$ for $t = 0$ giving $L = 7$ and $(c_6, c_7) = (2,1)$ for $t = 1$ giving $L = 2$, we get

$$396 = (\text{LHS of (3.1)}) \ge 21 \times 12 + 7(\tau_0 - 12) + 2\tau_1 + 21 = 404,$$

a contradiction. Hence, $a_1 \le 12$. □

Lemma 3.7. *The spectrum of a* $[65,5,42]_3$ *code satisfies*
(1) $a_i = 0$ *for all* $i \notin \{2,5,8,11,14,17,20,23\}$,
(2) $\lambda_2 \le 9$ *if* $a_{14} > 0$.

Proof. Let $\mathcal{C}$ be a $[65,5,42]_3$ code. Note that $\gamma_0 \le 2$ by Lemma 2.1. A w-solid with a t-plane satisfies

$$t \le \frac{w+4}{3} \tag{3.3}$$

by Lemma 2.2. If there exists a 3-solid, it follows from (3.3) that there exists no 3-plane, a contradiction. If there exists a 21-solid, it corresponds to a $[21,4,d_0]_3$ code

Table 2: The spectra of some ternary linear codes of dimension 5.

parameters	possible spectra	reference
$[6,5,2]_3$	$(a_0,a_1,a_2,a_3,a_4) = (11,30,45,20,15)$	[2]
$[8,5,3]_3$	$(a_0,a_1,a_2,a_3,a_4,a_5) = (4,22,30,32,23,10)$	[5]
	$(a_0,a_1,a_2,a_3,a_4,a_5) = (3,24,32,24,30,8)$	
	$(a_0,a_2,a_3,a_4,a_5) = (6,15,38,30,21,11)$	
$[9,5,4]_3$	$(a_0,a_1,a_2,a_3,a_4,a_5) = (1,18,36,12,36,18)$	[5]
$[10,5,5]_3$	$(a_1,a_2,a_4,a_5) = (10,45,30,36)$	[5]
$[11,5,6]_3$	$(a_2,a_5) = (55,66)$	[5]
$[15,5,8]_3$	$(a_1,a_3,a_4,a_6,a_7) = (6,15,42,25,33)$	[5]
	$(a_0,a_1,a_3,a_4,a_6,a_7) = (1,5,13,44,26,32)$	
	$(a_1,a_2,a_3,a_4,a_5,a_6,a_7) = (5,5,20,10,31,20,30)$	
$[16,5,9]_3$	$(a_1,a_4,a_7) = (6,57,58)$	[5]
$[18,5,10]_3$	$(a_0,a_2,a_4,a_5,a_6,a_7,a_8) = (1,9,18,18,12,36,27)$	[5]
	$(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8) = (2,8,2,14,20,11,38,26)$	
$[19,5,11]_3$	$(a_1,a_2,a_4,a_5,a_7,a_8) = (1,9,9,27,30,45)$	[5]
$[20,5,12]_3$	$(a_2,a_5,a_8) = (10,36,75)$	[5]
$[28,5,17]_3$	$(a_1,a_5,a_8,a_{10},a_{11}) = (1,18,18,39,45)$	[2]
	$(a_2,a_4,a_5,a_7,a_8,a_{10},a_{11}) = (1,4,14,4,14,32,52)$	
	$(a_2,a_4,a_5,a_7,a_8,a_{10},a_{11}) = (1,2,16,8,10,30,54)$	
$[29,5,18]_3$	$(a_2,a_5,a_8,a_{11}) = (1,18,18,84)$	[2]
$[37,5,23]_3$	$(a_2,a_7,a_8,a_{10},a_{11},a_{13},a_{14}) = (1,4,14,5,13,31,53)$	[2]
	$(a_1,a_8,a_{10},a_{11},a_{13},a_{14}) = (1,18,9,9,30,54)$	
$[38,5,24]_3$	$(a_2,a_8,a_{11},a_{14}) = (1,18,18,84)$	[2]
$[47,5,30]_3$	$(a_5,a_8,a_{11},a_{14},a_{17}) = (1,4,10,23,83)$	[2]
	$(a_2,a_{11},a_{14},a_{17}) = (1,18,18,84)$	
$[53,5,34]_3$	$(a_8,a_9,a_{10},a_{17},a_{18},a_{19}) = (1,2,8,12,52,46)$	[20]
	$(a_9,a_{10},a_{17},a_{18},a_{19}) = (4,7,13,50,47)$	
$[54,5,35]_3$	$(a_9,a_{10},a_{18},a_{19}) = (2,9,38,72)$	[20]
$[68,5,44]_3$	$(a_{14},a_{15},a_{23},a_{24}) = (1,15,39,66)$	[20]
	$(a_{14},a_{15},a_{23},a_{24}) = (4,12,36,69)$	
$[69,5,45]_3$	$(a_{15},a_{24}) = (16,105)$	[4]

with $d_0 \geq 21 - \lfloor (21+4)/3 \rfloor = 13$, which does not exist by Theorem 1.1. In this way, one can get $a_i = 0$ for all $i \notin \{0,1,2,5,8,9,10,11,14,15,17,18,19,20,23\}$. This procedure to rule out some possible multiplicities of hyperplanes using Theorems 1.1, 1.2 and Lemma 2.2 is called the **first sieve** [13]. Let Δ_w be a w-solid in PG(4,3). Lemma 2.2 (3) gives $\sum_j c_j = 3$ and

$$\sum_j (23-j)c_j = w + 4 - 3t. \tag{3.4}$$

One can determine $\gamma_2(\Delta_w)$ from (3.3) and Theorem 1.1. For example, Δ_{14} corresponds to a $[14,4,d_1]_3$ code with $d_1 \geq 14 - 6 = 8$ from (3.3). Since a $[14,4,9]_3$ code does not exist by Theorem 1.1, we have $d_1 = 8$ and $\gamma_2(\Delta_{14}) = 14 - d_1 = 6$.

Setting $(w,t) = (w, \gamma_2(\Delta_w))$ for $w \not\equiv 2 \pmod 3$, (3.4) has no solution. Hence $a_1 = a_9 = a_{10} = a_{11} = a_{15} = a_{18} = a_{19} = a_0 = 0$ and (1) follows.

Assume $a_{14} > 0$. Then, $a_2 = 0$ since the RHS of (3.4) is at most 18 for $w = 14$. From (2.2)-(2.4), one can get

$$15a_5 + 10a_8 + 6a_{11} + 3a_{14} + a_{17} = 30 + 3\lambda_2. \tag{3.5}$$

Assume $a_5 > 0$. From Table 1, the spectrum of a 5-solid is $(\tau_0,\tau_1,\tau_2,\tau_3) = (5,15,10,10)$. Setting $w = 5$, the maximum possible contributions of c_j's in (3.4) to the LHS of (3.5) are $(c_{14}, c_{23}) = (1, 2)$ for $t = 0$; $(c_{17}, c_{23}) = (1, 2)$ for $t = 1$; $(c_{20}, c_{23}) = (1, 2)$ for $t = 2$; $c_{23} = 3$ for $t = 3$. Hence we get

$$30 + 3\lambda_2 = (\text{LHS of (3.5)}) \leq 3\tau_0 + \tau_1 + 15 = 45,$$

giving $\lambda_2 \leq 5$. Similarly, using the possible spectra for a 8-solid from Table 1, we get $\lambda_2 \leq 6$ when $a_8 > 0$. Hence, assume $a_5 = a_8 = 0$. Using (3.4) and (3.5) with the possible spectra of a 14-solid from Table 1, one can get $\lambda_2 \leq 9$. □

Using Lemmas 2.2, 3.4 and Theorem 1.1, we get the following by the first sieve.

Lemma 3.8. *The spectrum of a* $[56, 5, 36]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{0, 1, 2, 5, 8\text{-}11, 14, 15, 17\text{-}20\}$.

Lemma 3.9. *The spectrum of a* $[77, 5, 50]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{0, 1, 5, 8, 9, 10, 14, 15, 17\text{-}20, 23\text{-}27\}$.

4 Proof of Theorem 1.3

Lemma 4.1. *There exists no* $[75, 6, 48]_3$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[75, 6, 48]_3$ code. We have $\gamma_0 \leq 2$ by Lemma 2.1. If a 2-point P in $\Sigma = \mathrm{PG}(5, 3)$ exists, then the projection of $\mathcal{M}_\mathcal{C}$ from P onto a hyperplane not on P gives a multiset for a $[73, 5, 48]_3$ code, which does not exist by Theorem 1.2, a contradiction. Hence, Σ has no 2-point. Let Π be a γ_4-hyperplane. It follows from Lemma 3.3 that Π has no t-solid for $t = 2, 3, 6$. An i-hyperplane Π_i with a t-solid satisfies

$$t \leq \frac{i + 6}{3} \tag{4.1}$$

by Lemma 2.2. Hence, using Theorem 1.2 and Lemma 2.2, we get $a_i = 0$ for all

$$i \notin \{0, 1, 6, 9, 10, 11, 15, 16, 18, 19, 20, 21, 24, 25, 27\}$$

by the first sieve. (2.6) for Π_i through a t-solid yields

$$\sum_{j=0}^{27} (27 - j)c_j = i + 6 - 3t \tag{4.2}$$

with $\sum_j c_j = 3$. From (4.1) and Theorem 1.2, we have $\gamma_3(\Pi_{25}) = 10$, but (4.2) has no solution for $(i, t) = (25, 10)$. Thus, $a_{25} = 0$. We obtain $a_i = 0$ for all $i \not\equiv 0 \pmod 3$ similarly. Hence we get $a_i = 0$ for all $i \notin \{0, 6, 9, 15, 18, 21, 24, 27\}$. From (2.2)-(2.4), one can get

$$36a_0 + 21a_6 + 15a_9 + 6a_{15} + 3a_{18} + a_{21} = 229. \tag{4.3}$$

Suppose $a_0 > 0$. Since Π_0 has spectrum $\tau_0 = 121$ and since the solution of (4.2) with $i = t = 0$ that maximizes the LHS of (4.3) is $(c_{21}, c_{27}) = (1, 2)$, we get

$$229 \leq \tau_0 + 36 = 157,$$

a contradiction. Hence $a_0 = 0$.

Suppose $a_6 > 0$. Then, Π_6 has spectrum $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4) = (11, 30, 45, 20, 15)$ from Table 2. Since the solutions of (4.2) with $i = 6$ that maximizes the LHS of (4.3) is $(c_{15}, c_{27}) = (1, 2)$ for $t = 0$; $(c_{18}, c_{27}) = (1, 2)$ for $t = 1$; $(c_{21}, c_{27}) = (1, 2)$ for $t = 2$; $(c_{24}, c_{27}) = (1, 2)$ for $t = 3$; $c_{27} = 3$ for $t = 4$, we get

$$229 \leq 6\tau_0 + 3\tau_1 + \tau_2 + 21 = 222,$$

a contradiction. Hence $a_6 = 0$.

Suppose $a_9 > 0$. Then, Π_9 has a 3-solid from Table 2, which contradicts that (4.2) has no solution for $t = 3$ since $c_{27} = 0$ by Lemma 3.3. Thus $a_9 = 0$. Setting $(i, t) = (18, 6)$, $(15, 6)$ in (4.2), one can get a contradiction since $c_{27} = 0$ for $t = 6$. Hence, $a_{15} = a_{18} = 0$. Now, we have $a_i = 0$ for all $i \notin \{21, 24, 27\}$.

Setting $i = 27$, (4.2) has no solution for $t = 0, 1$, which contradicts that Π has a 0-solid or a 1-solid by Lemma 3.3. This completes the proof of Lemma 4.1. □

As in the above proof to show $a_i = 0$ for $i = 0, 6$, we often obtain a contradiction by eliminating the maximum possible value of (2.5) (or some similar equation on the spectrum of $\mathcal{C}$) using (4.2) and the possible spectra of the putative i-hyperplane. We refer to this proof technique as the "i-**Max**" in what follows.

Lemma 4.2. *There exists no* $[77, 6, 49]_3$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[77, 6, 49]_3$ code. We first note that there is no 2-point in $\Sigma = \mathrm{PG}(5, 3)$ since there exists no $[75, 5, 49]_3$ code. An i-hyperplane Π_i with a t-solid satisfies $t \leq (i + 7)/3$ by Lemma 2.2. Let Π be a γ_4-hyperplane. From Table 2, Π has spectrum $(\tau_1, \tau_5, \tau_8, \tau_{10}, \tau_{11}) = (1, 18, 18, 39, 45)$ since the other possible spectra need a 2-point. Using Theorem 1.2 and Lemma 2.2, one can get $a_i = 0$ for all

$$i \notin \{0, 1, 2, 8, 9, 10, 11, 17, 18, 19, 20, 23, 24, 25, 26, 27, 28\}$$

by the first sieve. It follows from (2.5) and (2.6) that

$$\sum_{j=0}^{26} \binom{28 - i}{2} a_i = 3073, \tag{4.4}$$

$$\sum_{j=0}^{26} (27 - j) c_j = i + 7 - 3t \tag{4.5}$$

with $\sum_j c_j = 3$. Note that the spectrum of Π_i is $\tau_0 = 121$ for $i = 0$, $(\tau_0, \tau_1) = (81, 40)$ for $i = 1$, $(\tau_0, \tau_1, \tau_2) = (54, 54, 13)$ for $i = 2$ and see Table 2 for $i \in \{$8-11, 18-20$\}$. One can prove $a_i = 0$ by the i-Max for $i = 0, 1, 2, 8, 9, 10, 11, 20, 19, 18$ in this order using the possible spectra for Π_i, see [22] for the detail. Now, we have $a_i = 0$ for all $i \notin \{17, 23, 24, 25, 26, 27, 28\}$. Setting $i = 28$, (4.5) has no solution for $t = 1$, which contradicts that Π has a 1-solid. This completes the proof of Lemma 4.2. □

Lemma 4.3. *There exists no* $[102, 6, 66]_3$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[102, 6, 66]_3$ code. An i-hyperplane Π_i with a t-solid in $\Sigma = \mathrm{PG}(5, 3)$ satisfies $t \le (i + 6)/3$ by Lemma 2.2. Hence, a γ_4-hyperplane Π corresponds to a $[36, 5, 22]_3$ code. From Lemma 3.5, the spectrum of Π satisfies

$$\tau_4 = \tau_5 = \lambda_2(\Pi) = 0. \tag{4.6}$$

Hence Σ has no 2-point by Lemma 2.3. Using Theorem 1.2, Lemma 2.2 and (4.6), one can get $a_i = 0$ for all

$$i \notin \{0, 1, 2, 3, 12, 15, 16, 18, 19, 20, 21, 24, 25, 27, 28, 29, 30, 33, 34, 36\}$$

by the first sieve. It follows from (2.6) that

$$\sum_{j=0}^{35} (36 - j)c_j = i + 6 - 3t \tag{4.7}$$

with $\sum_j c_j = 3$. Setting $(i, t) = (34, 13)$, (4.7) has no solution, a contradiction. Similarly, one can rule out Π_i for $i \not\equiv 0 \pmod 3$ since (4.7) has no solution for such an i and $t = \gamma_3(\Pi_i)$. Hence, $a_i = 0$ for all $i \notin \{0, 3, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$. From (2.2)-(2.4), one can get

$$66a_0 + 55a_3 + 28a_{12} + 21a_{15} + 15a_{18} + 10a_{21} + 6a_{24} + 3a_{27} + a_{30} = 292. \tag{4.8}$$

Note that the spectrum of Π_i is $\tau_0 = 121$ for $i = 0$, $(\tau_0, \tau_1) = (81, 40)$ for $i = 1$, $(\tau_0, \tau_1, \tau_3) = (27, 81, 13)$ or $(\tau_0, \tau_1, \tau_2, \tau_3) = (36, 54, 27, 4)$ for $i = 3$. See Lemma 3.3 for $i = 27$ and Table 2 for $i = 15, 18$. One can prove $a_i = 0$ by the i-Max for $i = 0, 3, 18, 27, 15$ in this order using the possible spectra for Π_i.

Suppose $a_{12} > 0$. Setting $(i, t) = (12, 5)$, (4.7) has no solution, for $c_{36} = 0$ from (4.6), a contradiction. Hence, Π_{12} has spectrum $(\tau_0, \tau_3, \tau_6) = (3, 76, 42)$ by Lemma 3.1. Then, we can get a contradiction by the 12-Max. Thus $a_{12} = 0$.

Suppose $a_{21} > 0$ and let Π_{21} be a 21-hyperplane with spectrum $(\tau_0, \tau_1, \cdots, \tau_9)$. Recall that the solutions of (4.7) satisfy $c_{36} = 0$ for $t = 4, 5$ from (4.6). Setting $i = 21$ in (4.7), the maximum possible contribution of c_j's to the LHS of (4.8) are $(c_{21}, c_{24}, c_{36}) = (1, 1, 1)$ for $t = 0$; $(c_{21}, c_{30}, c_{33}) = (1, 1, 1)$ for $t = 1$; $(c_{21}, c_{30}, c_{36}) = (1, 1, 1)$ for $t = 2$; $(c_{21}, c_{33}, c_{36}) = (1, 1, 1)$ for $t = 3$; $(c_{30}, c_{33}) = (2, 1)$ for $t = 4$; $(c_{30}, c_{33}) = (1, 2)$ for $t = 5$; $(c_{30}, c_{33}, c_{36}) = (1, 1, 1)$ for $t = 6$; $(c_{30}, c_{36}) = (1, 2)$ for $t = 7$; $(c_{33}, c_{36}) = (1, 2)$ for $t = 8$; $c_{36} = 3$ for $t = 9$. Hence, from (4.8), we get

$$292 \le 16\tau_0 + 11\tau_1 + 11\tau_2 + 10\tau_3 + 2\tau_4 + \tau_5 + \tau_6 + \tau_7 + 10 \tag{4.9}$$

by the 21-Max. On the other hand, the equalities (2.2)-(2.4) yield the following:

$$\tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_6 + \tau_7 + \tau_8 + \tau_9 = 121, \tag{4.10}$$

$$\tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 + 5\tau_5 + 6\tau_6 + 7\tau_7 + 8\tau_8 + 9\tau_9 = 840, \tag{4.11}$$

$$\tau_2 + 3\tau_3 + 6\tau_4 + 10\tau_5 + 15\tau_6 + 21\tau_7 + 28\tau_8 + 36\tau_9 = 2730. \tag{4.12}$$

Then, $((4.10) \times 333 - (4.11) \times 69 + (4.12) \times 8)/15$ gives

$$\frac{111\tau_0}{5} + \frac{88\tau_1}{5} + \frac{203\tau_2}{15} + 10\tau_3 + 7\tau_4 + \frac{68\tau_5}{15} + \frac{13\tau_6}{5} + \frac{6\tau_7}{5} + \frac{\tau_8}{3} = \frac{1391}{5}. \quad (4.13)$$

Since (RHS of (4.9)) $\leq 10+$ (LHS of (4.13)) $= 10 + 1391/5 < 292$, a contradiction. Hence $a_{21} = 0$.

Now, we have $a_i = 0$ for all $i \notin \{24, 30, 33, 36\}$. Setting $i = 36$, (4.7) has no solution for $t = 0, 1, 3$. Hence, we may assume that the spectrum of Π is one of (d), (e), (f) in Lemma 3.5. Then, by the 36-Max, one can get a contradiction for each of the spectra. This completes the proof of Lemma 4.3. □

In the above proof, we proved $a_i = 0$ by the i-Max not using the possible spectra of an i-hyperplane for $i = 21$. We refer to this modified i-Max as "i-**Max-NS**" in what follows, which is effective when the possible spectra are too many or unknown.

Lemma 4.4. *There exists no $[104, 6, 67]_3$ code.*

Proof. Let $\mathcal{C}$ be a putative $[104, 6, 67]_3$ code. An i-hyperplane Π_i with a t-solid in $\Sigma = \mathrm{PG}(5, 3)$ satisfies $t \leq (i + 7)/3$ by Lemma 2.2. Since a γ_4-hyperplane Π corresponds to a $[37, 5, 23]_3$ code, the spectrum of Π satisfies

$$\tau_0 = \tau_3 = \tau_4 = \tau_5 = \tau_6 = \tau_9 = \tau_{12} = \lambda_2(\Pi) = 0 \quad (4.14)$$

from Table 2. Hence Σ has no 2-point by Lemma 2.3. If a 0-hyperplane Π_0 exists, then $\Pi_0 \cap \Pi$ is a 0-solid, a contradiction. Hence $a_0 = 0$. Using Theorem 1.2, Lemma 2.2 and (4.14), one can get $a_i = 0$ for all $i \notin \{1, 2, 11, 14\text{-}20, 23\text{-}29, 32\text{-}37\}$ by the first sieve. It follows from (2.5) and (2.6) that

$$\sum_{j=0}^{35} \binom{37-i}{2} = 3640, \quad (4.15)$$

$$\sum_{j=0}^{35} (36 - j)c_j = i + 6 - 3t \quad (4.16)$$

with $\sum_j c_j = 3$. One can prove that $a_i = 0$ for $i = 1, 2, 11, 16, 15, 18, 19, 20, 29, 28, 27$ in this order by the i-Max using the possible spectra of an i-hyperplane Π_i from Table 2 for $i \geq 11$ and Lemma 3.3, see the proof of Lemma 4.2 for the spectra of Π_1 and Π_2. Note that a putative Π_{28} has spectrum $(\tau_1, \tau_5, \tau_8, \tau_{10}, \tau_{11}) = (1, 18, 18, 39, 45)$ since $\gamma_0 = 1$.

Next, we shall prove $a_{14} = 0$ by the 14-Max-NS. Suppose $a_{14} > 0$ and let Π_{14} be a 14-hyperplane with spectrum $(\tau_0, \tau_1, \cdots, \tau_7)$. Recall that the solutions of (4.16) satisfy $c_{37} = 0$ for $t = 0, 3, 4, 5, 6$ and $c_{36} = 0$ for $t = 4, 5$ from (4.14) and Lemma 3.5. Setting $i = 14$ in (4.16), the maximum possible contribution of c_j's to the LHS of (4.8) are $(c_{23}, c_{32}, c_{35}) = (1, 1, 1)$ for $t = 0$; $(c_{23}, c_{33}, c_{37}) = (1, 1, 1)$ for $t = 1$; $(c_{23}, c_{36}, c_{37}) = (1, 1, 1)$ for $t = 2$; $(c_{32}, c_{35}) = (2, 1)$ for $t = 3$; $(c_{32}, c_{35}) = (1, 2)$ for

$t = 4$; $c_{35} = 3$ for $t = 5$; $c_{36} = 3$ for $t = 6$; $c_{37} = 3$ for $t = 7$. Hence, from (4.15), we get

$$3640 \leq 102\tau_0 + 97\tau_1 + 91\tau_2 + 21\tau_3 + 12\tau_4 + 3\tau_5 + 253 \tag{4.17}$$

by the 14-Max. On the other hand, the equalities (2.2)-(2.4) give the following:

$$\tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_6 + \tau_7 = 121, \tag{4.18}$$
$$\tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 + 5\tau_5 + 6\tau_6 + 7\tau_7 = 560, \tag{4.19}$$
$$\tau_2 + 3\tau_3 + 6\tau_4 + 10\tau_5 + 15\tau_6 + 21\tau_7 = 1183. \tag{4.20}$$

Then, $(4.18) \times 214 - (4.19) \times 68 + (4.20) \times 13$ gives

$$214\tau_0 + 146\tau_1 + 91\tau_2 + 49\tau_3 + 20\tau_4 + 4\tau_5 + \tau_6 + 11\tau_7 = 3193. \tag{4.21}$$

Since (RHS of (4.17)) $<$ 253+ (LHS of (4.21)) $= 3446 < 3640$, a contradiction. Hence $a_{14} = 0$. Then, we get $a_{24} = 0$ by the 24-Max using Lemma 3.2 and also $a_{23} = 0$ by the 23-Max-NS, see [22].

Suppose $a_{17} > 0$ and let Π_{17} be a 17-hyperplane with spectrum $(\tau_0, \tau_1, \cdots, \tau_8)$. Recall that the solutions of (4.16) satisfy $c_{37} = 0$ for $t = 0, 3, 4, 5, 6$ and $c_{25} = 0$ for $t = 2, 3$ from (4.14) and Table 2. Setting $i = 17$ in (4.16), the maximum possible contribution of c_j's to the LHS of (4.8) are $(c_{17}, c_{34}, c_{36}) = (1, 1, 1)$ for $t = 0$; $(c_{17}, c_{36}, c_{37}) = (1, 1, 1)$ for $t = 1$; $(c_{26}, c_{32}, c_{35}) = (1, 1, 1)$ for $t = 2$; $(c_{26}, c_{35}, c_{36}) = (1, , 1)$ for $t = 3$; $(c_{32}, c_{35}) = (2, 1)$ for $t = 4$; $(c_{32}, c_{35}) = (1, 2)$ for $t = 5$; $(c_{33}, c_{36}) = (1, 2)$ for $t = 6$; $(c_{34}, c_{37}) = (1, 2)$ for $t = 7$; $c_{37} = 3$ for $t = 8$. Hence, from (4.15), we get

$$3640 \leq 193\tau_0 + 190\tau_1 + 66\tau_2 + 58\tau_3 + 21\tau_4 + 12\tau_5 + 6\tau_6 + 3\tau_7 + 190 \tag{4.22}$$

by the 17-Max. On the other hand, the equalities (2.2)-(2.4) give the following:

$$\tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_6 + \tau_7 + \tau_8 = 121, \tag{4.23}$$
$$\tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 + 5\tau_5 + 6\tau_6 + 7\tau_7 + 8\tau_8 = 680, \tag{4.24}$$
$$\tau_2 + 3\tau_3 + 6\tau_4 + 10\tau_5 + 15\tau_6 + 21\tau_7 + 28\tau_8 = 1768. \tag{4.25}$$

Then, $(4.23) \times 234 - (4.24) \times 63 + (4.25) \times 10$ gives

$$234\tau_0 + 171\tau_1 + 118\tau_2 + 75\tau_3 + 42\tau_4 + 19\tau_5 + 6\tau_6 + 3\tau_7 + 10\tau_8 = 3154. \tag{4.26}$$

Since (RHS of (4.22)) $< 190 + 19\tau_1 +$ (LHS of (4.26)) $= 3344 + 19\tau_1$, we obtain $\tau_1 > (3640 - 3344)/19$, i.e., $\tau_1 \geq 16$, which contradicts Lemma 3.6. Hence $a_{17} = 0$.

Now, we have $a_i = 0$ for all $i \notin \{25, 26, 32, 33, \ldots, 37\}$. Recall that Π satisfies either $a_1 > 0$ or $a_2 > 0$ from Table 2, whilst (4.16) with $i = 37$ has no solution for $t = 1, 2$, a contradiction. This completes the proof of Lemma 4.4. $\square$

Lemma 4.5. *There exists no* $[227, 6, 150]_3$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[227, 6, 150]_3$ code. Let Π be a γ_4-hyperplane in $\Sigma = \mathrm{PG}(5, 3)$. We have $\gamma_0 \le 2$ by Lemma 2.1. An i-hyperplane Π_i with a t-solid satisfies $t \le (i+4)/3$ by Lemma 2.2. Using Theorem 1.2 and Lemmas 2.2 and 3.9, one can get $a_i = 0$ for all $i \notin \{0, 1, 11, 20, 38, 47, 53\text{-}56, 65, 68, 69, 74, 77\}$ by the first sieve. Lemma 2.2 (3) for Π_i gives $\sum_j c_j = 3$ and

$$\sum_j (77-j)c_j = 77c_0 + \cdots + 8c_{69} + 3c_{74} = i + 4 - 3t. \tag{4.27}$$

Since (4.27) has no solution for $(i, t) = (0, 0)$, $(1, 1)$, $(54, 19)$, $(55, 19)$, $(69, 24)$, we obtain $a_0 = a_1 = a_{54} = a_{55} = a_{69} = 0$. From (2.2)-(2.4), we get

$$77a_{11} + 57a_{20} + 26a_{38} + 15a_{47} + \frac{28a_{53}}{3} + 7a_{56} + 2a_{65} + a_{68} = 113 + 3\lambda_2. \tag{4.28}$$

One can prove $a_i = 0$ for $i \in \{11, 20, 38, 47, 53\}$ by the i-Max using the possible spectra for an i-hyperplane with $i \in \{11, 20, 38, 47, 53\}$ from Table 2.

Suppose $a_{68} > 0$ and let Π_{68} be a 68-hyperplane. It follows from Table 2 that Π_{68} has spectrum $(\tau_{14}, \tau_{15}, \tau_{23}, \tau_{24}) = (1, 15, 39, 66)$ with $\lambda_2' = 5$ or $(\tau_{14}, \tau_{15}, \tau_{23}, \tau_{24}) = (4, 12, 36, 69)$ with $\lambda_2' = 6$, where $\lambda_2' = \lambda_2(\Pi_{68})$, and the solutions of (4.27) that maximize the LHS of (4.28) are $(c_{56}, c_{68}, c_{77}) = (1, 1, 1)$ for $t = 14$; $(c_{56}, c_{74}) = (1, 2)$ for $t = 15$; $(c_{74}, c_{77}) = (1, 2)$ for $t = 23$; $c_{77} = 3$ for $t = 24$. For the former spectrum of Π_{68}, we get

$$128 = 113 + 3\lambda_2' \le 113 + 3\lambda_2 \le 8\tau_{14} + 7\tau_{15} + 1 = 114,$$

a contradiction. We can get a contradiction for the latter spectrum of Π_{68} as well. Thus, $a_{68} = 0$. Now, we have $a_i = 0$ for all $i \notin \{56, 65, 74, 77\}$ and (4.28) becomes

$$7a_{56} + 2a_{65} = 113 + 3\lambda_2. \tag{4.29}$$

We shall prove $a_{65} = 0$ by the 65-Max-NS. Suppose $a_{65} > 0$ and let Π_{65} be a 65-hyperplane with spectrum $(\tau_2, \tau_5, \cdots, \tau_{23})$, see Lemma 3.7. Setting $i = 65$, the solutions of (4.27) that maximize the LHS of (4.29) are $c_{56} = 3$ for $t = 2$; $(c_{56}, c_{65}) = (2, 1)$ for $t = 5$; $(c_{56}, c_{74}) = (2, 1)$ for $t = 8$; $(c_{56}, c_{65}, c_{74}) = (1, 1, 1)$ for $t = 11$; $(c_{56}, c_{74}) = (1, 2)$ for $t = 14$; $(c_{65}, c_{74}) = (1, 2)$ for $t = 17$; $c_{74} = 3$ for $t = 20$; $c_{77} = 3$ for $t = 23$. Hence we get

$$(\text{LHS of (4.29)}) \le 21\tau_2 + 16\tau_5 + 14\tau_8 + 9\tau_{11} + 7\tau_{14} + 2\tau_{17} + 2 \tag{4.30}$$

by the 65-Max. On the other hand, the equalities (2.2)-(2.4) for Π_{65} are

$$\tau_2 + \tau_5 + \tau_8 + \tau_{11} + \tau_{14} + \tau_{17} + \tau_{20} + \tau_{23} = 121, \tag{4.31}$$

$$2\tau_2 + 5\tau_5 + 8\tau_8 + 11\tau_{11} + 14\tau_{14} + 17\tau_{17} + 20\tau_{20} + 23\tau_{23} = 2600, \tag{4.32}$$

$$\tau_2 + 10\tau_5 + 28\tau_8 + 55\tau_{11} + 91\tau_{14} + 136\tau_{17} + 190\tau_{20} + 253\tau_{23} = 27040 + 27\lambda_2', \tag{4.33}$$

respectively, where $\lambda_2' = \lambda_2(\Pi_{65})$. Then, $((4.31) \times 299 - (4.32) \times 24 + (4.33))/9$ yields

$$28\tau_2 + 21\tau_5 + 15\tau_8 + 10\tau_{11} + 6\tau_{14} + 3\tau_{17} + \tau_{20} = 91 + 3\lambda_2'. \tag{4.34}$$

If $\tau_{14} = 0$, it follows from (4.30) and (4.34) that $111+3\lambda_2 \le 91+3\lambda_2'$, a contradiction. Hence, assume $\tau_{14} > 0$. $((4.31) \times 483 - (4.32) \times 43 + (4.33) \times 2)/9$ yields

$$\frac{133\tau_2}{3} + 32\tau_5 + \frac{65\tau_8}{3} + \frac{40\tau_{11}}{3} + 7\tau_{14} + \frac{8\tau_{17}}{3} + \frac{\tau_{20}}{3} = \frac{241}{3} + 6\lambda_2'. \tag{4.35}$$

Let $y = \lambda_2 - \lambda_2'$ (≥ 0). Then, from (4.30) and (4.35), we get

$$111 + 3(\lambda_2' + y) \le 241/3 + 6\lambda_2',$$

giving $\lambda_2' \ge 92/9$, which contradicts Lemma 3.7. Hence, $a_{65} = 0$.

Now, $a_i = 0$ for all $i \notin \{56, 74, 77\}$ and we have $a_{56} > 0$ from (4.29). From Lemma 2.1, we have $\gamma_1 \le 4$. If Π_{56} has a 2-point P, counting the multiplicities of lines through P which are not in Π_{56}, we get $n \le (4-2)(\theta_4 - \theta_3) + 56 = 218$, a contradiction. Hence, Π_{56} has no 2-point. From Lemma 3.8 and (4.27), the spectrum of Π_{56} satisfies $\tau_j = 0$ for all $j \notin \{5, 11, 17, 18, 19, 20\}$. One can get a contradiction by the 56-Max-NS as follows. By the 56-Max, we obtain

$$113 + 3\lambda_2 \le 14\tau_5 + 7\tau_{11} + 7. \tag{4.36}$$

On the other hand, (2.5) yields $105\tau_5 + 36\tau_{11} + 3\tau_{17} + \tau_{18} = 450$. Hence

$$113 + 3\lambda_2 \le (105\tau_5 + 36\tau_{11} + 3\tau_{17} + \tau_{18})/5 + 7 = 97,$$

a contradiction. This completes the proof of Lemma 4.5. $\square$

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1, k, d+1]_q$ code which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable*. We use the following well-known theorem.

Theorem 4.6 ([10, 11])**.** *Suppose $\mathcal{C}$ is an $[n, k, d]_q$ code with gcd$(d, q) = 1$. If $A_i > 0$ implies $i \equiv 0$ or $d \pmod q$, then $\mathcal{C}$ is extendable.*

We also need the following result about non-extendable code, which can be derived from Theorems 1.1, 1.2 in [17] and Theorem 3.13 in [25].

Lemma 4.7. *Let $\mathcal{C}$ be a non-extendable $[n, 6, d]_3$ code with gcd$(d, 3) = 1$. Then, for any $\pi \in \mathcal{F}_4$ with $m_{\mathcal{M}_\mathcal{C}}(\pi) \not\equiv n, n-d \pmod 3$, there are at most 54 solids Δ in π such that $m_{\mathcal{M}_\mathcal{C}}(\pi_j) \equiv n \pmod 3$ for $j = 1, 2$ and $m_{\mathcal{M}_\mathcal{C}}(\pi_3) \equiv n - d \pmod 3$, where π_1, π_2, π_3 are the hyperplanes through Δ other than π.*

Lemma 4.8. *There exists no $[226, 6, 149]_3$ code.*

Proof. Let $\mathcal{C}$ be a putative $[226, 6, 149]_3$ code. Then, $\mathcal{C}$ is not extendable since a $[227, 6, 150]_3$ code does not exist by Lemma 4.5. Let Π be a γ_4-hyperplane in $\Sigma = \mathrm{PG}(5, 3)$. An i-hyperplane with a t-solid satisfies $t \le (i+5)/3$ by Lemma 2.2. Recall that Π has no t-solid for $t \in \{2, 3, 4, 6, 7, 11, 12, 13, 16, 21, 22\}$ by Lemma 3.9. Using Theorem 1.2 and Lemmas 2.2, 3.9, one can get $a_i = 0$ for all

$$i \notin \{0, 1, 10, 11, 19, 20, 25, 37, 38, 46, 47, 49, 52\text{-}56, 64, 65, 67\text{-}70, 73, 74, 76, 77\}$$

by the first sieve. It follows from (2.5) and (2.6) that

$$\sum_{i=0}^{74} \binom{77-i}{2} a_i = 3768 + 81\lambda_2, \tag{4.37}$$

$$\sum_j (77-j)c_j = i + 5 - 3t \tag{4.38}$$

with $\sum_j c_j = 3$. Suppose $a_0 > 0$ and let Π_0 be a 0-hyperplane. Since Π_0 has spectrum $\tau_0 = 121$ and since the solution of (4.38) with $i = t = 0$ that maximizes the LHS of (4.37) is $(c_{73}, c_{76}, c_{77}) = (1, 1, 1)$, the 0-Max gives

$$3768 \le 3768 + 81\lambda_2 \le 6 \cdot 121 + 2926 = 3652,$$

a contradiction. Hence $a_0 = 0$.

Suppose $a_{54} > 0$ and let Π_{54} be a 54-hyperplane. Then, Π_{54} has spectrum $(\tau_9, \tau_{10}, \tau_{18}, \tau_{19}) = (2, 9, 38, 72)$ from Table 2. Setting $w = 54$ and $t = 19$, (4.38) has the unique solution $(c_{76}, c_{77}) = (2, 1)$, which contradicts Lemma 4.7, for $\tau_{19} = 72$. Hence $a_{54} = 0$. One can prove $a_{69} = 0$ similarly. Now, the spectrum of $\mathcal{C}$ satisfies $a_i = 0$ for all $i \not\equiv 1, 2 \pmod 3$. Applying Theorem 4.6, $\mathcal{C}$ is extendable, which contradicts Lemma 4.5. This completes the proof of Lemma 4.8. □

Now, Theorem 1.3 follows from Lemmas 4.1 - 4.5 and 4.8.

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Table 3. Values and bounds of $n_3(6, d)$ for $d \leq 360$.

d	g	n	d	g	n	d	g	n	d	g	n	d	g	n
1	6	6	37	59	1-2	73	112	114	109	167	168	145	220	1-2
2	7	7	38	60	1-2	74	113	115	110	168	169	146	221	1-2
3	8	9	39	61	1-2	75	114	116	111	169	170	147	222	1-2
4	10	10	40	63	1-2	76	116	118	112	171	172	148	224	1-2
5	11	11	41	64	1-2	77	117	119	113	172	173	149	225	**227**
6	12	12	42	65	1-2	78	118	120	114	173	174	150	226	**228**
7	14	15	43	67	1-3	79	120	122	115	175	176	151	228	1-2
8	15	17	44	68	2-3	80	121	123	116	176	177	152	229	231
9	16	18	45	69	2-3	81	122	124	117	177	178	153	230	232
10	19	20	46	72	1-2	82	127	0-1	118	180	181	154	233	234
11	20	21	47	73	1-2	83	128	0-1	119	181	182	155	234	235
12	21	22	48	74	**76**	84	129	0-1	120	182	183	156	235	236
13	23	24	49	76	**78**	85	131	0-1	121	184	185	157	237	238
14	24	25	50	77	79	86	132	133	122	185	186	158	238	239
15	25	26	51	78	80	87	133	134	123	186	187	159	239	240
16	27	29	52	80	82	88	135	136	124	188	189	160	241	241
17	28	30	53	81	83	89	136	137	125	189	190	161	242	242
18	29	31	54	82	84	90	137	138	126	190	191	162	243	243
19	32	1-2	55	86	1-2	91	140	0-2	127	193	1-2	163	248	248
20	33	1-2	56	87	1-2	92	141	0-2	128	194	1-2	164	249	249
21	34	36	57	88	1-2	93	142	1-2	129	195	1-2	165	250	250
22	36	38	58	90	1-2	94	144	1-2	130	197	199	166	252	252
23	37	39	59	91	1-2	95	145	1-2	131	198	200	167	253	253
24	38	40	60	92	1-2	96	146	1-2	132	199	201	168	254	254
25	40	42	61	94	1-2	97	148	1-2	133	201	203	169	256	257
26	41	43	62	95	97	98	149	1-2	134	202	204	170	257	258
27	42	44	63	96	98	99	150	1-2	135	203	205	171	258	259
28	46	0-1	64	99	1-2	100	153	154	136	207	1-2	172	261	261
29	47	48	65	100	1-2	101	154	155	137	208	1-2	173	262	262
30	48	49	66	101	**103**	102	155	156	138	209	1-2	174	263	263
31	50	51	67	103	**105**	103	157	158	139	211	1-2	175	265	266
32	51	52	68	104	106	104	158	159	140	212	1-2	176	266	267
33	52	53	69	105	107	105	159	160	141	213	1-2	177	267	268
34	54	54	70	107	109	106	161	162	142	215	1-2	178	269	270
35	55	55	71	108	110	107	162	163	143	216	1-2	179	270	271
36	56	56	72	109	111	108	163	164	144	217	1-2	180	271	272

Table 3 (continued).

d	g	n	d	g	n	d	g	n	d	g	n	d	g	n
181	274	275	217	328	328	253	383	383	289	436	437	325	491	491
182	275	276	218	329	329	254	384	384	290	437	438	326	492	492
183	276	277	219	330	330	255	385	385	291	438	439	327	493	493
184	278	279	220	332	332	256	387	387	292	440	441	328	495	495
185	279	280	221	333	333	257	388	388	293	441	442	329	496	496
186	280	281	222	334	334	258	389	389	294	442	443	330	497	497
187	282	1-2	223	336	336	259	391	391	295	444	445	331	499	0-1
188	283	285	224	337	337	260	392	392	296	445	446	332	500	0-1
189	284	286	225	338	338	261	393	393	297	446	447	333	501	0-1
190	288	288	226	341	341	262	396	396	298	450	0-1	334	504	504
191	289	289	227	342	342	263	397	397	299	451	0-1	335	505	505
192	290	290	228	343	343	264	398	398	300	452	0-1	336	506	506
193	292	292	229	345	345	265	400	400	301	454	0-1	337	508	0-1
194	293	293	230	346	346	266	401	401	302	455	456	338	509	0-1
195	294	294	231	347	347	267	402	402	303	456	457	339	510	0-1
196	296	296	232	349	349	268	404	404	304	458	459	340	512	513
197	297	297	233	350	350	269	405	405	305	459	460	341	513	514
198	298	298	234	351	351	270	406	406	306	460	461	342	514	515
199	301	0-1	235	354	354	271	410	410	307	463	0-1	343	517	0-1
200	302	303	236	355	355	272	411	411	308	464	465	344	518	0-1
201	303	304	237	356	356	273	412	412	309	465	466	345	519	0-1
202	305	306	238	358	358	274	414	414	310	467	468	346	521	0-1
203	306	307	239	359	359	275	415	415	311	468	469	347	522	523
204	307	308	240	360	360	276	416	416	312	469	470	348	523	524
205	309	310	241	362	362	277	418	418	313	471	472	349	525	526
206	310	311	242	363	363	278	419	419	314	472	473	350	526	527
207	311	312	243	364	364	279	420	420	315	473	474	351	527	528
208	314	314	244	370	370	280	423	424	316	476	476	352	531	531
209	315	315	245	371	371	281	424	425	317	477	477	353	532	532
210	316	316	246	372	372	282	425	426	318	478	478	354	533	533
211	318	318	247	374	374	283	427	428	319	480	480	355	535	535
212	319	319	248	375	375	284	428	429	320	481	481	356	536	536
213	320	320	249	376	376	285	429	430	321	482	482	357	537	537
214	322	322	250	378	378	286	431	432	322	484	484	358	539	539
215	323	323	251	379	379	287	432	433	323	485	485	359	540	540
216	324	324	252	380	380	288	433	434	324	486	486	360	541	541