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Abstract. We prove that a non-trivial minimal blocking set with respect to hyperplanes in PG$(r, 2)$, $r \geq 3$, is a skeleton contained in some s-flat with odd $s \geq 3$. We also consider non-trivial minimal blocking sets with respect to lines and planes in PG$(r, 2)$, $r \geq 3$. Especially, we show that there are exactly two non-trivial minimal blocking sets with respect to lines and six non-trivial minimal blocking sets with respect to planes up to projective equivalence in PG$(4, 2)$. A characterization of an elliptic quadric in PG$(5, 2)$ as a special non-trivial minimal blocking set with respect to planes meeting every hyperplane in a non-trivial minimal blocking sets with respect to planes is also given.

1 Introduction

We denote by PG(r, q) the projective geometry of dimension r over the field of q elements $\mathbb{F}_q$. A j-flat is a projective subspace of dimension j in PG(r, q). In this paper, Π_k stands for a k-flat in in PG(r, q). We set $\Pi_k = \emptyset$ for $k < 0$. The 0-flats, 1-flats, 2-flats, 3-flats and $(r-1)$-flats are called *points, lines, planes, solids* and *hyperplanes*, respectively. A set of points in PG(r, q) meeting every $(r-k)$-flat is called a *k-blocking set* or a blocking set with respect to $(r-k)$-flats [4]. A 1-blocking set is simply called a *blocking set*. But we only use '1-blocking set' to avoid confusion. A k-flat in PG(r, q) is the smallest k-blocking set [5] and a k-blocking set containing a k-flat in PG(r, q) is called *trivial*. A k-blocking set $\mathcal{B}$ is *minimal* if $\mathcal{B} \setminus \{P\}$ is no longer a k-blocking set for any point P of $\mathcal{B}$.

For an integer $r \geq 3$ and a prime power $q \geq 3$, a smallest non-trivial 1-blocking set $\mathcal{B}_0$ in a plane δ in PG(r, q) is also a smallest non-trivial 1-blocking set in PG(r, q). The speciality for the binary case is that a non-trivial 1-blocking set in PG$(2, 2)$ does not exist.

Denote by Cone$(\Pi_k, \mathcal{B})$ (or simply $\Pi_k\mathcal{B}$) a cone with vertex a k-flat Π_k and base $\mathcal{B}$ in an s-flat Δ skew to Π_k. Note that the cone is just $\mathcal{B}$ if Π_k is empty.

Govaerts and Storme proved the following.

Theorem 1.1 ([16]). (a) *Any smallest non-trivial* 1*-blocking set in PG*$(r, 2)$*,* $r \geq 3$*, is an elliptic quadric in a solid in PG*$(r, 2)$*.*

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(b) *Every non-trivial minimal* 2*-blocking set in PG*$(3,2)$ *is the complement of an elliptic quadric.*

(c) *Any smallest non-trivial* k*-blocking set in PG*$(r,2)$, $r \geq 3$, *with* $2 \leq k \leq r-1$ *is Cone*$(\Pi_{k-3}, \mathcal{T})$ *where* $\mathcal{T}$ *is the set of 10 points consisting of the complement of an elliptic quadric in a solid* Δ.

An elliptic quadric in PG$(3,2)$ is a set of five points no four of which are coplanar, that is the only non-trivial minimal 1-blocking set in PG$(3,2)$ up to projective equivalence. A natural question is to classify all non-trivial minimal k-blocking sets in PG$(r,2)$ up to projective equivalence for $1 \leq k \leq r-1$.

In this paper, the point P in PG$(r,2)$ with coordinate vector $(p_0, p_1, \ldots, p_r)$ is denoted by $(p_0, p_1, \ldots, p_r)$ or simply $p_0p_1 \ldots p_r$, and the number of 1's in $\{p_0, p_1, \ldots, p_r\}$ is called the *weight* of P. The hyperplane defined by the equation $a_0x_0 + a_1x_1 + \cdots + a_rx_r = 0$ is denoted by $[a_0a_1 \ldots a_r]$. For two distinct points $P(p_0, p_1, \ldots, p_r)$ and $Q(q_0, q_1, \ldots, q_r)$ in PG$(r,2)$, we denote the point $(p_0 + q_0, p_1 + q_1, \ldots, p_r + q_r)$ by $P + Q$.

Let $\boldsymbol{e}_i = 0 \cdots 010 \cdots 0$ be the point of PG$(r,2)$ the only i-th entry of which is 1. We denote by $\mathbf{1}$ the point $11 \cdots 1$ and let $\mathcal{I}_r := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{r+1}, \mathbf{1}\}$ in PG$(r,2)$ with odd $r \geq 3$. Note that $\mathcal{I}_3 = \{1000, 0100, 0010, 0001, 1111\}$ is an elliptic quadric in PG$(3,2)$. It is easy to see that $\mathcal{I}_r$ is a non-trivial 1-blocking set in PG$(r,2)$ since r is odd. Since $\mathcal{I}_r$ meets the hyperplane $[\boldsymbol{e}_j + \mathbf{1}]$ in the point $\boldsymbol{e}_j$ and meets the hyperplane $[11 \cdots 1]$ in the point $\mathbf{1}$, $\mathcal{I}_r$ is minimal. Thus $\mathcal{I}_r$ is a non-trivial minimal 1-blocking set in PG$(r,2)$ for odd $r \geq 3$.

Let $P_1, P_2, \ldots, P_{r+1}$ be $r+1$ points of PG$(r,2)$ in general position. We call the $(r+2)$-set $\{P_1, P_2, \ldots, P_{r+1}, \sum_{i=1}^{r+1} P_i\}$ a *skeleton* in PG$(r,2)$, which is also called a 'frame' [2]. Obviously, a skeleton in PG$(r,2)$ is projectively equivalent to $\mathcal{I}_r$. We prove the following.

Theorem 1.2. *Let* S *be a non-trivial minimal* 1*-blocking set in PG*$(r,2)$, $r \geq 3$. *Then,* S *is projectively equivalent to* $\mathcal{I}_s$ *in some* s*-flat of PG*$(r,2)$ *with odd* $s \geq 3$.

Corollary 1.3. *There are exactly* $\lfloor (r-1)/2 \rfloor$ *non-trivial minimal* 1*-blocking sets up to projective equivalence in PG*$(r,2)$, $r \geq 3$.

Next, let us give two examples of non-trivial minimal $(r-1)$-blocking sets in PG$(r,2)$.

Example 1.4. (1) From Theorem 1.1, Cone$(\Pi_{r-4}, \mathcal{T})$ with an $(r-4)$-flat Π_{r-4} and $\mathcal{T}$ in a solid Δ skew to Π_{r-4} is the smallest non-trivial $(r-1)$-blocking set of size $11 \cdot 2^{r-3} - 1$ in PG$(r,2)$ for $r \geq 3$, say *of type* A_1, where $\mathcal{T}$ is the complement of a skeleton (an elliptic quadric) in Δ.

(2) Take two hyperplanes H_1, H_2 and a line l skew to $H_1 \cap H_2$ in PG$(r,2)$ for $r \geq 3$. Let $Q_i = H_i \cap l$ for $i = 1, 2$ and take the point $P = Q_1 + Q_2$ on l. Then, $S = (H_1 \setminus \{Q_1\}) \cup (H_2 \setminus \{Q_2\}) \cup \{P\}$ forms a non-trivial minimal $(r-1)$-blocking set of size $3 \cdot 2^{r-1} - 2$ in PG$(r,2)$, say *of type* A_2.

We note that in PG$(3,2)$ the non-trivial minimal 2-blocking sets of type A_1 and type A_2 are the same. Any smallest non-trivial $(r-1)$-blocking set in PG$(r,2)$, $r \geq 3$, is Cone$(\Pi_{r-4}, \mathcal{T})$ of size $11 \cdot 2^{r-3} - 1 = 2^{r+1} - 1 - 5 \cdot 2^{r-3}$, where $\mathcal{T}$ is a 10-set in some solid Δ such that $\Delta \setminus \mathcal{T}$ is a skeleton by Theorem 1.1(c). As for the second and third smallest ones, we show the following.

Theorem 1.5. (a) *Any second smallest non-trivial* $(r-1)$*-blocking set in* $PG(r,2)$*,* $r \geq 4$*, is* $Cone(\Pi_{r-5}, S_{22})$ *of size* $2^{r+1} - 1 - 9 \cdot 2^{r-4}$*, where* S_{22} *is a non-trivial minimal* 3*-blocking set of type* A_2 *in a* 4*-flat.*

(b) *Any third smallest non-trivial* $(r-1)$*-blocking set in* $PG(r,2)$*,* $r \geq 5$*, has size* $2^{r+1} - 1 - 17 \cdot 2^{r-5}$*.*

As a consequence of Theorem 1.2, there is only one non-trivial minimal 1-blocking set up to projective equivalence in PG$(4,2)$, which is a skeleton in a solid. We also classify non-trivial minimal k-blocking sets in PG$(4,2)$ up to projective equivalence for $k = 2, 3$.

Theorem 1.6. *There are exactly two non-trivial minimal* 3*-blocking sets in* $PG(4,2)$*. One is of type* A_1 *with size* 21*, that is,* $Cone(P, \mathcal{T})$ *with a point* P *and the complement* $\mathcal{T}$ *of a skeleton in* Δ*, where* Δ *is a solid not containing* P*. The other is of type* A_2 *with size* 22*, consisting of two solids* Δ_1, Δ_2 *with two points* $Q_i \in \Delta_i \setminus (\Delta_1 \cap \Delta_2)$*,* $i = 1, 2$*, deleted plus one point* $Q_1 + Q_2$*.*

For t flats $\chi_1, \dots, \chi_t$, we denote by $\langle \chi_1, \dots, \chi_t \rangle$ the smallest flat containing $\chi_1, \dots, \chi_t$. From Theorem 1.1, we get (a) of the following theorem.

Theorem 1.7. (a) *Let* S_{10} *be the set of 10 points in a solid* Δ *in* $PG(4,2)$ *which is the complement of a skeleton in* Δ*. Then,* S_{10} *is the smallest non-trivial* 2*-blocking set in* $PG(4,2)$*.*

(b) *Let* $S_{11} = \mathrm{Cone}(P, K)$ *with a point* P *and a skeleton* K *in a solid* Δ *not containing* P*. Then,* S_{11} *is a non-trivial minimal* 2*-blocking set with size* 11 *in* $PG(4,2)$*.*

(c) *Take two planes* δ_1, δ_2 *meeting in a point* P *in* $PG(4,2)$ *and a point* $Q_i \in \delta_i \setminus \{P\}$ *for* $i = 1, 2$*. Let* $S_{12} = (\delta_1 \setminus \{Q_1\}) \cup (\delta_2 \setminus \{Q_2\}) \cup \{Q_1 + Q_2\}$*. Then,* S_{12} *is a non-trivial minimal* 2*-blocking set with size* 12 *in* $PG(4,2)$*.*

(d) *Take three points* Q_1, Q_2, Q_3 *not on a line and a line* l *which is skew to the plane* $\langle Q_1, Q_2, Q_3 \rangle$ *in* $PG(4,2)$*. Let* $\delta_i = \langle Q_i, l \rangle$ *for* $i = 1, 2, 3$ *and let* $P = Q_1 + Q_2 + Q_3$*. Then,* $S_{13} = \{P\} \cup \bigcup_{i=1}^{3} (\delta_i \setminus \{Q_i\})$ *is a non-trivial minimal* 2*-blocking set with size* 13 *in* $PG(4,2)$*.*

(e) *Take a skeleton* $\{Q_1, Q_2, Q_3, Q_4, P = \sum_{i=1}^{4} Q_i\}$ *in a solid* Δ *and a point* R_1 *out of* Δ*. Let* $l_1, \dots, l_4$ *be the lines defined by* $l_1 = \{P, R_1, R_1' = P + R_1\}$ *and*

$$l_j = \{P, R_j = R_{j-1} + Q_{j-1}, R_j' = R_{j-1}' + Q_{j-1}\}, \ j = 2, 3, 4.$$

Then, $S_{13}' = \bigcup_{i=1}^{4} (l_i \cup \{P + Q_i\})$ *is a non-trivial minimal* 2*-blocking set with size* 13 *in* $PG(4,2)$*.*

(f) *A parabolic quadric* $\mathcal{P}_4$ *is a non-trivial minimal* 2*-blocking set with size* 15 *in* $PG(4,2)$*.*

Theorem 1.8. *There are exactly six non-trivial minimal* 2*-blocking sets in* $PG(4,2)$ *up to projective equivalence, which are described in Theorem 1.7.*

An elliptic quadric $\mathcal{E}_5$ in PG$(5,2)$ meets a hyperplane in a 11-set projectively equivalent to S_{11} in Theorem 1.7 or a parabolic quadric $\mathcal{P}_4$. Hence we get the following.

Corollary 1.9. *An elliptic quadric $\mathcal{E}_5$ in PG$(5,2)$ is a non-trivial minimal* 3*-blocking set meeting every hyperplane in a non-trivial minimal* 2*-blocking set.*

We prove that the converse is also valid:

Theorem 1.10. *Let S be a non-trivial minimal* 3*-blocking set in PG$(5,2)$ meeting every hyperplane in a non-trivial minimal* 2*-blocking set. Then, S is an elliptic quadric $\mathcal{E}_5$ in PG$(5,2)$.*

We prove Theorem 1.2 in Section 2, Theorems 1.5 and 1.6 in Section 3 and Theorems 1.7, 1.8 and 1.10 in Section 4.

2 Non-trivial minimal 1-blocking sets in PG$(r,2)$

For a set S in PG(r,q), we denote by S^c the complement of S in PG(r,q). The following is well known, which is straightforward from the definition.

Lemma 2.1. *A set S is a non-trivial k-blocking set in PG(r,q) if and only if S^c is a non-trivial $(r-k)$-blocking set in PG(r,q).*

Lemma 2.2. *Every non-trivial* 1*-blocking set in PG$(3,2)$ is a skeleton (an elliptic quadric).*

Proof. Let S be a non-trivial 1-blocking set in PG$(3,2)$. By Lemma 2.1, the complement S^c is a non-trivial 2-blocking set in PG$(3,2)$. Then, S^c contains the complement of an elliptic quadric $\mathcal{E}_3^c$ by Theorem 1.1(b), whence S is contained in $\mathcal{E}_3$. Since $\mathcal{E}_3$ is the smallest non-trivial 1-blocking set in PG$(3,2)$, we have $S=\mathcal{E}_3$. □

Lemma 2.3. *Every non-trivial minimal* 1*-blocking set in PG$(4,2)$ is a skeleton of some solid.*

Proof. Let S be a non-trivial minimal 1-blocking set in PG$(4,2)$. Assume that S is contained in a solid Δ. Then, S is a non-trivial minimal 1-blocking set in Δ, which is a skeleton of Δ by Lemma 2.2. Next, assume that S is not contained in a solid. Note that S contains no skeleton of a solid because of the minimality. Without loss of generality, we may assume that S contains the 5-set $K=\{10000, 01000, 00100, 00010, 00001\}$. Then, S contains a point of even weight since the solid $[11111]$ contains no point of K. On the other hand, S contains no point of weight 2 (resp. 4) since S contains no line (resp. no skeleton of a solid), a contradiction. □

Proof of Theorem 1.2. We prove Theorem 1.2 by induction on r. The theorem is valid for $r=3,4$ by Lemmas 2.2, 2.3, respectively. We first assume $r=2m-1$ with $m\geq 3$ and that our assertion holds for at most $r-1$ dimensions. Let S be a non-trivial minimal 1-blocking set in PG$(r,2)$. If S is contained in a hyperplane H, then S forms a non-trivial minimal 1-blocking set of H, which is projectively equivalent to $\mathcal{I}_s$ in some s-flat of H with odd $s\geq 3$ from the induction hypothesis. So, we assume that S is not contained in a hyperplane. Without loss of generality, we may assume that S contains the $2m$-set $K=\{\boldsymbol{e}_1,\cdots,\boldsymbol{e}_{2m}\}$. Then, S contains a point of even weight since the hyperplane $H_1=[11\cdots 1]$ contains no point of K. Suppose that S contains a point $P=(p_1,\ldots,p_{2m})$ with weight $2t$ for some $t<m$ and let $p_j=1$ for $j=u_1,\ldots,u_{2t}$. Then, S contains the

$(2t+1)$-set $\{\boldsymbol{e}_{u_1}, \ldots, \boldsymbol{e}_{u_{2t}}, P\}$ which is projectively equivalent to $\mathcal{I}_{2t-1}$. This contradicts the minimality of S. Hence S contains the point $\mathbf{1} = 11\cdots1$, giving $S = \mathcal{I}_{2m-1}$. One can prove our assertion similarly for the case $r = 2m$ with $m \geq 3$. Actually, we get a contradiction when we assume that S is not contained in a hyperplane since every point of PG$(2m, 2)$ with even weight has a 0 entry. □

3 Non-trivial minimal $(r-1)$-blocking sets in PG$(r, 2)$

In this section, we consider non-trivial minimal blocking sets with respect to lines in PG$(r, 2)$. A t-set T in PG(r, q) is called a *t-cap* if T meets any line in at most two points. A t-cap T is *complete* if it is not contained in a $(t+1)$-cap. For $q = 2$, it is well known that a largest complete cap in PG$(r, 2)$ is the complement of a hyperplane. The following is obvious from the definitions.

Lemma 3.1 ([4]). *A t-set T in PG$(r, 2)$ is a complete t-cap if and only if the complement T^c is a minimal $(r-1)$-blocking set in PG$(r, 2)$.*

Much attention has been given to the complete caps in PG$(r, 2)$ from coding theory to study binary quasi-perfect codes. An *$[n, k, d]_q$ code* is a linear code of length n, dimension k and minimum weight d over $\mathbb{F}_q$. Let $\mathcal{C}$ be an $[n, n-r-1, 4]_2$ code with parity check matrix H with size $(r+1) \times n$ and let T be the n-set in PG$(r, 2)$ consisting the n columns of H. Then, it can be shown that T is a complete cap if and only if $\mathcal{C}$ has covering radius 2. If the code $\mathcal{C}$ of minimum distance 4 has covering radius 2, $\mathcal{C}$ is called *quasi-perfect*. See [6] and [12] for binary quasi-perfect linear codes and caps in binary projective spaces.

It follows from Lemma 3.1 that the known results on complete caps in PG$(r, 2)$ can be seen as results on minimal $(r-1)$-blocking sets in PG$(r, 2)$, see [1, 6, 7, 8, 9, 10, 11, 12, 15, 21, 22, 23] and the references therein for complete caps in PG$(r, 2)$.

An n-cap in PG$(r, 2)$ is called *large* if $n \geq 2^{r-1}+1$, *critical* if $n = 2^{r-1}+1$, and *small* if $n \leq 2^{r-1}$ [7]. The following is known for critical complete caps in PG$(r, 2)$ for $r \leq 6$.

Theorem 3.2 ([10, 11, 12, 22]). (a) *Every complete 5-cap in PG$(3, 2)$ is projectively equivalent to $\mathcal{I}_3 = \{1000, 0100, 0010, 0001, 1111\}$.*

(b) *Every complete 9-cap in PG$(4, 2)$ is projectively equivalent to $C_9 = \{01000, 00100, 00010, 00001, 01111, 10100, 10010, 10001, 10111\}$.*

(c) *There are exactly five inequivalent complete 17-caps in PG$(5, 2)$ up to projective equivalence.*

(d) *There are exactly 42 inequivalent complete 33-caps in PG$(6, 2)$ up to projective equivalence.*

Let T_k be a k-cap in a hyperplane H of PG$(r, 2)$ and let P be a point out of H. Then, $T_{2k} = T_k \cup \{P + Q \mid Q \in T_k\}$ forms a $2k$-cap in PG$(r, 2)$. It is also known that the cap T_{2k} is complete in PG$(r, 2)$ if and only if T_k is complete in H. This construction of T_{2k} from T_k is called the *doubling construction* or *Plotkin construction* [6, 12]. This means that the minimal $(r-1)$-blocking set $T_{2k}^c = \text{PG}(r, 2) \setminus T_{2k}$ is obtained as Cone(P, T_k^c).

All exact possible sizes of large complete caps and the structure of complete n-caps with $n > 2^{r-1} - 1$ is known as follows.

Theorem 3.3 ([12]). (a) *A complete t-cap in $PG(r,2)$ with $t > 2^{r-1}$ exists if and only if $t = 2^{r-1} + 2^{r-1-g}$ with $g \in \{0, 2, 3, \dots, r-1\}$.*

(b) *In $PG(r,2)$, for $g = 2, 3, \dots, r-2$, each complete $(2^{r-1}+2^{r-1-g})$-cap can be obtained by $(r-1-g)$-fold application of the doubling construction to a complete (2^g+1)-cap in $PG(g+1,2)$.*

Hence, every large complete cap can be obtained from some critical complete cap by the doubling construction. Theorems 3.2 and 3.3 yield the following.

Theorem 3.4 ([12]). (a) *In $PG(r,2)$, $r \geq 3$, the second largest complete caps are $5 \cdot 2^{r-3}$-caps, which are projectively equivalent to the cap obtained by $(r-3)$-fold application of the doubling construction to $\mathcal{I}_3$.*

(b) *In $PG(r,2)$, $r \geq 4$, the third largest complete caps are $9 \cdot 2^{r-4}$-caps, which are projectively equivalent to the cap obtained by $(r-4)$-fold application of the doubling construction to C_9.*

(c) *In $PG(r,2)$, $r \geq 5$, the fourth largest complete caps are $17 \cdot 2^{r-5}$-caps.*

The part (a) of Theorem 3.4 implies Theorem 1.1(c) for $k = r-1$. Taking two hyperplanes $H_1 = [00111]$, $H_2 = [01111]$ and two points $Q_1 = 01000 \in H_1$ and $Q_2 = 01111 \in H_2$, one can see that the complement of C_9 in PG$(4,2)$ is a non-trivial minimal 3-blocking set of type A_2 in Example 1.4. Hence, Theorem 1.5 follows from the parts (b) and (c) of Theorem 3.4.

Every non-trivial minimal 2-blocking sets in PG$(3,2)$ is the complement of a skeleton (an elliptic quadric) by Lemmas 2.1 and 2.2. As for small n-caps with $n \leq 2^{r-1}$ in PG$(r,2)$, the following is known for $r \leq 6$, see [15] for $r \geq 7$.

Theorem 3.5 ([13, 14, 21, 22]). (a) *A small complete cap does not exist in $PG(r,2)$ for $r \leq 4$.*

(b) *In $PG(5,2)$, there are only small complete 13-caps.*

(c) *In $PG(6,2)$, the possible sizes of small complete caps are 21, 22, 24, 25, 26.*

Now, let S be a non-trivial minimal 3-blocking set in $\Sigma =$ PG$(4,2)$. It follows from Theorem 3.5(a) that $|S^c| \geq 2^3 + 1$, i.e., $|S| \leq 22$. If $|S| = 22$, $S = S_{22}$ in Theorem 1.5. If $|S| = 21$, S has the smallest size from Theorem 1.1(c). Thus, we obtain Theorem 1.6.

Table 1 gives the number of non-trivial minimal $(r-1)$-blocking sets in PG$(r,2)$ up to projective equivalence for $r \leq 6$. The classification of complete caps in PG$(r,2)$ for $r = 5, 6$ is obtained by an exhaustive computer search, see [10, 22].

4 Non-trivial minimal $(r-2)$-blocking sets in PG$(r,2)$

For a given set S, a line l is called an *i-line* of S if $|S \cap l| = i$. An *i-plane*, *i-solid* and so on are defined similarly. We denote by a_i the number of i-hyperplanes. The list of the values a_i is called the *spectrum* of S. For example, the spectrum of a skeleton K in PG$(3,2)$ is $(a_1, a_3) = (5, 10)$ and there is a unique 1-plane of K through P for any point P of K.

Table 1: The number of non-trivial minimal $(r-1)$-blocking sets in $\mathrm{PG}(r,2)$

r	Size	#
3	10	1
4	21	1
	22	1
5	43	1
	45	1
	46	5
	50	1

r	Size	#
6	87	1
	91	1
	93	5
	94	42
	96	2
	98	3
	99	1

r	Size	#
6	100	4
	101	2
	102	13
	103	6
	105	2
	106	5

In this section, we consider non-trivial minimal blocking sets with respect to planes. We first give some examples of non-trivial minimal $(r-2)$-blocking sets in $\mathrm{PG}(r,q)$ with $3 \le r \le 5$, see [19] and [20] for quadrics in $\mathrm{PG}(r,q)$.

Example 4.1. Let q be a prime power.

(1) An elliptic quadric $\mathcal{E}_3$ in $\mathrm{PG}(3,q)$ is a non-trivial minimal 1-blocking set of size q^2+1 since $\mathcal{E}_3$ has spectrum $(a_1, a_{q+1}) = (q^2+1, q^3+q)$ and since each point of $\mathcal{E}_3$ is on a 1-plane, see [18]. Recall that the $q+1$ points of $\mathcal{E}_3$ in a $(q+1)$-plane forms a $(q+1)$-arc, which is a $(q+1)$-set no three of which are collinear.

(2) Take an elliptic quadric $\mathcal{E}_3$ in a solid Δ and a point P out of Δ in $\mathrm{PG}(4,q)$. Let $\Pi_0\mathcal{E}_3$ be the cone with vertex P and base $\mathcal{E}_3$. It follows from the spectrum of $\mathcal{E}_3$ that $\Pi_0\mathcal{E}_3$ has spectrum

$$(a_{q+1}, a_{q^2+1}, a_{q^2+q+1}) = (q^2+1, q^4, q^3+q).$$

Let H be a solid. If H contains the vertex P, then H meets $\Pi_0\mathcal{E}_3$ in a line (resp. non-coplanar $q+1$ lines) through P when $H \cap \Delta$ is a 1-plane (resp. $(q+1)$-plane). Otherwise, H meets $\Pi_0\mathcal{E}_3$ in an elliptic quadric. Hence, $\Pi_0\mathcal{E}_3$ is a non-trivial minimal 2-blocking set in $\mathrm{PG}(4,q)$.

(3) A parabolic quadric $\mathcal{P}_4$ in $\mathrm{PG}(4,q)$ has spectrum

$$(a_{q^2+1}, a_{q^2+q+1}, a_{q^2+2q+1}) = (\frac{q^4-q^2}{2}, (q+1)(q^2+1), \frac{q^4+q^2}{2})$$

and an i-solid Δ meets $\mathcal{P}_4$ in an elliptic quadric, a cone with vertex a point and base a conic, a hyperbolic quadric for $i = q^2+1, q^2+q+1, q^2+2q+1$, respectively. So, possible planes are 1-, $(q+1)$- and $(2q+1)$-planes. For any point P in $\mathcal{P}_4$, one can find a solid through P meeting $\mathcal{P}_4$ in an elliptic quadric. Hence, $\mathcal{P}_4$ is a non-trivial minimal 2-blocking set of size $(q+1)(q^2+1)$.

(4) An elliptic quadric $\mathcal{E}_5$ in $\mathrm{PG}(5,q)$ is a non-trivial minimal 3-blocking set of size $(q+1)(q^3+1)$ meeting every hyperplane in a non-trivial minimal 2-blocking set since $\mathcal{E}_5$ meets every hyperplane in a parabolic quadric $\mathcal{P}_4$ or a cone $\Pi_0\mathcal{E}_3$.

From now on, we consider the case when $q = 2$. Let S_n be a non-trivial minimal 2-blocking set of size n in PG(4, 2). We denote the numbers of i-planes by b_i. Simple counting arguments yield the following.

Lemma 4.2. (a) $\sum_{i=1}^{6} b_i = 155$.

(b) $\sum_{i=1}^{6} ib_i = 35n$.

(c) $\sum_{i=2}^{6} i(i-1)b_i = 7n(n-1)$.

Proof. Recall that the number of lines in PG(4, q) is $(q^2+1)\theta_4$ and that the number of planes through a fixed point in PG(4, q) is $(q^2+1)\theta_2$, where $\theta_j = (q^{j+1}-1)/(q-1)$, see [19]. Hence (1) and (2) hold. Counting the number of $(\{P, Q\}, \delta)$ with distinct points P, Q and a plane δ containing P and Q in PG(4, 2), one can obtain (3). □

A given set S in PG(4, 2) is a non-trivial minimal 2-blocking set if and only if S satisfies that $b_0 = b_7 = 0$ and that every point of S is on a 1-plane. We first prove Theorem 1.7.

Proof of Theorem 1.7.

(a) Let $S_{10} = \Delta \setminus \mathcal{E}_3$, where Δ is a solid and $\mathcal{E}_3$ is a skeleton of Δ. S_{10} is a smallest non-trivial minimal 2-blocking set by Theorem 1.1(c) with

$$(a_4, a_6, a_{10}) = (20, 10, 1),\ (b_1, b_2, b_3, b_4, b_6) = (40, 60, 40, 10, 5).$$

(b) Take a skeleton $K = \{P_1, P_2, P_3, P_4, P_1+P_2+P_3+P_4\}$ in a solid Δ and a point P out of Δ. Let $S_{11} = \mathrm{Cone}(P, K)$. Note that S_{11} is uniquely determined by 5 points in general position: P_1, P_2, P_3, P_4, P. It follows from Example 4.1(2) that S_{11} is a non-trivial minimal 2-blocking set in PG(4, 2) with spectrum $(a_3, a_5, a_7) = (5, 16, 10)$. Let H be a solid. If H contains the vertex P, then $H \cap S_{11}$ is a line (resp. non-coplanar three lines) through P when $H \cap \Delta$ is a 1-plane (resp. 3-plane). Otherwise, $H \cap S_{11}$ is a skeleton. Hence, by Lemma 4.2, we have $(b_1, b_3, b_5) = (50, 95, 10)$.

(c) Take a skeleton $K = \{P_1, P_2, P_3, P_4, P_0 = P_1 + P_2 + P_3 + P_4\}$ in a solid Δ and a point P out of Δ again. Take two lines l_1, l_2 and two planes δ_1, δ_2 as

$$l_1 = \{P_1, P_2, Q_1 = P_1 + P_2\},\ l_2 = \{P_3, P_4, Q_2 = P_3 + P_4\},$$

$\delta_1 = \langle l_1, P\rangle$, $\delta_2 = \langle l_2, P\rangle$ and let $S_{12} = (\delta_1 \setminus \{Q_1\}) \cup (\delta_2 \setminus \{Q_2\}) \cup \{P_0\}$. S_{12} is also uniquely determined by 5 points in general position: P_1, P_2, P_3, P_4, P. Obviously, S_{12} satisfies $b_0 = b_7 = 0$. Indeed, one can calculate

$$(a_4, a_5, a_6, a_7, a_8) = (5, 12, 4, 4, 6), (b_1, b_2, b_3, b_4, b_5, b_6) = (26, 40, 54, 25, 8, 2).$$

Let l_0 be a 0-line of S_{12} in Δ containing none of Q_1, Q_2 (there are four such lines). Then, $\langle l_0, P\rangle$ is a 1-plane at P. Let l_i' be the line through Q_i on δ_i other than the two lines $l_i, \langle Q_i, P\rangle$ for $i = 1, 2$. For the line $l = \langle Q_1, Q_2\rangle$, the union of the five planes $\langle l, l_1\rangle, \langle l, l_1'\rangle, \langle l, l_2\rangle, \langle l, l_2'\rangle, \langle l, P\rangle$ includes S_{12}. Hence, the other two planes through l are 1-planes at P_0. For any point $P' \in \delta_i \setminus \{P, Q_i\}$, one can find two 1-planes at P' through the line $\langle P', Q_j\rangle$ for $\{i, j\} = \{1, 2\}$ similarly. Thus, S_{12} is minimal.

(d) Take three non-collinear points Q_1, Q_2, Q_3 and a line $l = \{P_1, P_2, P_1 + P_2\}$ which is skew to the plane $\delta_0 = \langle Q_1, Q_2, Q_3 \rangle$. Let $P = Q_1 + Q_2 + Q_3$ and $S_{13} = (\{P\} \cup \delta_1 \cup \delta_2 \cup \delta_3) \setminus \{Q_1, Q_2, Q_3\}$, where $\delta_i = \langle l, Q_i \rangle$ for $i = 1, 2, 3$. S_{13} is uniquely determined by the 5 points P_1, P_2, Q_1, Q_2, Q_3 in general position. It can be checked that

$$(a_3, a_5, a_7, a_9) = (1, 12, 15, 3),\ (b_1, b_2, b_3, b_4, b_5, b_6) = (22, 27, 60, 34, 9, 3).$$

Obviously, δ_0 is a 1-plane at P and that $\langle P', Q_j, Q_k \rangle$ is a 1-plane at any point $P' \in (\delta_i \cap S_{13}) \setminus l$ for $\{i, j, k\} = \{1, 2, 3\}$. Let m be the line on δ_0 not meeting $\{Q_1, Q_2, Q_3, P\}$. For any point R on L, $\langle R, m \rangle$ is a 1-plane at R. Hence, S_{13} is minimal.

(e) Take a skeleton $K = \{Q_1, Q_2, Q_3, Q_4, P = Q_1 + Q_2 + Q_3 + Q_4\}$ in a solid Δ and a point R_1 out of Δ. Let $R_j = R_{j-1} + Q_{j-1}$ for $j = 2, 3, 4$ and take lines $L_1 = \{P, R_1, R'_1 = P + R_1\}$, $L_j = \{P, R_j, R'_j = P + R_j = R'_{j-1} + Q_{j-1}\}$ for $j = 2, 3, 4$. Let $S'_{13} = \bigcup_{i=1}^{4} (L_i \cup \{P_i = P + Q_i\})$. S'_{13} is also uniquely determined by 5 points Q_1, Q_2, Q_3, Q_4, R_1 in general position. One can calculate that

$$(a_5, a_7, a_9) = (15, 12, 4),\ (b_1, b_2, b_3, b_4, b_5, b_6) = (15, 44, 50, 32, 10, 4).$$

Since $\Delta \cap S'_{13} = \{P_1, P_2, P_3, P_4, P\}$ is a skeleton, each of the points P_1, P_2, P_3, P_4, P is on a 1-plane. The plane $\langle R_1, Q_2, Q_3 \rangle$ is a 1-plane through R_1 and the plane $\langle R'_1, Q_2, Q_3 \rangle$ is a 1-plane through R'_1. One can find a 1-plane through each of $R_2, R_3, R_4, R'_2, R'_3, R'_4$ similarly. Thus, S'_{13} is minimal.

(f) A parabolic quadric in PG(4, 2) is a non-trivial minimal 2-blocking set of size 15 with $(a_5, a_7, a_9) = (6, 15, 10)$ and $(b_1, b_3, b_5) = (15, 95, 45)$ by Lemma 4.2, see Example 4.1(3).

□

Theorem 1.8 can be proved with the aid of a computer as follows.

Proof of Theorem 1.8. Let S be a non-trivial minimal 2-blocking set in $\mathrm{PG}(4, 2)$. Since S is smallest when $|S| = 10$ and such a set is the complement of a skeleton in a solid by Theorem 1.1(c), we may assume that $11 \le |S| \le 20$ by Lemma 2.1 and that $\pi \setminus S$ contains no skeleton for any solid π. Let Δ be a solid meeting S in s points. We have $s \le 12$ since S contains no plane. If $s = 11$ or 12, one can find a point of $S \cap \Delta$ which is not on a 1-plane, a contradiction. Assume $s = 10$. If $\Delta \setminus S$ contains no line, then the 5-set $\Delta \setminus S$ is a skeleton in Δ, a contradiction. Suppose $\Delta \setminus S$ consists of a line l and two points Q_1, Q_2. If Q_2 is on the plane $\langle l, Q_1 \rangle$, then there is no 1-plane of S through a point of $(\Delta \cap S) \setminus \langle l, Q_1 \rangle$, a contradiction. Hence $Q_2 \notin \langle l, Q_1 \rangle$. Take a plane δ in Δ through the line $\langle Q_1, Q_2 \rangle$ meeting l at Q, say. Then, there is no 1-plane of S through the point $Q_1 + Q_2 + Q \in \Delta \cap S$, a contradiction again. Thus, we may assume that S meets every solid in at most 9 points. Recall that S satisfies $b_0 = b_7 = 0$. We show that $b_5 + b_6 > 0$. Suppose $b_5 = b_6 = 0$. If $b_4 = 0$, then S meets every solid in at most 5 points and we have $|S| \le (5 - 3)3 + 3 = 9$, a contradiction. Hence there is a 4-plane, say δ. Let π be a solid through δ with $|S \cap \pi| = n > 4$. Taking the n points of $S \cap \pi$ as columns of a generator matrix, one can get an $[n, 4, n-4]_2$ code $\mathcal{C}$, see [3]. Since no $[9, 4, 5]_2$ codes exist [17], we have $n \le 8$. If $n = 7$ or 8, then $\mathcal{C}$ is a Hamming $[7, 4, 3]_2$ code or an extended

Hamming $[8,4,4]_2$ code, and $\mathcal{C}$ has a codeword of weight n, which implies that π contains a 0-plane, a contradiction. Hence a solid π through δ satisfies $|S \cap \pi| \leq 6$, and we have $|S| \leq (6-4)3+4 = 10$, a contradiction again. Thus $b_5 + b_6 > 0$.

Assume $b_6 > 0$ and let $\Delta_1, \Delta_2, \Delta_3$ be the solids through a 6-plane δ_0. Since S meets these solids in at most 9 points, we have $|S| \leq (9-6)3+6 = 15$. Without loss of generality, we may assume that $\Delta_1 = [00001]$, $\Delta_2 = [00010]$, $\Delta_3 = [00011]$ and

$$S \cap \delta_0 = \{10000, 01000, 11000, 00100, 10100, 01100\}.$$

Let $t_i = |S \cap \Delta_i| - 6$. By an exhaustive computer search for t_i points from Δ_i, we obtain 576 sets which are projectively equivalent to S_{12} when $(t_1, t_2, t_3) = (2, 2, 2)$, 256 sets and 768 sets which are projectively equivalent to S_{13} and S'_{13}, respectively, when $(t_1, t_2, t_3) = (3, 3, 1)$. Assuming $b_6 = 0$ and $b_5 > 0$, a similar exhaustive computer search found non-trivial minimal 2-blocking sets projectively equivalent to either S_{11} or a parabolic quadric $\mathcal{P}_4$. □

Finally, we prove Theorem 1.10.

Proof of Theorem 1.10. Let S be a non-trivial minimal 3-blocking set in $\mathrm{PG}(5,2)$ meeting every hyperplane in a non-trivial minimal 2-blocking set. It follows form Theorem 1.8 and from the proof of Theorem 1.7 that we have $|S \cap H| \in \{10, 11, 12, 13, 15\}$ for any hyperplane H and $3 \leq |S \cap \Delta| \leq 10$ for any solid Δ. Suppose that a 10-solid Δ_{10} exists. Since the hyperplanes through Δ_{10} are 10-hyperplanes, we have $|S| = 10$. Then, one can find a 0-plane, a contradiction. Suppose that a 4-solid Δ_4 exists. Since the hyperplanes through Δ_4 are 12-hyperplanes, we have $|S| = (12-4)3+4 = 28$. On the other hand, the hyperplanes through a fixed 8-solid are also 12-hyperplanes, and $|S| = (12-8)3+8 = 20$, a contradiction. Suppose that there is a 5-solid Δ_5 such that $S \cap \Delta_5$ is not a skeleton. Since such a 5-solid exists only for S_{13} and S'_{13}, we have $|S| = (13-5)3+5 = 29$. Take a 9-solid δ_9 in a 13-hyperplane. Since $S \cap \delta_9$ is not a hyperbolic quadric, there is no 15–hyperplane through δ_9, whence $|S| = (13-9)3+9 = 21$, a contradiction. Hence, S meets every hyperplane in a cone $\Pi_0\mathcal{E}_3$ or a parabolic quadric $\mathcal{P}_3$, and S has size $(15-9)3+9 = 27$. Such a set S is an elliptic quadric $\mathcal{E}_5$ by Theorem 1.97 in [20]. □

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