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Abstract

In this paper we present the intraday model of Robinson Crusoe where total consumption in a day is composed of two parts : consumption in AM (breakfast) and consumption in PM (dinner). We show the intraday fluctuation of consumption in response to various changes in exogenous variables. We also show the possibilities of *no breakfast optimum* and *no dinner optimum* as a result of rational behavior of Robinson Crusoe.

1. Introduction

The Robinson Crusoe model of optimal consumption and labor supply in a day is often used to study efficient allocation of resources, given his preference and production technology[1]. Both microeconomics and macroeconomics use such a model as an easily understandable, useful apparatus. [See, e. g., Hirshleifer (1984) and Varian (1984, 1990) for microeconomics, while see Barro (1990) for macroeconomics.] Especially, the analysis of dynamic resource allocation applys such a representative agent model in the infinite time horizon framework. [See, e. g., Sargent (1987), Blanchard and Fischer (1989), and Azariadis (1993).]

However, the Robinson Crusoe model implicitly ignores an important time concept concerning the production technology. ***First comes input, second***

1) Here the Robinson Crusoe model is such as : max $u(c, 1-n)$, subject to $c=f(n)$, where $u(.)$ is utility function, $f(.)$ is production function, c is consumption, and n is work effort. Crusoe is supposed to choose optimal consumption and labor supply in order to maximize his utility, given production technology.

comes output. This is a fundamental time concept on production technology even within a day. Also usual frameworks of the model implicitly assume that Robinson Crusoe eats exogenously determined amount of breakfast, including no breakfast case. But we think that the choice concerning breakfast should be analyzed endogenously, and simultaneously with optimal decision on labor supply and dinner.

The purpose of this paper is to present the intraday model of Robinson Crusoe where total consumption in a day is composed of two parts, i. e., breakfast and dinner, and to show the intraday fluctuation in consumption in response to various changes in exogenous variables. Also we show the possibilities of ***no breakfast optimum*** and ***no dinner optimum*** as a result of rational behavior of Robinson Crusoe.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 carrys out comparative static analysis. Section 4 shows the possibilities of no breakfast optimum and no dinner optimum. Section 5 gives concluding remarks.

2. The Model

Let us consider a day of Robinson Crusoe. A day is composed of two parts, i. e., AM and PM[2]. At the beginning of AM, he gets up with exogenously given initial endowment. This endowment is consumable and investable. Also this endowment is assumed to be storable from AM to PM within a day, but non-storable beyond a day. Crusoe's decision problem is how much he consumes in breakfast and dinner, and how much he supplys labor service for production. For simplicity, we assume that he does not eat a lunch. His optimization problem can be formulated as follows :

2) The idea of division of one period into two parts can be found in, e. g., cash-in-advance models of monetary economy. In a typical cash-in-advance model, asset market is open in the first part of the period, while goods market is open in the second part of the period. On the other hand, in our model, input for production is introduced in the first part of the period, while output appears in the second part of the period.

$$\max\ u(c_1+\beta c_2,\ 1-n), \tag{1}$$
$$\text{s.t. } c_1+k=y, \tag{2}$$
$$c_2=f(k,\ n). \tag{3}$$

c_1 is consumption in AM, i. e., breakfast, while c_2 is consumption in PM, i. e., dinner. $c_1+\beta c_2$ is total consumption in a day. β is a parameter which measures relative weight between c_1 and c_2, and $0<\beta\leqq 1$. n is work effort such that $0\leqq n<1$, so $1-n$ is leisure he enjoys. As usual, total time endowment is normalized to unity. y is the initial endowment such that $y>0$. k is physical capital for production such that $k\geqq 0$[3]. $f(k,\ n)$ is a production function with constant returns to scale. In the following we specify the log-linear utility function and the Cobb-Douglas production function to obtain closed-form solutions.

$$u(c_1+\beta c_2,\ 1-n)=\ln(c_1+\beta c_2)+\gamma\ln(1-n),\ 0<\gamma<1, \tag{4}$$

$$f(k,\ n)=Ak^{\alpha}n^{1-\alpha},\ A>0,\ 0<\alpha<1. \tag{5}$$

γ is the relative weight of the utility from enjoying leisure to the utility from consumption. A is a fixed parameter representing production technology. α is a positive constant indicating capital share. Forming the Lagrangian for this problem, we have :

$$L=\ln(c_1+\beta c_2)+\gamma\ln(1-n)+\lambda_1(y-c_1-k)+\lambda_2(Ak^{\alpha}n^{1-\alpha}-c_2), \tag{6}$$

where λ_1 and λ_2 are the Lagrange multipliers attached to budget constraints (2) and (3), respectively. The Kuhn-Tucker conditions are as follows[4] :

$$\partial L/\partial c_1=1/(c_1+\beta c_2)-\lambda_1\leqq 0,\ c_1\geqq 0,\ c_1[\partial L/\partial c_1]=0, \tag{7}$$

$$\partial L/\partial c_2=\beta/(c_1+\beta c_2)-\lambda_2\leqq 0,\ c_2\geqq 0,\ c_2[\partial L/\partial c_2]=0, \tag{8}$$

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3) In our model, all investment are used as capital input for production. Therefore, we treat the term "investment" as interchangeable with term "capital".

4) Since the utility function (4) is concave, the Kuhn-Tucker conditions are necessary and sufficient conditions for utility maximization.

$$\partial L/\partial n = -\gamma/(1-n) + \lambda_2(1-\alpha)Ak^{\alpha}n^{-\alpha} \leqq 0, \quad n \geqq 0, \quad n[\partial L/\partial n] = 0, \tag{9}$$

$$\partial L/\partial k = -\lambda_1 + \lambda_2 \alpha A k^{\alpha-1} n^{1-\alpha} \leqq 0, \quad k \geqq 0, \quad k[\partial L/\partial k] = 0. \tag{10}$$

As the utility function is log-linear form, $c_1 + \beta c_2$ must be positive. This means that at least one of c_1 and c_2 is positive. We can consider three possible cases : (i) $c_1 > 0$ and $c_2 > 0$, (ii) $c_1 = 0$ and $c_2 > 0$, and (iii) $c_1 > 0$ and $c_2 = 0$. Case (i) is an interior solution. Case (ii) is a corner solution, which we call ***no breakfast optimum***. Case (iii) is another corner solution, which we call ***no dinner optimum***. In the following we treat case (i) as the benchmark case. Cases (ii) and (iii) are examined as peculiar optimums in section 4.

In case (i) where $c_1 > 0$ and $c_2 > 0$, it must be that $n > 0$ and $k > 0$ from (3) and (5). Using the Kuhn-Tucker conditions (7)–(10) with equality and budget constraints (2) and (3) (with (5)), we obtain optimal consumption, work effort, and investment.

$$c_1 = \frac{-(\alpha\beta A)^{1/(1-\alpha)}}{(1+\gamma)} + \frac{(1-\alpha+\gamma)y}{(1-\alpha)(1+\gamma)}, \tag{11}$$

$$c_2 = \frac{1/\alpha\beta)(\alpha\beta A)^{1/(1-\alpha)}}{(1+\gamma)} - \frac{(\gamma/\beta)y}{(1-\alpha)(1+\gamma)}, \tag{12}$$

$$n = \frac{1}{(1+\gamma)} - \frac{(\alpha\gamma)y}{(1-\alpha)(1+\gamma)(\alpha\beta A)^{1/(1-\alpha)}}, \tag{13}$$

$$k = \frac{(\alpha\beta A)^{1/(1-\alpha)}}{(1+\gamma)} - \frac{(\alpha\gamma)y}{(1-\alpha)(1+\gamma)}. \tag{14}$$

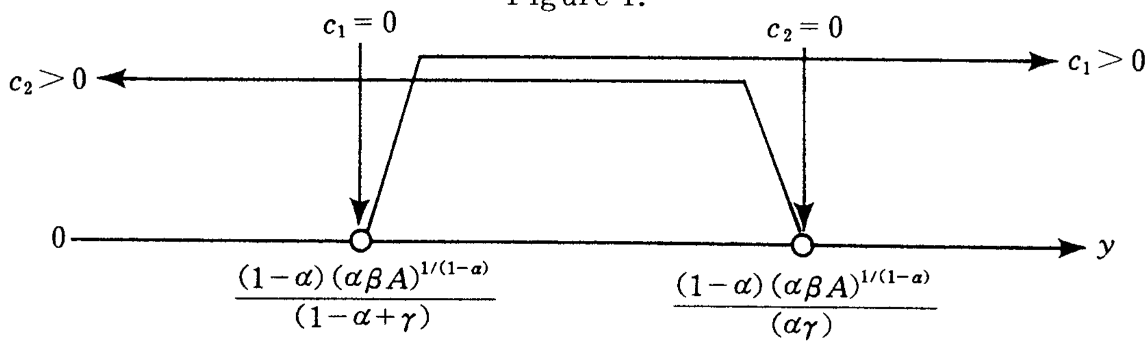
Combining (11) and (12), we get the total consumtion in a day as follows.

$$c_1 + \beta c_2 = \frac{(1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}}{\alpha(1+\gamma)} + \frac{y}{(1+\gamma)}. \tag{15}$$

By parameter assumptions, $c_1 + \beta c_2$ is definitely positive. From (11), $c_1 > 0$ requires $y > (1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}/(1-\alpha+\gamma)$. On the other hand, from (12), $c_2 > 0$ needs $y < (1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}/(\alpha\gamma)$, which is also the condition both for $n > 0$ from (13), and for $k > 0$ from (14), respectively. Thus, in the following we assume that the initial endowment y satisfies a constraint (16) below. [See Figure 1.]

$$(1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}/(1-\alpha+\gamma)<y<(1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}/(\alpha\gamma). \quad (16)$$

Figure 1.



3. Comparative Statics

In this section we explore various implications of the model by comparative static analysis. We study the effects on optimal allocation of change in the initial endowment, progress in production technology, and shift in intraday consumption preference.

3.1. Change in the Initial Endowment

The change in the initial endowment is represented by an increase in y. Differentiating equations (11)–(15) with respect to y, we have the following comparative statics results.

$$\frac{\partial c_1}{\partial y}=\frac{(1-\alpha+\gamma)}{(1-\alpha)(1+\gamma)}>0, \quad (17)$$

$$\frac{\partial c_2}{\partial y}=\frac{-(\gamma/\beta)}{(1-\alpha)(1+\gamma)}<0, \quad (18)$$

$$\frac{\partial(c_1+\beta c_2)}{\partial y}=\frac{1}{(1+\gamma)}>0, \quad (19)$$

$$\frac{\partial n}{\partial y}=\frac{-(\alpha\gamma)}{(1-\alpha)(1+\gamma)(\alpha\beta A)^{1/(1-\alpha)}}<0, \quad (20)$$

$$\frac{\partial k}{\partial y}=\frac{-(\alpha\gamma)}{(1-\alpha)(1+\gamma)}<0. \quad (21)$$

The rise in y increases c_1 and decreases c_2. However, the total consumption $c_1+\beta c_2$ definitely increases. Both work effort n and investment k decrease. (17)–(21) suggest that Crusoe increases breakfast and decreases investment, other things being equal, when the initial endowment increases. Also he decreases work effort. By the decrease of input for production, dinner becomes less. In spite of decrease of dinner, however, total consumption in a day increases as the change of breakfast dominates that of dinner. Moreover, the leisue that Crusoe enjoys increases as a result of the decrease of work effort.

3.2. Progress in Production Technology

The progress in production technology is reflected in an increase in A. Differentiating equations (11)–(15) with respect to A, we have the comparative statics results as follows.

$$\frac{\partial c_1}{\partial A}=\frac{-(\alpha\beta A)^{1/(1-\alpha)}}{(1-\alpha)(1+\gamma)A}<0, \tag{22}$$

$$\frac{\partial c_2}{\partial A}=\frac{(\alpha\beta A)^{\alpha/(1-\alpha)}}{(1-\alpha)(1+\gamma)}>0, \tag{23}$$

$$\frac{\partial(c_1+\beta c_2)}{\partial A}=\frac{\beta(\alpha\beta A)^{\alpha/(1-\alpha)}}{(1+\gamma)}>0, \tag{24}$$

$$\frac{\partial n}{\partial A}=\frac{(\alpha\gamma/A)y}{(1-\alpha)^2(1+\gamma)(\alpha\beta A)^{1/(1-\alpha)}}>0, \tag{25}$$

$$\frac{\partial k}{\partial A}=\frac{(\alpha\beta A)^{1/(1-\alpha)}}{(1-\alpha)(1+\gamma)A}>0. \tag{26}$$

The progress in production technology generates a decrease in c_1 and an increase in c_2. Change in total consumption $c_1+\beta c_2$ is definitely positive. Also, changes in work effort n and investment k are positive. (22)–(26) imply that, other things being equal, Crusoe decreases breakfast and increases both investment and work effort in response to the progress in production technology. Thus the volume of dinner increases. Because the degree of increase in breakfast dominates that of decrease in dinner, the total consumption within a day increases. Also, by the increase of work effort, Crusoe enjoys less leisure.

3.3. Shift in Intraday Consumption Preference

We consider the shift in intraday consumption preference by a rise in β. Differentiating equations (11)–(15) with respect to β, we have the following comparative statics results.

$$\frac{\partial c_1}{\partial \beta}=\frac{-(\alpha\beta A)^{1/(1-\alpha)}}{(1-\alpha)(1+\gamma)\beta}<0, \tag{27}$$

$$\frac{\partial c_2}{\partial \beta}=\frac{(\alpha\beta A)^{1/(1-\alpha)}+(\gamma/\beta)y}{(1-\alpha)(1+\gamma)\beta}>0, \tag{28}$$

$$\frac{\partial (c_1+\beta c_2)}{\partial \beta}=\frac{A(\alpha\beta A)^{\alpha/(1-\alpha)}}{(1+\gamma)}>0, \tag{29}$$

$$\frac{\partial n}{\partial \beta}=\frac{(\alpha\gamma/\beta)y}{(1-\alpha)^2(1+\gamma)(\alpha\beta A)^{1/(1-\alpha)}}>0, \tag{30}$$

$$\frac{\partial k}{\partial \beta}=\frac{(\alpha\beta A)^{1/(1-\alpha)}}{(1-\alpha)(1+\gamma)\beta}>0. \tag{31}$$

The increase in β brings about a decrease in c_1, but an increase in c_2. Total consumption $c_1+\beta c_2$ definitely increases. The effects on work effort and investment are positve. (27)–(31) show that Crusoe decreases breakfast and increases both investment and work effort by the shift of intraday consumption preference, other things being equal. Thus the dinner he eats increases. The intraday total consumption increases because the degree of increase of dinner dominates that of decrease of breakfast. Moreover, the leisure that Crusoe enjoys decreases by the increase of the work effort. As easily recognized, the effects of the shift in intraday consumption preference, described by a rise in β, on the optimal allocation of consumption, work effort, and investment are similar to those of the progress in production technology.

4. Peculiar Equilibrium

In this section we show two peculiar equilibriums. They are ***no breakfast optimum*** and ***no dinner optimum***. From the Kuhn-Tucker conditions and budget constraints, we can obtain such equilibriums as a result of rational behavior of Robinson Crusoe.

4.1. No Breakfast Optimum

If y is equal to $[(1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}]/(1-\alpha+\gamma)$, then $c_1=0$. [See Figure 1.] At the same time, k, n, and c_2 become as follows :

$$k=y, \tag{32}$$

$$n=(1-\alpha)/(1-\alpha+\gamma), \tag{33}$$

$$c_2=Ay^{\alpha}[(1-\alpha)/(1-\alpha+\gamma)]^{1-\alpha}. \tag{34}$$

Equation (32) means that Robinson Crusoe uses all initial endowment only for investment. Equation (33) indicates the level of work effort, which is determined by α and γ. Work effort does not depend on β, A, and y. Note that n is not unity. Even if there is no breakfast, Robinson Crusoe enjoys leisure which is $1-n=\gamma/(1-\alpha+\gamma)$. Equation (34) implies the level of dinner when there is no breakfast. Equations (32)-(34) suggest that ***no breakfast optimum*** emerges as a result of rational behavior of Robinson Crusoe, when the initial endowment y is equal to the minimum critical value which is determined by preference and production parameters, β, γ, A, and α.

4.2. No Dinner Optimum

If y is equal to $[(1-\alpha)(\alpha\beta A)^{1/(1-\alpha)}]/(\alpha\gamma)$, then $c_2=0$. [See Figure 1.] At the same time, c_1, k, and n become as follows :

$$c_1=y, \tag{35}$$

$$k=0, \tag{36}$$

$$n=0. \tag{37}$$

Equation (35) shows that all initial endowment is consumed as breakfast. Equation (36) means that there is no investment. Equation (37) implies no work effort. Thus, Robinson Crusoe enjoys all time endowment as leisure. Equations (35)−(37) suggest that *no dinner optimum*, where he consumes all initial endowment as breakfast and enjoys maximal leisure, appears as a result of rational behavior of Robinson Crusoe, when the initial endowment y is equeal to the maximum critical value which is determined by preference and production parameters, β, γ, A, and α.

5. Concluding Remarks

In this paper we study the intraday model of Robinson Crusoe where total consumption in a day is composed of two parts : consumption in AM (breakfast) and consumption in PM (dinner). We show the intraday fluctuation of consumption in response to various changes in exogenous variables. Comparative statics results are summarized in Table 1. Also we show the possibilities of ***no breakfast optimum*** and ***no dinner optimum*** as a result of rational behavior of Robinson Crusoe.

Table 1.

	c_1	c_2	$c_1+\beta c_2$	n	k
y	+	−	+	−	−
A	−	+	+	+	+
β	−	+	+	+	+

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