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Abstract. If you release a slinky hung in midair from its initial stationary position, you can observe something interesting: the bottom of the slinky is at rest in midair for a while. Among physicists, this phenomenon is well known and has been discussed from the physical point of view. In this paper we deal with the problem of a freely falling slinky from the purely mathematical point of view. Through the rigorous mathematical process we obtained an explicit formula which describes the motion of the freely falling slinky and it turns out that this is a solution to an inhomogeneous wave equation. Our work gives the mathematical justification of the results obtained by physicists.

1 Introduction

A slinky is a loosely coiled spring-like toy which was invented by Richard T. James in 1943 and put on the market in the United State in 1945. Betty James, the inventor's wife, name the toy "slinky" because its movement is quite smooth and elegant ([8]). The way children play with a slinky is, for example, as follows: put the slinky at the head of the stairs and let go of it, and it will go down the stairs step by step autonomously in a humorous way. They have fun watching its behavior. Apart from children's play the slinky is used in scientific experiments in school to observe wave phenomena such as longitudinal waves or transverse waves.

Figure 1: A rainbow-colored plastic slinky.

Another amazing phenomenon about the slinky is the following: suspend the slinky in the air with its either end held by hand until it is stabilized and drop it. Surprisingly, its bottom stays at rest for a while as you see in Figure 2. According to physicists the reason why this counterintuitive phenomenon happens is explained as follows: the information that the topmost end of the slinky is released travels

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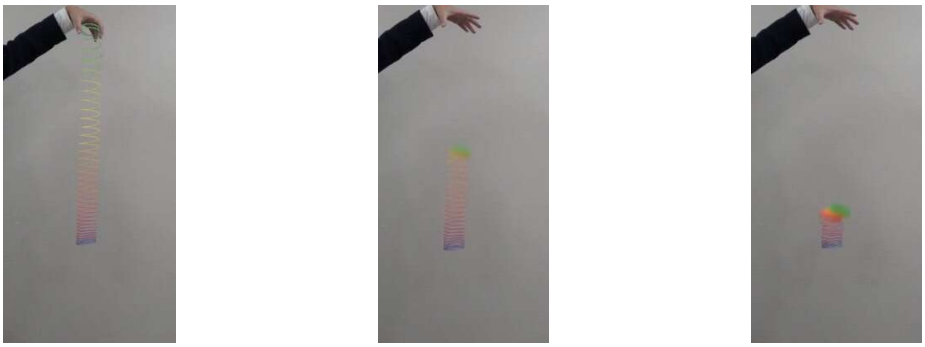
through it in a finite speed and it takes some time until this information arrives at its bottommost end, so the bottom levitates in the air. In physicists' circles this interesting phenomenon is well known and has been discussed in a lot of papers ([1], [2], [3], [4], [5], [6]). In particular, [3] compares two real slinkies with different parameters, namely metallic and plastic ones, and discusses the fitness to the data of the model. Work on this matter done by mathematicians is, however, quite few. In fact, as far as we know only [7] is an available article. Physicists' arguments are based on physical principles and full of technical terms. So it is hard for untrained mathematicians to follow their discussions. The aim of this paper is to understand the behavior of a freely falling slinky by analyzing a simplified model in a mathematically rigorous way. Although most of the results obtained in this paper may not be new to physicists, mathematical analysis of the phenomenon is fertile and interesting in its own right.



(a) initial state **(b)** middle state **(c)** final state

Figure 2: A slinky in free fall.

Let A and B be the end points of the slinky. Let $\mathrm{P}(\xi)$ be the point whose distance from A is ξ when the slinky is not stretched nor compressed. Suspend the slinky with A fixed as the upper end in the gravitational field with constant acceleration g due to gravity. Wait until the slinky is in equilibrium and let go of it at time $t = 0$. We take a vertical x-axis whose positive direction is downward. Moreover we assume that A is located at origin O at $t = 0$. Let L, K and M be the equilibrium length, spring constant and mass of the slinky respectively. A real slinky is a three-dimensional helix consisting of several turns with a certain diameter. Moreover it cannot be compressed further when a turn of the slinky is in contact with another. The motion of the slinky is fully analyzed by the study of its vertical displacement and its horizontal rotation. But in order to simplify the situation, we ignore the horizontal motion and we assume that the slinky is a simple one-dimensional elastic spring, for which Hooke's law is in force beyond its elastic limits in infinitesimal

parts Let $u(\xi, t)$ be the vertical position of $\mathrm{P}(\xi)$ at time t. What we want to know is the behavior of $u(\xi, t)$. To this end we need to derive the equation of motion for $u(\xi, t)$. In deriving such an equation we discretize the slinky, because we want to make this paper accessible to anyone unfamiliar with continuum dynamics. Another reason is that we want to know what happens at an infinitesimal level.

The organization of the present paper is as follows: in Section 2 we set up a discrete oscillating system and derive a system of linear differential equations with constant coefficients which governs the dynamics of the system. It turns out that the coefficient matrix is a real symmetric. Moreover its eigenvalues and corresponding eigenvectors are concretely calculated in Proposition 3.1. In Section 3 thanks to this result we obtain the explicit formula of the solutions $x_i(t)$ in Theorem 3.4. In Section 4 by expanding the formula in Maclaurin series we analyze the infinitesimal behavior of $x_i(t)$ near $t = 0$. The result is stated in Theorem 4.3. In Section 5 by passing to the limit we obtain $u(\xi, t)$. In Theorem 5.3, $u(\xi, t)$ is expressed as $x(\xi, \tau)$ in terms of dimensionless variables $\lambda = \dfrac{\xi}{L}$ and $\tau = \dfrac{\tau}{T}$. Finally in Section 6 we study the behavior of $u(\xi, t)$. Furthermore, in Theorem 6.1 we present the partial differential equation $u(\xi, t)$ satisfies.

2 Setting of the discrete model

As explained in the introduction, we regard a slinky as a one-dimensional elastic spring and study its behavior in a free fall state by using its discrete model. As a discrete model we consider an oscillating system of $n + 1$ weights connected with n springs. We assume that every weight has mass m_{w} and every spring is massless and has equilibrium length l_{s} and spring constant k_{s}. First the topmost weight is held by an observer at origin O and the other weights are hung beneath it and held stationary (see Figure 3). Then he drops the weight at time $t = 0$ and weights start to fall freely. For $i = 0, 1, \ldots, n$, let $x_i(t)$ be the position of the i-th weight counted from the top at time t. Note that $x_0(0) = 0$ by assumption. For notational simplicity $x_i(t)$ will be sometimes denoted by x_i. In order to make sure that this model is reasonable enough, we carried out a preliminary experiment. Figure 4 shows some pictures taken in the experiment. According to these, it seems that freely falling weights

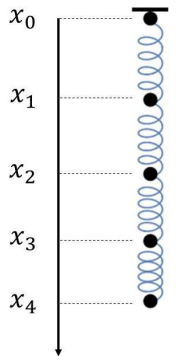
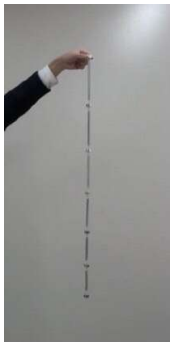


Figure 3: The initial state in the case of $n = 4$.

behave like a freely falling slinky.


(a) initial state


(b) middle state


(c) final state

Figure 4: 7 weights connected with 6 springs in free fall.

We derive differential equations which describe the motion of the discrete oscillating system in free fall by applying Newton's second law to each weight. To this end let us consider what forces act on it. In the course of free fall the gravitational force $m_{\mathrm{w}}g$ acts on each weight. The way the tension force of spring acts depends on where the weight is located. On the topmost weight labeled $i = 0$ the lower spring exerts a tension $k_{\mathrm{s}}(x_1 - x_0 - l_{\mathrm{s}})$, so we have

$$m_{\mathrm{w}}\frac{d^2x_0}{dt^2} = k_{\mathrm{s}}(x_1 - x_0 - l_{\mathrm{s}}) + m_{\mathrm{w}}g.$$

On the weight labeled i $(1 \leq i \leq n-1)$ the upper and lower springs exert tensions $-k_{\mathrm{s}}(x_i - x_{i-1} - l_{\mathrm{s}})$ and $k_{\mathrm{s}}(x_{i+1} - x_i - l_{\mathrm{s}})$ respectively, so we have

$$m_{\mathrm{w}}\frac{d^2x_i}{dt^2} = k_{\mathrm{s}}(x_{i+1} - 2x_i + x_{i-1}) + m_{\mathrm{w}}g.$$

On the bottommost weight labeled $i = n$ the upper spring exerts a tension $-k_{\mathrm{s}}(x_n - x_{n-1} - l_{\mathrm{s}})$, so we have

$$m_{\mathrm{w}}\frac{d^2x_n}{dt^2} = k_{\mathrm{s}}(l_{\mathrm{s}} - x_n + x_{n-1}) + m_{\mathrm{w}}g.$$

These results lead to the conclusion that variables x_i satisfy the following system of linear differential equations

$$\begin{cases} \dfrac{d^2x_0}{dt^2} &= \alpha(x_1 - x_0 - l_{\mathrm{s}}) + g, \\ \dfrac{d^2x_i}{dt^2} &= \alpha(x_{i+1} - 2x_i + x_{i-1}) + g \quad (1 \leq i \leq n-1), \\ \dfrac{d^2x_n}{dt^2} &= \alpha(l_{\mathrm{s}} - x_n + x_{n-1}) + g \end{cases} \tag{2.1}$$

where $\alpha = \dfrac{k_{\mathrm{s}}}{m_{\mathrm{w}}}$. To solve (2.1) uniquely, we must set up appropriate initial conditions. Considering all weights stay at rest at time $t = 0$, it is clear $\dfrac{dx_i}{dt}(0) = 0$ for $i = 0, 1, \ldots, n$.

Next we find $x_i(0)$. At time $t = 0$ the topmost weight is in equilibrium because it is supported with an external force exerted by the observer. On the other hand, the other weights are in equilibrium by the gravitational forces and tension forces of springs. Hence at $t = 0$ we can put $\dfrac{d^2x_i}{dt^2} = 0$ in (2.1) except $i = 0$, so we have the following system of linear equations

$$\begin{cases} x_{i+1}(0) - 2x_i(0) + x_{i-1}(0) = -\dfrac{g}{\alpha} \quad (1 \leq i \leq n-1), & \text{(2.2a)} \\ l_{\mathrm{s}} - x_n(0) + x_{n-1}(0) = -\dfrac{g}{\alpha}. & \text{(2.2b)} \end{cases}$$

Summing (2.2a) over $i = 1, 2, \ldots, n-1$ and (2.2b), we get $x_1(0) = l + \dfrac{n}{\alpha}g$ because we assume $x_0(0) = 0$. Summing (2.2a) over $i = 1, 2, \ldots, j-1$, we have

$$x_j(0) - x_{j-1}(0) = x_1(0) - (j-1)\frac{g}{\alpha}$$

for $j = 1, 2, \ldots, n$. Now it is easy to get $x_i(0) = il_{\mathrm{s}} + \dfrac{g}{2\alpha}i(2n - i + 1)$ for $i = 0, 1, 2, \ldots, n$.

To summarize, the initial conditions are

$$\begin{cases} \dfrac{dx_i}{dt}(0) & = 0, \\ x_i(0) & = il_{\mathrm{s}} + \dfrac{g}{2\alpha}i(2n - i + 1) \end{cases} \tag{2.3}$$

for $i = 0, 1, \ldots, n$.

Remark 2.1 In order to make our discrete model simple and easy to analyze, we assumed the following ideal situation. Adjacent weights interact each other via the tension force of the massless spring between them and it obeys Hooke's law perfectly beyond the equilibrium length, e.g., the tension $k_{\mathrm{s}}(x_{i+1} - x_i - l_{\mathrm{s}})$ exerted by the lower spring on the i-th weight is still in force even if $x_{i+1} - x_i - l_{\mathrm{s}} < 0$. Moreover, the weights are transparent, i.e., during free fall they pass through without colliding each other.

3 Explicit formula of the solutions

We rewrite the system of linear differential equations (2.1) in a vector form. To this end we define $(n+1)$-dimensional vectors $\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}$ and a square matix A of order $n+1$ by

$$\boldsymbol{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \ \boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

Then (2.1) can be expressed as

$$\frac{d^2\boldsymbol{x}}{dt^2} = \alpha A\boldsymbol{x} + \alpha l_{\mathrm{s}}\boldsymbol{b} + g\boldsymbol{a}. \tag{3.1}$$

Clearly the square matrix A is a real symmetric matrix, so by the general theory of Linear algebra A has $n+1$ real eigenvalues counted with multiplicity and can be diagonalized by an ortogonal matrix. Moreover we have explicite eigenvalues and eigenvectors of A.

Proposition 3.1 It holds that

(1) A has $n+1$ distinct nonpositive eigenvalues $-\mu_0^2 > -\mu_1^2 > \cdots > -\mu_n^2$ where

$$\mu_j = 2\sin\frac{\theta_j}{2}, \quad \theta_j = \frac{\pi}{n+1}j \quad (0 \le j \le n).$$

(2) The i-th component p_{ij} of the normalized eigenvector $\boldsymbol{p}_j$ for the eigenvalue $-\mu_j^2$ is given by

$$p_{ij} = \begin{cases} \dfrac{1}{\sqrt{n+1}} & (j=0), \\ \sqrt{\dfrac{2}{n+1}}\cos(2i+1)\dfrac{\theta_j}{2} & (1 \le j \le n). \end{cases} \tag{3.2}$$

Proof Let $\boldsymbol{y} = {}^t(y_0, y_1, \ldots, y_n) \neq \mathbf{0}$ be an eigenvector of A and λ its corresponding eigenvalue. Then the eigenequation $(A - \lambda E)\boldsymbol{y} = \mathbf{0}$ holds . For notational convenience put $-2\nu = -2 - \lambda$. Since

$$(A - \lambda E)\boldsymbol{y} = \begin{pmatrix} 1-2\nu & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2\nu & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2\nu & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2\nu & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2\nu & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1-2\nu \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},$$

it holds that

$$\begin{cases} (1-2\nu)y_0 + y_1 & = 0, \\ y_0 - 2\nu y_1 + y_2 & = 0, \\ \qquad \vdots & \\ y_{n-2} - 2\nu y_{n-1} + y_n & = 0, \\ y_{n-1} + (1-2\nu)y_n & = 0. \end{cases} \tag{3.3}$$

We define auxiliary variables y_{-1} and y_{n+1} by $y_{-1} = y_0$ and $y_{n+1} = y_n$. Then the variables $y_i, -1 \leq i \leq n+1$ satisfy the following linear recurrence relation with constant coefficients

$$y_i - 2\nu y_{i+1} + y_{i+2} = 0 \ \ (-1 \leq i \leq n-1). \tag{3.4}$$

We solve (3.4) under the boundary conditions

$$y_{-1} = y_0, \tag{3.5}$$
$$y_{n+1} = y_n. \tag{3.6}$$

The characteristic polynomial of (3.4) is $\rho^2 - 2\nu\rho + 1$ and its roots are $\rho = \nu \pm \sqrt{\nu^2 - 1}$. Put

$$\rho_1 = \nu + \sqrt{\nu^2 - 1},$$
$$\rho_2 = \nu - \sqrt{\nu^2 - 1}.$$

Note that $\rho_1 + \rho_2 = 2\nu$ and $\rho_1\rho_2 = 1$.

Case 1 : $\nu^2 \neq 1$.

In this case, $\rho_1 \neq \rho_2$. Then the solution of (3.4) can be expressed as

$$y_i = \beta_1 \rho_1^i + \beta_2 \rho_2^i \quad (-1 \leq i \leq n+1) \tag{3.7}$$

with unknown constants β_1 and β_2. By (3.5) we have $\beta_1 \rho_1^{-1} + \beta_2 \rho_2^{-1} = \beta_1 + \beta_2$. Since $\rho_1^{-1} - 1 \neq 0$ and $\rho_2 = \rho_1^{-1}$,

$$\beta_1 = \frac{1 - \rho_2^{-1}}{\rho_1^{-1} - 1}\beta_2 = \frac{1 - \rho_1}{\rho_1^{-1} - 1}\beta_2 = \rho_1 \beta_2. \tag{3.8}$$

By (3.6) it holds that $\beta_1 \rho_1^{n+1} + \beta_2 \rho_2^{n+1} = \beta_1 \rho_1^n + \beta_2 \rho_2^n$ which implies

$$\beta_1 \rho_1^n (\rho_1 - 1) = \beta_2 \rho_2^n (-\rho_2 + 1).$$

Using (3.8) and $\rho_2 = \rho_1^{-1}$ we have $\beta_2 \rho_1^{n+1}(\rho_1 - 1) = \beta_2 \rho_1^{-n-1}(-1 + \rho_1)$. Since $\beta_2 \neq 0$ and $\rho_1 \neq 1$ we get

$$\rho_1^{2(n+1)} = 1. \tag{3.9}$$

If $|\nu| > 1$, then ρ_1 is real and $|\rho_1| \neq 1$, which contradicts (3.9). Therefore $|\nu| < 1$. Put $\nu = \cos\theta$ $(0 < \theta < \pi)$ and $\epsilon = e^{\sqrt{-1}\theta}$. Then

$$\rho_1 = \nu + \sqrt{\nu^2 - 1} = \cos\theta + \sqrt{-1}\sin\theta = e^{\sqrt{-1}\theta} = \epsilon.$$

By (3.9) we have $\epsilon^{2(n+1)} = e^{2\sqrt{-1}(n+1)\theta} = 1$. Hence θ must be one of n values $\theta_j = \dfrac{\pi}{n+1} j \quad (j = 1, 2, \cdots n)$. Denote ν and λ for $\theta = \theta_j$ by ν_j and λ_j respectively. Then

$$\lambda_j = -2(1 - \nu_j) = -2(1 - \cos\theta_j) = -\mu_j^2$$

where $\mu_j = 2\sin\dfrac{\theta_j}{2}$. By (3.7) and (3.8) we have

$$y_i = \beta_1 \rho_1^i + \beta_2 \rho_2^i = \beta_2 \epsilon \epsilon^i + \beta_2 \epsilon^{-i} = \beta_2(\epsilon^{i+1} + \epsilon^{-i}).$$

Set $\beta_2 = \dfrac{\gamma}{2}\epsilon^{-\frac{1}{2}}$, then

$$y_i = \frac{\gamma}{2}(\epsilon^{i+\frac{1}{2}} + \epsilon^{-i-\frac{1}{2}}) = \gamma\,\mathrm{Re}(\epsilon^{i+\frac{1}{2}}) = \gamma\cos(2i+1)\frac{\theta_j}{2}.$$

We find $\gamma > 0$ satisfying $||\boldsymbol{y}|| = 1$.

$$\begin{aligned} ||\boldsymbol{y}||^2 &= \gamma^2 \sum_{i=0}^{n} \cos^2(2i+1)\frac{\theta_j}{2} = \gamma^2 \sum_{i=0}^{n} \frac{1}{2}\Big(1 + \cos(2i+1)\theta_j\Big) \\ &= \gamma^2 \Big(\frac{n+1}{2} + \frac{1}{2}\sum_{i=0}^{n} \cos(2i+1)\theta_j\Big). \end{aligned}$$

By (1) in Lemma 3.5 at the end of this section, we have

$$\sum_{i=0}^{n} \cos(2i+1)\theta_j = \frac{1}{2}\frac{\sin 2(n+1)\theta_j}{2\sin\theta_j} = 0,$$

so $||\boldsymbol{y}||^2 = \dfrac{n+1}{2}\gamma^2$. Thus $\gamma = \sqrt{\dfrac{2}{n+1}}$. Let $\boldsymbol{p}_j$ be the normalized eigenvector for λ_j. Then the i-th commponent p_{ij} of $\boldsymbol{p}_j$ is given by

$$p_{ij} = \sqrt{\frac{2}{n+1}}\cos(2i+1)\frac{\theta_j}{2}.$$

Case 2 : $\nu = 1$.

In this case, (3.3) can be easily solved and we obtain $y_0 = y_1 = y_2 = \cdots = y_n$. Therefore the eigenvalue λ_0 and the corresponding normalized eigenvector $\boldsymbol{p}_0$ are clealy

$$\lambda_0 = 0, \quad \boldsymbol{p}_0 = \frac{1}{\sqrt{n+1}}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Case 3 : $\nu = -1$.

In this case, the solution of (3.4) is expressed as $y_i = (\beta_1' + \beta_2' i)(-1)^i$ with unknown constants β_1' and β_2'. The solution satisfying the boundary conditions (3.5) and (3.6) must satisfy $\beta_1' = \beta_2' = 0$, thereby $y_i = 0$ which contradicts $\boldsymbol{y} \neq \boldsymbol{0}$.

This completes the proof of the theorem. ■

Let $P = (\boldsymbol{p}_0\, \boldsymbol{p}_1\, \cdots\, \boldsymbol{p}_n)$. Then P is an orthogonal matrix and $P^{-1} = {}^tP$, so we have

$$P^{-1}AP = \begin{pmatrix} -\mu_0^2 & 0 & \cdots & 0 & 0 \\ 0 & -\mu_1^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\mu_{n-1}^2 & 0 \\ 0 & 0 & \cdots & 0 & -\mu_n^2 \end{pmatrix}.$$

Multiplying the both sides of (3.1) by P^{-1} from left, we have

$$\frac{d^2P^{-1}\boldsymbol{x}}{dt^2} = \alpha(P^{-1}AP)P^{-1}\boldsymbol{x} + \alpha l_{\mathrm{s}}P^{-1}\boldsymbol{b} + gP^{-1}\boldsymbol{a}. \tag{3.10}$$

For simplicity, put

$$\boldsymbol{z} = P^{-1}\boldsymbol{x}, \tag{3.11}$$
$$\boldsymbol{c} = \alpha l_{\mathrm{s}} P^{-1}\boldsymbol{b} + gP^{-1}\boldsymbol{a}, \tag{3.12}$$
$$B = P^{-1}AP.$$

Then (3.10) is turned into

$$\frac{d^2\boldsymbol{z}}{dt^2} = \alpha B\boldsymbol{z} + \boldsymbol{c}.$$

Considering each commponent of the above equatin, we get a single second order linear differential equation with constant coefficients

$$\frac{d^2 z_j}{dt^2} = -\alpha\mu_j^2 z_j + c_j \tag{3.13}$$

for $j = 0, 1, \ldots, n$.

Case 1 : $j \neq 0$.

Since $\mu_j \neq 0$, (3.13) can be rewritten as

$$\frac{d^2}{dt^2}\Big(z_j(t) - \frac{c_j}{\alpha\mu_j^2} \Big) = -\alpha\mu_j^2\Big(z_j(t) - \frac{c_j}{\alpha\mu_j^2} \Big).$$

Therefore we have

$$z_j(t) - \frac{c_j}{\alpha\mu_j^2} = A_j \cos\sqrt{\alpha}\mu_j t + B_j \sin\sqrt{\alpha}\mu_j t$$

with unknown constans A_j and B_j. By the initial condition $\dfrac{dz_j}{dt}(0) = 0$, it is immediate that $B_j = 0$. Hence we have

$$z_j(t) = A_j \cos\sqrt{\alpha}\mu_j t + \frac{c_j}{\alpha\mu_j^2}. \tag{3.14}$$

By putting $t = 0$ we get $A_j = z_j(0) - \dfrac{c_j}{\alpha\mu_j^2}$.

Case 2 : $j = 0$.

Since $\mu_0 = 0$, (3.13) is simply $\dfrac{d^2 z_0}{dt^2} = c_0$, which implies $z_0(t) = \dfrac{1}{2}c_0 t^2 + C_1 t + C_2$ with unknown constants C_1 and C_2.

By (3.11) and the initial condition $\dfrac{d}{dt}\boldsymbol{x}(0) = 0$, we have $C_1 = 0$. Since $z_0(0) = C_2$, we obtain

$$z_0(t) = \frac{1}{2}c_0 t^2 + z_0(0). \tag{3.15}$$

With these calculations done, we come to the conclusion that we need to calculate c_j and $z_j(0)$ for $j = 0, 1, \ldots, n$ in order to solve (3.13).

Lemma 3.2 For $j = 0, 1, \ldots, n$, we have

$$c_j = \begin{cases} g\sqrt{n+1} & j = 0 \ , \\ \alpha l_{\rm s}\sqrt{\dfrac{2}{n+1}}\big((-1)^j - 1 \big) \cos\dfrac{\theta_j}{2} & (1 \le j \le n) \ . \end{cases} \tag{3.16}$$

Proof By (3.12) and $P^{-1} = {}^tP = {}^t(\boldsymbol{p}_0\, \boldsymbol{p}_1 \cdots \boldsymbol{p}_n)$, we have

$$c_j = \alpha l_{\rm s}(\boldsymbol{p}_j, \boldsymbol{b}) + g(\boldsymbol{p}_j, \boldsymbol{a}) \tag{3.17}$$

where $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^d$. The first inner product is calculated as follows :

$$(\boldsymbol{p}_j, \boldsymbol{b}) = p_{0j}(-1) + p_{nj} = p_{nj} - p_{0j}. \tag{3.18}$$

For $j = 0$

$$(\boldsymbol{p}_0, \boldsymbol{b}) = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+1}} = 0 \ .$$

For $1 \le j \le n$

$$\begin{aligned} (\boldsymbol{p}_j, \boldsymbol{b}) &= \sqrt{\frac{2}{n+1}}\Big(\cos{(2n+1)}\frac{\theta_j}{2} - \cos\frac{\theta_j}{2} \Big) \\ &= \sqrt{\frac{2}{n+1}}((-1)^j - 1)\cos\frac{\theta_j}{2} \ . \end{aligned}$$

because $(2n+1)\dfrac{\theta_j}{2} = \pi j - \dfrac{\theta_j}{2}$. Therefore

$$(\boldsymbol{p}_j, \boldsymbol{b}) = \begin{cases} 0 & (j = 0) \ , \\ \sqrt{\dfrac{2}{n+1}}\big((-1)^j - 1 \big) \cos\dfrac{\theta_j}{2} & (1 \le j \le n) \ . \end{cases} \tag{3.19}$$

Since $\boldsymbol{a} = \sqrt{n+1}\,\boldsymbol{p}_0$, the second inner product is calculated as follows:

$$(\boldsymbol{p}_j, \boldsymbol{a}) = \sqrt{n+1}(\boldsymbol{p}_j, \boldsymbol{p}_0) = \begin{cases} \sqrt{n+1} & (j = 0) \ , \\ 0 & (1 \le j \le n) \ . \end{cases} \tag{3.20}$$

By (3.17), (3.19) and (3.20) , we have

$$c_j = \begin{cases} g\sqrt{n+1} & (j = 0) \ , \\ \alpha l_{\rm s}\sqrt{\dfrac{2}{n+1}}\big((-1)^j - 1 \big) \cos\dfrac{\theta_j}{2} & (1 \le j \le n) \ . \end{cases}$$

This completes the proof of the lemma. ■

The next lemma gives an explicit formula for $z_j(0)$.

Lemma 3.3 For $j = 1, 2, \ldots, n$, $z_j(0)$ is given by

$$z_j(0) = \sqrt{\frac{2}{n+1}}\Big(l_{\rm s}(-1+(-1)^j) - \frac{g}{\alpha}(n+1) \Big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}}.$$

Proof By (3.11) we have $z_j(0) = (\boldsymbol{p}_j, \boldsymbol{x}(0))$, which is calculated as follows:

$$\begin{aligned} z_j(0) &= \sum_{i=0}^{n} p_{ij}\ x_i(0) = \sum_{i=0}^{n} \sqrt{\frac{2}{n+1}} \cos(2i+1)\frac{\theta_j}{2}\ \Big(i l_{\rm s} + \frac{g}{2\alpha} i(2n+1-i) \Big) \\ &= \sqrt{\frac{2}{n+1}}\Big(\big(l_{\rm s} + \frac{g}{2\alpha}(2n+1) \big) \sum_{i=0}^{n} i\cos(2i+1)\frac{\theta_j}{2} - \frac{g}{2\alpha}\sum_{i=0}^{n} i^2\cos(2i+1)\frac{\theta_j}{2} \Big). \end{aligned} \tag{3.21}$$

Here we use (2) and (3) in Lemma 3.5 with $\theta = \dfrac{\theta_j}{2}$. Observing $2(n+1)\theta = \pi j$, we have

$$\sum_{i=0}^{n} i\cos(2i+1)\frac{\theta_j}{2} = \big(-1+(-1)^j\big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}} \tag{3.22}$$

and

$$\sum_{i=0}^{n} i^2\cos(2i+1)\frac{\theta_j}{2} = \big(1+(2n+1)(-1)^j\big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta}{2}}. \tag{3.23}$$

By (3.21), (3.22) and (3.23) we finally get

$$\begin{aligned} z_j(0) &= \sqrt{\frac{2}{n+1}}\left\{ \Big(l_{\rm s} + \frac{g}{2\alpha}(2n+1) \Big)\big(-1+(-1)^j\big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}} \right. \\ &\qquad \left. - \frac{g}{2\alpha}\big(1+(2n+1)(-1)^j\big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta}{2}} \right\} \\ &= \sqrt{\frac{2}{n+1}}\Big(l_{\rm s}(-1+(-1)^j) - \frac{g}{\alpha}(n+1) \Big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}}, \end{aligned}$$

which proves the lemma. ■

Now we are in a position to solve the equations (3.1).

Theorem 3.4 The solutions of (3.1) satisfying the initial condition (2.3) are given by

$$x_i(t) = il_{\rm s} + \frac{g}{2\alpha} i(2n+1-i) + \frac{1}{2}gt^2 + \frac{g}{2\alpha}\sum_{j=1}^{n} \cos(2i+1)\frac{\theta_j}{2}\ \frac{\cos\frac{\theta_j}{2}}{\sin^2\frac{\theta_j}{2}}(\,1-\cos\sqrt{\alpha}\mu_j t\,)$$

for $i = 0, 1, \cdots, n$.

Proof By using Lemmas 3.2 and 3.3 we can calculate A_j $(1 \le j \le n)$ as follows:

$$\begin{aligned} A_j &= z_j(0) - \frac{c_j}{\alpha\mu_j^2} \\ &= \sqrt{\frac{2}{n+1}}\Big(\, l_{\rm s}(-1+(-1)^j) - \frac{g}{\alpha}(n+1)\,\Big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}} - \sqrt{\frac{2}{n+1}}l_{\rm s}\big(\,(-1)^j-1\,\big)\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}} \\ &= -\frac{g}{\alpha}\sqrt{2(n+1)}\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}}. \end{aligned} \tag{3.24}$$

By (3.11), we have $\boldsymbol{x}(t) = P\boldsymbol{z}(t)$. Hence

$$\begin{aligned} x_i(t) &= p_{i0}\, z_0(t) + \sum_{j=1}^{n} p_{ij}\, z_j(t) \\ &= \frac{1}{\sqrt{n+1}}\Big(\frac{1}{2}c_0t^2 + z_0(0)\Big) + \sum_{j=1}^{n} p_{ij}\Big(A_j\cos\sqrt{\alpha}\mu_j t + \frac{c_j}{\alpha\mu_j^2}\Big) \\ &= \frac{1}{\sqrt{n+1}}z_0(0) + \sum_{j=1}^{n} p_{ij}\Big(A_j + \frac{c_j}{\alpha\mu_j^2}\Big) + \frac{1}{2}gt^2 + \sum_{j=1}^{n} p_{ij}A_j(\cos\sqrt{\alpha}\mu_j t - 1) \\ &= x_i(0) + \frac{1}{2}gt^2 + \sum_{j=1}^{n} p_{ij}A_j(\cos\sqrt{\alpha}\mu_j t - 1). \end{aligned}$$

By (2.3) and (3.24) we obtain

$$\begin{aligned} x_i(t) &= il_{\rm s} + \frac{g}{2\alpha}i(2n+1-i) + \frac{1}{2}gt^2 \\ &\quad + \sum_{j=1}^{n}\sqrt{\frac{2}{n+1}}\cos(2i+1)\frac{\theta_j}{2}\Big(-\frac{g}{\alpha}\sqrt{2(n+1)}\frac{\cos\frac{\theta_j}{2}}{4\sin^2\frac{\theta_j}{2}}\Big)(\cos\sqrt{\alpha}\mu_j t - 1) \\ &= il_{\rm s} + \frac{g}{2\alpha}i(2n+1-i) + \frac{1}{2}gt^2 \\ &\quad + \frac{g}{2\alpha}\sum_{j=1}^{n}\cos(2i+1)\frac{\theta_j}{2}\,\frac{\cos\frac{\theta_j}{2}}{\sin^2\frac{\theta_j}{2}}(1-\cos\sqrt{\alpha}\mu_j t). \end{aligned}$$

This completes the proof of the theorem. ■

We close this section by collecting some formulas needed in this section. The proof of the next lemma is easily given by induction on i, so we omit it.

Lemma 3.5 For $n = 0, 1, 2, \ldots$, we have the following formulas

$$(1)\quad \sum_{i=0}^{n} \cos(2i+1)\theta = \frac{\sin 2(n+1)\theta}{2\sin\theta}.$$

$$(2)\quad \sum_{i=0}^{n} i\cos(2i+1)\theta = \frac{1}{4\sin^2\theta}\Big(-\cos\theta + \cos\theta\cos 2(n+1)\theta + (2n+1)\sin\theta\sin 2(n+1)\theta \Big).$$

$$(3)\quad \sum_{i=0}^{n} i^2\cos(2i+1)\theta = \frac{1}{4\sin^3\theta}\Big(\big(2n(n+1)\sin^2\theta - \cos^2\theta \big)\sin 2(n+1)\theta + \sin\theta\cos\theta\big(1 + (2n+1)\cos 2(n+1)\theta \big) \Big).$$

4 Infinitesimal behavior of the solutions near $t = 0$

We investigate the infinitesimal behavior of $x_i(t)$ near $t = 0$ by expanding $x_i(t)$ in the Maclaurin series.

By Theorem 3.4 we have

$$\begin{aligned} x_i(t) &= x_i(0) + \frac{1}{2}gt^2 + \frac{g}{2\alpha}\sum_{j=1}^{n} \cos(2i+1)\frac{\theta_j}{2}\,\frac{\cos\frac{\theta_j}{2}}{\sin^2\frac{\theta_j}{2}}\Big(1 - \sum_{k=0}^{\infty}(-1)^k\frac{(\sqrt{\alpha}\mu_j t)^{2k}}{(2k)!}\Big) \\ &= x_i(0) + \frac{1}{2}gt^2 + \frac{g}{2\alpha}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\alpha^k}{(2k)!}D_{i,k}t^{2k}, \end{aligned} \tag{4.1}$$

where

$$D_{i,k} = \sum_{j=1}^{n} \cos(2i+1)\frac{\theta_j}{2}\,\frac{\cos\frac{\theta_j}{2}}{\sin^2\frac{\theta_j}{2}}\,\mu_j^{2k}.$$

The following elementary lemma plays an important role to calculate $D_{i,k}$.

Lemma 4.1 Let $z \in \mathbb{C}, |z| = 1$, then it holds that

$$\mathrm{Re}\Big(\frac{1}{1-z}\Big) = \frac{1}{2}. \tag{4.2}$$

Proof Since

$$\frac{1}{1-z} = \frac{1-\overline{z}}{1-z-\overline{z}+z\overline{z}} = \frac{1-\overline{z}}{2-2(z+\overline{z}\,)} = \frac{1-\overline{z}}{2-2\,\mathrm{Re}(z)},$$

we have

$$\mathrm{Re}\Big(\frac{1}{1-z}\Big) = \frac{1-\mathrm{Re}(\,\overline{z}\,)}{2-2\,\mathrm{Re}(z)} = \frac{1-\mathrm{Re}(z)}{2-2\,\mathrm{Re}(z)} = \frac{1}{2}.$$

■

Lemma 4.2 The explicit value of $D_{i,k}$ is give by the followings

(1) For $i = 0$ and $k = 1$, we have

$$D_{0,1} = 2n.$$

(2) For $i = 1, 2, \cdots, n$, we have

$$D_{i,k} = \begin{cases} -2 & (k=1), \\ 0 & (2 \le k \le i), \\ (-1)^i 2(n+1) & (k = i+1). \end{cases}$$

Proof Since $\mu_i = 2\sin\dfrac{\theta_j}{2}$, we have

$$D_{i,k} = 2^{2k} \sum_{j=1}^{n} \cos(2i+1)\frac{\theta_j}{2} \cos\frac{\theta_j}{2} \sin^{2(k-1)}\frac{\theta_j}{2}.$$

Define ϵ by $\epsilon = e^{\sqrt{-1}\frac{\pi}{2(n+1)}}$. Note that $\epsilon^{2(n+1)} = -1$. By using $\cos\theta = \dfrac{1}{2}(\, e^{\sqrt{-1}\theta} + e^{-\sqrt{-1}\theta}\,)$ and $\sin\theta = \dfrac{1}{2\sqrt{-1}}(\, e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta}\,)$, we can rewrite $D_{i,k}$ as

$$\begin{aligned} D_{i,k} &= 4^k \sum_{j=1}^{n} \Big(\frac{\epsilon^{j(2i+1)} + \epsilon^{-j(2i+1)}}{2}\Big)\Big(\frac{\epsilon^j + \epsilon^{-j}}{2}\Big)\Big(\frac{\epsilon^j - \epsilon^{-j}}{2\sqrt{-1}}\Big)^{2(k-1)} \\ &= (-1)^{k-1} \sum_{j=1}^{n} (\,\epsilon^{j(2i+2)} + \epsilon^{-2ij} + \epsilon^{2ij} + \epsilon^{-j(2i+2)}\,)(\,\epsilon^j - \epsilon^{-j}\,)^{2(k-1)} \\ &= (-1)^{k-1} \sum_{j=1}^{n} \sum_{l=\pm i, \pm(i+1)} \epsilon^{2jl} (\,\epsilon^j - \epsilon^{-j}\,)^{2(k-1)}. \end{aligned} \tag{4.3}$$

Case 1 : $k = 1$.

In this case, we have

$$D_{i,1} = \sum_{j=1}^{n} (\epsilon^{2ij} + \epsilon^{-2ij} + \epsilon^{j(2i+2)} + \epsilon^{-j(2i+2)}) = 2 \sum_{j=1}^{n} \mathrm{Re}(\epsilon^{2ij} + \epsilon^{j(2i+2)})$$
$$= 2 \sum_{j=0}^{n} \mathrm{Re}(\epsilon^{2ij} + \epsilon^{j(2i+2)}) - 4. \tag{4.4}$$

If $i \neq 0$, then

$$D_{i,1} = 2 \,\mathrm{Re}\Big(\frac{1 - \epsilon^{2i(n+1)}}{1 - \epsilon^{2i}} + \frac{1 - \epsilon^{2(i+1)(n+1)}}{1 - \epsilon^{2i+2}} \Big) - 4$$
$$= 2 \,\mathrm{Re}\Big(\frac{1 - (-1)^{i}}{1 - \epsilon^{2i}} + \frac{1 - (-1)^{i+1}}{1 - \epsilon^{2i+2}} \Big) - 4.$$

By Lemma 4.1, we have

$$D_{i,1} = 1 - (-1)^{i} + 1 - (-1)^{i+1} - 4 = -2. \tag{4.5}$$

If $i = 0$

$$D_{0,1} = 2 \sum_{j=0}^{n} \mathrm{Re}(1 + \epsilon^{2j}) - 4$$
$$= 2(n+1) + 2 \,\mathrm{Re}\Big(\frac{2}{1 - \epsilon^{2}} \Big) - 4 = 2(n+1) + 2 - 4 = 2n.$$

Case 2 : $2 \leq k \leq i$.

Clearly $(\epsilon^{j} - \epsilon^{-j})^{2(k-1)} = 0$ for $j = 0$ and by the binomial theorem, we have

$$(\epsilon^{j} - \epsilon^{-j})^{2(k-1)} = \sum_{r=0}^{2(k-1)} {}_{2(k-1)}C_r (\epsilon^{j})^{r} (-\epsilon^{-j})^{2(k-1)-r}$$
$$= \sum_{r=0}^{2(k-1)} {}_{2(k-1)}C_r (-1)^{r} \epsilon^{2j(r-k+1)},$$

where ${}_{2(k-1)}C_r$ is the binomial coefficient. Therefore, using the fact that $D_{i,k}$ is real, we have

$$D_{i,k} = (-1)^{k-1} \sum_{l = \pm i, \pm(i+1)} \sum_{r=0}^{2(k-1)} {}_{2(k-1)}C_r (-1)^{r} \,\mathrm{Re} \sum_{j=0}^{n} \epsilon^{2j(l+r-k+1)}. \tag{4.6}$$

Set $m = l + r - k + 1$. Note that if $0 < |m| < 2(n+1)$, then we have $\epsilon^{2m} = e^{\sqrt{-1}\frac{m}{n+1}\pi} \neq 1$.

Since $0 \le r \le 2(k-1)$, $l-k+1 \le m \le l+k-1$. From this it is easy to see

$$i-k+1 \le |m| \le i+k \tag{4.7}$$

for $l = \pm i, \pm(i+1)$. Using $k \le i$, we have $1 \le |m| \le 2i \le 2n$, which implies $\epsilon^{2m} \ne 1$. Hence by Lemma 4.1

$$\begin{aligned}\mathrm{Re}\sum_{j=0}^{n}\epsilon^{2j(l+r-k+1)} &= \mathrm{Re}\,\frac{1-\epsilon^{2(l+r-k+1)(n+1)}}{1-\epsilon^{2(l+r-k+1)}} = \mathrm{Re}\,\frac{1-(-1)^{(l+r-k+1)}}{1-\epsilon^{2(l+r-k+1)}} \\ &= \frac{1}{2}\big(1-(-1)^{(l+r-k+1)}\big). \end{aligned}\tag{4.8}$$

Plugging this into (4.6), we obtain

$$\begin{aligned} D_{i,k} &= (-1)^{k-1}\frac{1}{2}\sum_{l=\pm i,\pm(i+1)}\sum_{r=0}^{2(k-1)} {}_{2(k-1)}C_r\big((-1)^r-(-1)^{(l-k+1)}\big) \\ &= (-1)^{k-1}\frac{1}{2}\sum_{l=\pm i,\pm(i+1)}\Big(\sum_{r=0}^{2(k-1)} {}_{2(k-1)}C_r(-1)^r-(-1)^{(l-k+1)}\sum_{r=0}^{2(k-1)} {}_{2(k-1)}C_r\Big) \\ &= (-1)^{k-1}\frac{1}{2}\sum_{l=\pm i,\pm(i+1)}(-1)^{l-k}2^{2(k-1)} \\ &= -2^{2k-3}\sum_{l=\pm i,\pm(i+1)}(-1)^l = 0. \end{aligned}$$

Case 3 : $k = i+1$.

Remark that (4.6) is valid for $k = i+1$. Set

$$I_{i,l} = \sum_{r=0}^{2i} {}_{2i}C_r(-1)^r\,\mathrm{Re}\sum_{j=0}^{n}\epsilon^{2j(l+r-i)},$$

then we have $D_{i,i+1} = (-1)^i \sum_{l=\pm i,\pm(i+1)} I_{i,l}$. The inequality (4.7) holds as well for $k = i+1$, so it holds that $0 \le m \le 2i+1 \le 2n+1$. It is easily verified that $m = 0$

if and only if $l = i, r = 0$ or $l = -i, r = 2(k-1)$. Hence it follows from (4.8) that

$$\begin{aligned}
&\mathrm{Re}\sum_{j=0}^{n}\epsilon^{2j(l+r-i)} \\
&= \begin{cases} n+1 & l = i, r = 0 \text{ or } l = -i, r = 2i, \\ \dfrac{1}{2}\big(1-(-1)^r\big) & l = i, 0 < r \le 2i \text{ or } l = -i, 0 \le r < 2i, \\ \dfrac{1}{2}\big(1+(-1)^r\big) & l = \pm(i+1), 0 \le r \le 2i. \end{cases}
\end{aligned}$$

By means of this, we can calculate $I_{i,l}$ for $l = \pm i, \pm(i+1)$ as follows. First, since

$$I_{i,i} = n+1+\frac{1}{2}\sum_{r=1}^{2i}{}_{2i}C_r(-1)^r\big(1-(-1)^r\big)$$

and

$$I_{i,-i} = n+1+\frac{1}{2}\sum_{r=0}^{2i-1}{}_{2i}C_r(-1)^r\big(1-(-1)^r\big),$$

we have

$$I_{i,i} = I_{i,-i} = n+1+\frac{1}{2}\sum_{r=0}^{2i}{}_{2i}C_r\big((-1)^r-1\big) = n+1-2^{2i-1}.$$

Next we have

$$I_{i,i+1} = I_{i,-i-1} = \frac{1}{2}\sum_{r=0}^{2i}{}_{2i}C_r\big((-1)^r+1\big) = 2^{2i-1}.$$

Therefore we obtain

$$D_{i,i+1} = (-1)^i(I_{i,i}+I_{i,-i}+I_{i,i+1}+I_{i,-i-1}) = (-1)^i 2(n+1).$$

This concludes of the proof of the lemma. ■

Theorem 4.3 For $i = 0, 1, \ldots, n$, $x_i(t)$ has the following Maclaurin series expansion.

$$x_i(t) = x_i(0)+\frac{n+1}{(2(i+1))!}g\alpha^i\,t^{2(i+1)}+\cdots$$

Proof Recall (4.1):

$$x_i(t) = x_i(0)+\frac{1}{2}gt^2+\frac{g}{2\alpha}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\alpha^k}{(2k)!}D_{i,k}\,t^{2k}.$$

For $1 \leq i \leq n$, (2) in Lemma 4.2 yields

$$\begin{aligned} x_i(t) &= x_i(0) + \frac{1}{2}gt^2 - \frac{1}{2}gt^2 + \frac{g}{2\alpha}(-1)^{i+2}\frac{\alpha^{i+1}}{(2i+2)!}(-1)^i 2(n+1)\, t^{2(i+1)} \\ &\quad + \sum_{k=i+2}^{\infty}(-1)^{k+1}\frac{\alpha^k}{(2k)!}D_{i,k}\, t^{2k} \\ &= x_i(0) + \frac{n+1}{(2(i+1))!}g\alpha^i\, t^{2(i+1)} + \cdots . \end{aligned} \tag{4.9}$$

For $i = 0$, (1) in Lemma 4.2 yields

$$\begin{aligned} x_0(t) &= x_0(0) + \frac{1}{2}gt^2 + \frac{n}{2}gt^2 + \frac{g}{2\alpha}\sum_{k=2}^{\infty}(-1)^{k+1}\frac{\alpha^k}{(2k)!}D_{0,k}\, t^{2k} \\ &= x_0(0) + \frac{n+1}{2}gt^2 + \cdots , \end{aligned}$$

which coincides with (4.9) when $i = 0$. ■

Theorem 3.4 indicates that the greater the number i is, the closer contact to $x_i(0)$ $x_i(t)$ has near $t = 0$. This is confirmed in Figure 5 in the case of $n = 5$. The trajectories were calculated based on the formulas given in Theorem 3.4. So we can think of this as a mathematical explanation of the phenomenon shown in Figure 4.

Remark 4.4 Vanderbei in [7] showed that the Maclaurin series of $x_i(t)$ starts from the term of order at least $2(i+1)$ without relying on the diagonalization of A as we did. Instead, he used a quite slick method.

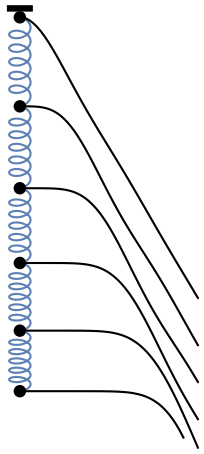


Figure 5: A computer-generated picture of 6 weights connected with 5 springs in free fall.

5 Passing to the limit

In this section, first we establish a correspondence between the parameters and variable of the slinky and the ones of its discrete model. Then by letting $n \to \infty$ in $x_i(t)$, we obtain $u(\xi, t)$ which describes the motion of freely falling slinky. We need to express parameters m_{w}, l_{s}, k_{s} by M, L, K and n. Considering the setting

in Section 2, it is natural to set $m_{\rm w} = \dfrac{M}{n+1}$, $l_{\rm s} = \dfrac{L}{n}$ and $k_{\rm s} = nK$. Hence we have

$$\alpha = \frac{k_{\rm s}}{m_{\rm w}} = n(n+1)\frac{K}{M} = \frac{n(n+1)}{T^2} \tag{5.1}$$

where $T = \sqrt{\dfrac{M}{K}}$ is a parameter with dimension of time.

Next we consider the correspondence between i and ξ. Recall that i is the variable which indicates the position of a weight and takes values in the integers $0 \le i \le n$. On the other hand ξ is the variable which indicates the position of a point of the slinky and takes values in the real numbers $0 \le \xi \le L$. So it is natural to set $i = \Big[n\dfrac{\xi}{L} \Big]$. It is convenient to use a dimensionless variable λ defined by $\lambda = \dfrac{\xi}{L}$ instead of ξ. Then we have the relation $i = [n\lambda]$.

Lemma 5.1 It holds that

$$\lim_{n\to\infty} \Big(il_{\rm s} + \frac{g}{2\alpha} i(2n+1-i) \Big) = L\lambda + \frac{1}{2} gT^2(2-\lambda)\lambda.$$

Proof Since

$$il_{\rm s} = L \cdot \frac{[n\lambda]}{n} \to L\lambda,$$

$$\frac{g}{2\alpha} i(2n+1-i) = \frac{1}{2} gT^2 \cdot \frac{[n\lambda]}{n} \cdot \frac{2n+1-[n\lambda]}{n+1} \to \frac{gT^2}{2}(2-\lambda)\lambda$$

as $n \to \infty$, the statement is proved. ■

Let $\varphi_2(x) = (x-1)^2 - \dfrac{1}{3}$. We extend $\varphi_2(x)$ restricted on $[0,2)$ to a periodic function $\overline{\varphi}_2(x)$ defined on $\mathbb{R}$, namely, $\overline{\varphi}_2(x) = \varphi_2\Big(x - 2\Big[\dfrac{x}{2} \Big] \Big)$. Since $\overline{\varphi}_2(x)$ is a continuous piecewise smooth periodic function with period 2, Fourier analysis tells us that we have the following Fourier series

$$\sum_{j=1}^{\infty} \frac{1}{j^2} \cos \pi j x = \frac{\pi^2}{4} \overline{\varphi}_2(x) \tag{5.2}$$

for $x \in \mathbb{R}$.

Proposition 5.2 It holds that

$$\begin{aligned} &\lim_{n\to\infty} \frac{g}{2\alpha} \sum_{j=1}^{n} \cos{(2i+1)\frac{\theta_j}{2}} \frac{\cos\dfrac{\theta_j}{2}}{\sin^2\dfrac{\theta_j}{2}} (\, 1 - \cos\sqrt{\alpha}\mu_j t\,) \\ &= \frac{1}{2} gT^2 \Big(\overline{\varphi}_2(\lambda) - \frac{1}{2}\overline{\varphi}_2\Big(\lambda + \frac{t}{T} \Big) - \frac{1}{2}\overline{\varphi}_2\Big(\lambda - \frac{t}{T} \Big) \Big). \end{aligned} \tag{5.3}$$

Proof Set

$$b_{n,j} = 1_{j\le n}\cos(2i+1)\frac{\theta_j}{2}\cos\frac{\theta_j}{2}\cdot\frac{1-\cos\sqrt{\alpha}\mu_j t}{\alpha\sin^2\frac{\theta_j}{2}}$$

where $1_{j\le n}$ is the indicator of the condition $j\le n$ taking 1 if the condition is satisfied and 0 otherwise. Then the left-hand side of (5.3) can be expressed as $\lim_{n\to\infty}\frac{g}{2}\sum_{j=1}^{\infty}b_{n,j}$. Since

$$i=[n\lambda],\ \theta_j=\frac{\pi}{n+1}j,\ \mu_j=2\sin\frac{\theta_j}{2}\text{ and }\alpha=\frac{n(n+1)}{T^2},$$

it is easy to see

$$(2i+1)\frac{\theta_j}{2}\to\lambda\pi j,\quad \sqrt{\alpha}\mu_j\to\frac{\pi j}{T}$$

as $n\to\infty$. Hence we have

$$\lim_{n\to\infty}b_{n,j}=\cos\lambda\pi j\cdot\frac{4T^2}{\pi^2 j^2}\Big(1-\cos\frac{\pi j}{T}t\Big).$$

Next we show that we can choose a constant C_j such that $|b_{n,j}|\le C_j$ for all $j\ge 1$ and $\sum_{j=1}^{\infty}C_j<\infty$. Clearly we have

$$|b_{n,j}|\le\frac{1-\cos\sqrt{\alpha}\mu_j t}{\alpha\sin^2\frac{\theta_j}{2}}\le\frac{2}{\alpha\sin^2\frac{\theta_j}{2}}.$$

Here we use the inequality $\frac{2}{\pi}\le\frac{\sin\theta}{\theta}\le 1$ for $0\le\theta\le\frac{\pi}{2}$. Since $0<\frac{\theta_j}{2}<\frac{\pi}{2}$ for $1\le j\le n$, we can apply this inequality to the estimate above, so that

$$|b_{n,j}|\le\frac{2\pi^2}{\alpha\theta_j^2}=\frac{2(n+1)}{n}T^2\cdot\frac{1}{j^2}\le\frac{2T^2}{j^2}. \tag{5.4}$$

Since $b_{n,j}=0$ for $j>n$, (5.4) holds trivially. Set $C_j=\frac{2T^2}{j^2}$. Then C_j satisfies the desired properties. Therefore by the dominated convergence theorem we have

$$\begin{aligned}\lim_{n\to\infty}\frac{g}{2}\sum_{j=1}^{\infty}b_{n,j}&=\frac{2gT^2}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\cos\pi\lambda j\Big(1-\cos\pi j\frac{t}{T}\Big)\\&=\frac{2gT^2}{\pi^2}\sum_{j=1}^{\infty}\Big(\frac{1}{j^2}\cos\pi\lambda j-\frac{1}{j^2}\cos\pi\lambda j\cos\pi j\frac{t}{T}\Big).\end{aligned}$$

Note that we can rewrite

$$\frac{1}{j^2}\cos\pi\lambda j\cos\pi j\frac{t}{T}=\frac{1}{2j^2}\Big(\cos\pi\Big(\lambda+\frac{t}{T}\Big)j+\cos\pi\Big(\lambda-\frac{t}{T}\Big)j\Big).$$

By means of (5.2) we finally obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{g}{2}\sum_{j=1}^{\infty}b_{n,j}&=\frac{2gT^2}{\pi^2}\Big(\frac{\pi^2}{4}\overline{\varphi}_2(\lambda)-\frac{1}{2}\Big\{\frac{\pi^2}{4}\overline{\varphi}_2\Big(\lambda+\frac{t}{T}\Big)+\frac{\pi^2}{4}\overline{\varphi}_2\Big(\lambda-\frac{t}{T}\Big)\Big\}\Big)\\&=\frac{1}{2}gT^2\Big(\overline{\varphi}_2(\lambda)-\frac{1}{2}\overline{\varphi}_2\Big(\lambda+\frac{t}{T}\Big)-\frac{1}{2}\overline{\varphi}_2\Big(\lambda-\frac{t}{T}\Big)\Big).\end{aligned}$$

This completes the proof of the theorem. ■

Here we introduce another dimensionless variable τ defined by $\tau=\dfrac{t}{T}$. From Theorem 3.4, Lemma 5.1 and Proposition 5.2 it follows that $x_i(\tau T)$ with $i=[n\lambda]$ converges as $n\to\infty$ for $0\le\lambda\le 1$ and $\tau\ge 0$.

Theorem 5.3 $x(\lambda,\tau)=\lim\limits_{n\to\infty}x_{[n\lambda]}(\tau T)$ is expressed as

$$x(\lambda,\tau)=L\lambda+\frac{1}{3}gT^2+\frac{1}{2}gT^2\tau^2-\frac{1}{4}gT^2\Big(\overline{\varphi}_2(\lambda+\tau)+\overline{\varphi}_2(\lambda-\tau)\Big)\qquad(5.5)$$

for $0\le\lambda\le 1$ and $\tau\ge 0$.

Proof Putting Theorem 3.4, Lemma 5.1 and Proposition 5.2 together, we get

$$\begin{aligned}&x(\lambda,\tau)\\&=L\lambda+\frac{gT^2}{2}(2-\lambda)\lambda+\frac{gT^2}{2}\tau^2+\frac{gT^2}{2}\Big(\overline{\varphi}_2(\lambda)-\frac{1}{2}\overline{\varphi}_2(\lambda+\tau)-\frac{1}{2}\overline{\varphi}_2(\lambda-\tau)\Big)\\&=L\lambda+\frac{gT^2}{2}\Big((2-\lambda)\lambda+\tau^2+\overline{\varphi}_2(\lambda)-\frac{1}{2}\overline{\varphi}_2(\lambda+\tau)-\frac{1}{2}\overline{\varphi}_2(\lambda-\tau)\Big).\end{aligned}$$

Note that $0\le\lambda\le 1$ implies $\overline{\varphi}_2(\lambda)=\lambda^2-2\lambda+\dfrac{2}{3}$, so

$$x(\lambda,\tau)=L\lambda+\frac{1}{3}gT^2+\frac{1}{2}gT^2\tau^2-\frac{1}{4}gT^2\Big(\overline{\varphi}_2(\lambda+\tau)+\overline{\varphi}_2(\lambda-\tau)\Big).$$

This concludes the proof of the theorem. ■

Definition 5.4

(1) For $\lambda\in\mathbb{R}$ and $\tau\ge 0$, we redefine $x(\lambda,\tau)$ by (5.5).

(2) For $\xi\in\mathbb{R}$ and $t\ge 0$, we define $u(\xi,t)=x\Big(\dfrac{\xi}{L},\dfrac{t}{T}\Big)$.

6 Analysis of free fall of the slinky

By making use of the explicit formula of $x(\lambda,\tau)$ given in (5.5), we show that after the release of the slinky the lower a point of the slinky is located the longer it stays at rest. If $0 \le \tau \le \lambda$, then $\lambda \pm \tau$ are in the interval $[0,2]$, so that we get $\overline{\varphi}_2(\lambda+\tau) = (\lambda+\tau)^2 - 2(\lambda+\tau) + \dfrac{2}{3}$ and $\overline{\varphi}_2(\lambda-\tau) = (\lambda-\tau)^2 - 2(\lambda-\tau) + \dfrac{2}{3}$, which yield

$$\overline{\varphi}_2(\lambda+\tau) + \overline{\varphi}_2(\lambda-\tau) = 2\tau^2 + 2\lambda^2 - 4\lambda + \frac{4}{3}.$$

From Theorem 5.3 it follows that if $0 \le \tau \le \lambda$, then $x(\lambda,\tau) = L\lambda + \dfrac{1}{2}gT^2\lambda(2-\lambda)$, which is free of τ. Since $u(\xi,t) = x\Big(\dfrac{\xi}{L}, \dfrac{t}{T}\Big)$, this means that the point labeled ξ of the slinky stays at rest if $0 \le t \le \dfrac{T}{L}\xi$.

From this we can interpret the physical meaning of parameters $T = \sqrt{\dfrac{M}{K}}$ and $V = \dfrac{L}{T}$ as follows: T is the time span during which the bottommost point labeled $\xi = L$ stays at rest after the release of and V is the speed at which the information that the uppermost point of the slinky is released propagates through the slinky. As we can see in Figure 6, after staying at its original position for a while each points start to move along a piecewise linear curve.

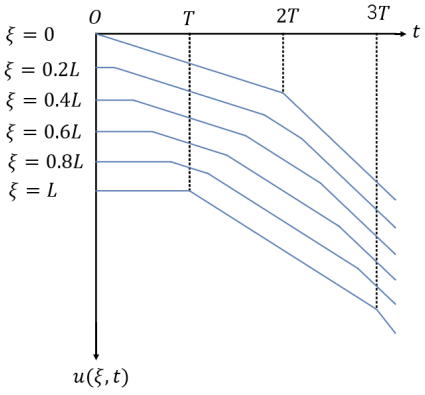


Figure 6: The trajectories of 6 points of the slinky labeled $\xi = 0$, $0.2L$, $0.4L$, $0.6L$, $0.8L$ and L.

Before concluding this paper we consider the partial differential equation $u(\xi,t)$ satisfies. Recall

$$u(\xi,t) = \xi + \frac{1}{3}gT^2 + \frac{1}{2}gt^2 - \frac{1}{4}gT^2\Big(\overline{\varphi}_2\Big(\frac{\xi}{L}+\frac{t}{T}\Big) + \overline{\varphi}_2\Big(\frac{\xi}{L}-\frac{t}{T}\Big)\Big).$$

Since $\overline{\varphi}_2(\lambda)$ is not differentiable at $\lambda = 2k$ $(k \in \mathbb{Z})$, we differentiate $u(\xi,t)$ except $\dfrac{\xi}{L} \pm \dfrac{t}{T} = 2k$ $(k \in \mathbb{Z})$ with respect to ξ and t twice. Then we have

$$\frac{\partial^2}{\partial\xi^2}u(\xi,t) = -\frac{1}{4}gT^2\Big(\frac{1}{L^2}\overline{\varphi}_2''\Big(\frac{\xi}{L}+\frac{t}{T}\Big) + \frac{1}{L^2}\overline{\varphi}_2''\Big(\frac{\xi}{L}-\frac{t}{T}\Big)\Big) \tag{6.1}$$

and

$$\frac{\partial^2}{\partial t^2}u(\xi,t) = g - \frac{1}{4}gT^2\Big(\frac{1}{T^2}\overline{\varphi}_2'\Big(\frac{\xi}{L}+\frac{t}{T}\Big) - \frac{1}{T^2}\overline{\varphi}_2'\Big(\frac{\xi}{L}-\frac{t}{T}\Big)\Big). \tag{6.2}$$

Multiply (6.1) by $V^2 = L^2/T^2$ and subtract it from (6.2), and we obtain a inhomogeneous wave equation $\dfrac{\partial^2 u}{\partial t^2} - V^2\dfrac{\partial^2 u}{\partial \xi^2} = g$ in

$$D = \Big\{ (\xi,t) \,\Big|\, \xi \in \mathbb{R}, t \geq 0,\ \frac{\xi}{L} \pm \frac{t}{T} \neq 2k\ (k \in \mathbb{Z}) \Big\}. \tag{6.3}$$

Since

$$\frac{\partial}{\partial t}u(\xi,t) = gt - \frac{1}{4}gT^2\Big(\frac{1}{T}\overline{\varphi}_2'\Big(\frac{\xi}{L}+\frac{t}{T}\Big) - \frac{1}{T}\overline{\varphi}_2'\Big(\frac{\xi}{L}-\frac{t}{T}\Big)\Big) \text{ in } D,$$

we have $\dfrac{\partial}{\partial t}u(\xi,0) = 0$ for $\xi \neq 2kL$ $(k \in \mathbb{Z})$ and $u(\xi,0) = \xi + \dfrac{1}{3}gT^2 - \dfrac{1}{2}gT^2\overline{\varphi}_2\Big(\dfrac{\xi}{L}\Big)$.

In conclusion, we have the following:

Theorem 6.1 $u(\xi,t)$ satisfies the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - V^2\frac{\partial^2 u}{\partial \xi^2} = g \qquad \text{in } D \tag{6.4}$$

subject to the initial conditions:

$$\begin{cases} \dfrac{\partial}{\partial t}u(\xi,0) = 0 & \xi \neq 2kL\ (k \in \mathbb{Z}), \\ u(\xi,0) = \xi + \dfrac{1}{3}gT^2 - \dfrac{1}{2}gT^2\overline{\varphi}_2\Big(\dfrac{\xi}{L}\Big) & \xi \in \mathbb{R}. \end{cases} \tag{6.5}$$

Remark 6.2 Through the following intuitive argument, we can "prove" Theorem 6.1 by letting $n \to \infty$ in (2.1) and (2.3). By definition, we have $\alpha \sim \dfrac{1}{T^2}n^2$ and $i \sim \dfrac{\xi}{L}n$. If $\Delta\xi$ is taken to be $i+1 \sim \dfrac{\xi+\Delta\xi}{L}n$, then $\alpha \sim V^2\dfrac{1}{\Delta\xi^2}$. Define $\Delta x_i = x_{i+1} - x_i$ and $\Delta^2 x_i = \Delta x_i - \Delta x_{i-1}$. Then the second equation in (2.1) is approximately expressed as $\dfrac{d^2x_i}{dt^2} = V^2\dfrac{\Delta^2 x_i}{\Delta\xi^2} + g$. Letting $n \to \infty$ in this equation, we formally obtain (6.4). Since $u(\xi,0) = \xi + \dfrac{1}{2}gT^2\dfrac{\xi}{L}\Big(2 - \dfrac{\xi}{L}\Big)$ for $0 \leq \xi \leq L$, it is easy to see that we can formally obtain (6.5) for $0 < \xi < L$ by letting $n \to \infty$ in (2.3).

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