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Abstract

This paper introduces and investigates best-response improvement paths under a hierarchy. A best-response improvement path under a hierarchy is a sequence of action profiles such that, at each step, one player is selected among players who are 'closest' to the previous deviator under the hierarchy, the selected player deviates to a best-response action, and the others does not change their actions. It is shown that in a nested $\mathcal{H}$-indicator best-response game proposed in the paper, every best-response improvement path under hierarchy $\mathcal{H}$ leads to a Nash equilibrium of the game.

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1 Introduction

This paper proposes and studies an adaptation process under a hierarchy. When analyzing a situation where myopic players interact with others in the long run, we often consider an adaptive process (Milgrom and Roberts [7, 8], Young [17], Kandori and Rob [3], Monderer and Shapley [10], Milchtaich [6], Friedman and Mezzetti [2], Kukushkin [4], Kukushkin, *et al.* [5], Park [13]). In the literature an adaptation process takes place in an unbiased way within the whole people. In the real world players are segmented in different groups (firms, cities, etc.) within which interaction is most frequent. These groups also are segmented in upper level groups (industries, countries, etc.) within which interaction is less frequent than that

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within the lower level groups, and so on. There is a hierarchy of interaction levels and it is sometimes appropriate to suppose that an adaptation process is restricted by the hierarchy.

To capture such an adaptation process we define a (unilateral) best-response (improvement) path under a hierarchy. Recall that in the standard definition of a best response path, a sequence of action profiles is formulated in such a way that in each step, one player is selected from the set of all players and his action is changed so that it would be a best response to the current action profile of other players. Let a hierarchy be denoted by a nested sequence of partitions of the player set. A best-response path under a hierarchy is a best-response path such that, in each step, a player is selected among players who are 'closest' from the player whose action is changed in previous step under a distance on players defined by the hierarchy.

We provide a sufficient condition for the convergence of a best-response path under a hierarchy in terms of nested indicator best-response potentials (nested indicator BR-potentials). Nested indicator BR-potentials generalize the best-response potentials (BR-potentials) introduced by Voorneveld [16], applying the idea of 'nesting' developed by Uno [14]. A BR-potential is a real valued function on the set of action profiles that 'incorporates' information about every players' best-response. It is known that every maximizer of such a potential is a Nash equilibrium of the game. An indicator BR-potential is a BR-potential such that its maximizer equals the set of all Nash equilibria of the game. It is as if such potentials are payoff functions of one representative agent who chooses strategies for all players.

In considering a nested indicator BR-potential game, we think of a representative agent for a subset T of players instead of all of them. Suppose that there is a partition $\mathcal{T}$ of players such that, for each member T of $\mathcal{T}$, there is such a representative agent whose payoff function is f_T.[1] Then the collection of f_T's can be seen as a new game, where each member T in $\mathcal{T}$ is regarded as a single player. That is, the original game is reduced to a game with a smaller number of players.

Notice that such reduction can be nested: the new game among step 1 representative agents may be reduced to a game with an even smaller number of players, by considering a step 2 representative agent for step 1 representative agents, and then a representative agent of these, and so on. We say that a game is a nested indicator BR-potential game if the game can be reduced to a game with one representative agent through this process.

We show that in nested indicator BR-potential games every best-response path under a hierarchy leads to a Nash equilibrium (Theorem 8). We call such a property the finite best-response path property under a hierarchy (FBRP under a hierarchy).

In the literature the FBRP under a hierarchy relates to the finite best-response path property (FBRP) and the weak finite best-response path property (weak FBRP) discussed

[1] This idea also has appeared as q-potential in Monderer [9].

by Milchtaich [6], Kukushkin [4], and Kukushkin, *et al.* [5]. A best-response (improvement) path is a best-response path under the only two-level hierarchy, where, for each player, the distance between any two players is the same. A game has the FBRP if every best-response path leads to a Nash equilibrium; a game has the weak FBRP if, for every action profile, there exists a best-response path from the action profile leading to a Nash equilibrium. Games with the FBRP under a hierarchy locate between those with the FBRP and those with the weak FBRP. This implies that our condition of Theorem 8 is also a sufficient condition for the weak FBRP (Corollary 9).

2 Hierarchical Adaptations

A strategic form game with ordinal preferences consists of a finite set $N = \{1, \dots, n\}$ of players, a finite set A_i of actions for $i \in N$ and an ordinal payoff function $g_i : A \to \mathbb{R}$ for $i \in N$, where $A := \prod_{i \in N} A_i$. Since we fix the set A of action profiles, we denote a strategic form game $(N, (A_i)_{i \in N}, (g_i)_{i \in N})$ simply by $\mathbf{g}^N := (g_i)_{i \in N}$. For notational convenience, we write $a = (a_i)_{i \in N} \in A$; for $i \in N$, $A_{-i} = \prod_{j \neq i} A_j$ and $a_{-i} = (a_j)_{j \neq i} \in A_{-i}$; and for $T \subseteq N$, $A_T = \prod_{i \in T} A_i$, $a_T = (a_i)_{i \in T} \in A_T$, $A_{-T} = \prod_{i \in N \setminus T} A_i$, and $a_{-T} = (a_i)_{i \in N \setminus T} \in A_{-T}$. We write $(a_T, a_{-T}) \in A_T \times A_{-T}$. We write (a_i, a_{-i}) instead of $(a_{\{i\}}, a_{-\{i\}})$ for simplicity. For $i \in N$, let BR_i be i's best-response correspondence, i.e., for $i \in N$ and $a_{-i} \in A_{-i}$, $BR_i(a_{-i}) := \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. Given a strategic form game $\mathbf{g}^N$, for any $T \subseteq N$ and any $a_{-T} \in A_{-T}$, let $\mathbf{g}^N|_{a_{-T}}$ be the game restricted by a_{-T}, where, for each $i \in N \setminus T$, i's action set is replaced by $\{a_i\}$. We often regard such a restricted game $\mathbf{g}^N|_{a_{-T}}$ as a m-person game, where m is the number of elements of T.

For $T \subseteq N$, a *hierarchy* of T is a nested sequence $(\mathcal{T}^k)_{k=0}^K$ of partitions of T, i.e., $(\mathcal{T}^k)_{k=0}^K$ is an increasingly coarser sequence of partition of T such that $\mathcal{T}^0 = \{\{i\} | i \in T\}$ and $\mathcal{T}^K = \{T\}$. We call a hierarchy of N simply a hierarchy.

We introduce the best-response improvement path under a hierarchy.

Definition 1. Let $\mathcal{H} := (\mathcal{T}^k)_{k=0}^K$ be a hierarchy. A sequence $(a^0, a^1, \dots)$ of action profiles is a *best-response improvement path under hierarchy* $\mathcal{H}$ if, for each $m = 1, 2, \dots$, there exist a player $i(m) \in N$ and an integer $k(m)$ with $0 \leq k(m) \leq K$ such that

- for $T \in \mathcal{T}^{k(m)-1}$ with $i(m-1) \in T$, for each $i \in T$, $a_i^{m-1} \in BR_i(a_{-i}^{m-1})$; and
- for each k with $k(m) \leq k \leq K$, for $T \in \mathcal{T}^k$ with $i(m-1) \in T$, $i(m) \in T$, $a_{i(m)}^m \in BR_{i(m)}(a_{-i(m)}^{m-1})$, $g_{i(m)}(a^m) > g_{i(m)}(a^{m-1})$, $a_{-i(m)}^m = a_{-i(m)}^{m-1}$.

A game has the $\mathcal{H}$*-finite best-response path property* ($\mathcal{H}$-FBRP) if there exists no infinite best-response improvement path under hierarchy $\mathcal{H}$.

It is clear that we can replace $g_{i(m)}(a^m) > g_{i(m)}(a^{m-1})$ in the above definition by $a^{m-1}_{i(m)} \notin BR_{i(m)}(a^{m-1}_{-i(m)})$.

In the above definition, for each step m, $k(m)$ means to be the least level k of hierarchy such that there exists a player i in a set $T \in \mathcal{T}^k$ such that player i does not take i's best-response action and the deviator $i(m-1)$ in the previous step also belongs to the set T.

A best-response improvement path under a hierarchy $\mathcal{H}$ can be interpreted as follows. A hierarchy can be represented by an acyclic graph, where $\{T | T \in \mathcal{T}^k \text{ for } k = 0, \ldots, K\}$ is the set of nodes and, for each $k = 1, \ldots, K$, each $T \in \mathcal{T}^{k-1}$ and each $T' \in \mathcal{T}^k$, if $T \subseteq T'$ then there is an edge between T and T'.[2] We can define a distance between players (or nodes) under the hierarchy as the minimum number of edges between the players in the acyclic graph. Given the distance, a best-response improvement path under a hierarchy $\mathcal{H}$ is a best-response improvement path such that at each step one player is selected among the players that did not take a best-response action at the previous step who is closest from the previous deviator. In a game with the $\mathcal{H}$-FBRP, every best-response improvement path under hierarchy $\mathcal{H}$ leads to a Nash equilibrium in a finite number of steps.

If a hierarchy has only two levels, i.e., $\mathcal{H} = (\{i | i \in N\}, \{N\})$, then a best-response improvement path under hierarchy $\mathcal{H}$ is called a *best-response improvement path* and the $\mathcal{H}$-FBRP is called the *finite best-response path property* (FBRP) in the literature. A game has *the weak finite best-response path property* (weak FBRP) if, for any action profile a, there exists a best-response improvement path that leads to a Nash equilibrium. It is clear that the FBRP implies the $\mathcal{H}$-FBRP for any hierarchy $\mathcal{H}$; and these imply the weak FBRP.

3 Nested Indicator Potential Games

This section introduce the nested indicator best-response potential (BR-potential) games. To do so, firstly, we introduce the partition $\mathcal{T}$-BR-potentials by a construction similar to Monderer [9] and Uno [14].[3] Let $\mathcal{T}$ be a partition of N. A partition $\mathcal{T}$-BR-potential of a game $\mathbf{g}^N$ is a tuple $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ where, for each $T \in \mathcal{T}$ and each $i \in T$, i's best-response against the other players' actions a_{-i} in the alternative game where i's payoff function is given by f_T is equivalent to that in the original game $\mathbf{g}^N$:

Definition 2. Let $\mathcal{T}$ be a partition of N. A tuple $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ is a *partition*

[2] A graph is acyclic if there is no cycle in graph.

[3] The idea of partition $\mathcal{T}$ potentials is same as that of q-potential defined by Monderer [9] independently and earlier than Uno [14]: a game $\mathbf{g}^N$ is a q-potential game if, and only if, $\mathbf{g}^N$ has a partition $\mathcal{T}$-potential, where q refers to the number of elements in $\mathcal{T}$. For convenience to define nested potentials we use the partition $\mathcal{T}$-potentials.

$\mathcal{T}$*-BR-potential* of $\mathbf{g}^N$ if, for each $T \in \mathcal{T}$, $f_T : A \to \mathbb{R}$ satisfies that, for each $i \in T$,

$$\arg\max_{a_i \in A_i} f_T(a_i, a_{-i}) = BR_i(a_{-i}) \tag{1}$$

for all $a_{-i} \in A_{-i}$. If there exists a partition $\{N\}$-BR-potential $(\{N\}, (A), (f))$ in $\mathbf{g}^N$ then f is a *BR-potential* of $\mathbf{g}^N$, and $\mathbf{g}^N$ is a *BR-potential game* defined by Voorneveld [16].[4]

Voorneveld [16] characterized the pure strategy Nash equilibria in a BR-potential game:

Proposition 3. *Let* $\mathbf{g}^N := (g_i)_{i \in N}$ *be a BR-potential game with a BR-potential* f*. The pure strategy Nash equilibria of* $\mathbf{g}^N$ *coincide with those of* $(f)_{i \in N}$*, where* $(f)_{i \in N}$ *is an* n *person game in which each player's payoff function equals* f*.*

Proposition 3 implies that in every BR-potential game we can find all pure strategy Nash equilibria by locally maximizing a BR-potential.

Notice that we can regard each partition $\mathcal{T}$ BR-potential $\mathbf{f}^{\mathcal{T}}$ as a strategic form game, where $\mathcal{T}$ is the player set; for each $T \in \mathcal{T}$, A_T is the action set of T; and for each $T \in \mathcal{T}$, f_T is the payoff function of T.

For each $T \subset N$ and any $a_{-T} \in A_{-T}$, let $E_T(a_{-T})$ be the set of pure strategy Nash equilibria of the game $\mathbf{g}^N|_{a_{-T}}$ restricted by a_{-T} of $\mathbf{g}^N$:

$$E_T(a_{-T}) := \{a_T \in A_T |\ a_T \text{ is a pure strategy Nash equilibria of } \mathbf{g}^N|_{a_{-T}}\}.$$

We introduce the partition $\mathcal{T}$-indicator BR-potential. A partition $\mathcal{T}$-indicator BR-potential is a partition $\mathcal{T}$-BR-potential of $\mathbf{g}^N$ such that, for each representative agent $T \in \mathcal{T}$ and any action profile a_{-T} of players outside T, the best-response correspondence of representative agent T against a_{-T} equals the set of all Nash equilibria of $\mathbf{g}^N|_{a_{-T}}$:

Definition 4. Let $\mathcal{T}$ be a partition of N. A tuple $(\mathcal{T}, (A_T)_{T \in \mathcal{T}}, (f_T)_{T \in \mathcal{T}})$ is a *partition* $\mathcal{T}$*-indicator BR-potential* of $\mathbf{g}^N$ if $(\mathcal{T}, (A_T)_{T \in \mathcal{T}}, (f_T)_{T \in \mathcal{T}})$ is a partition $\mathcal{T}$-BR-potential of $\mathbf{g}^N$ such that, for each $T \in \mathcal{T}$,

$$\arg\max_{a_T \in A_T} f_T(a_T, a_{-T}) = E_T(a_{-T}) \tag{2}$$

for all $a_{-T} \in A_{-T}$.

It is clear that the notion of partition $\mathcal{T}$-indicator BR-potential is strictly stronger than that of the partition $\mathcal{T}$-BR-potentials. However, in a game with a partition $\mathcal{T}$-BR-potential, we can always find a partition $\mathcal{T}$-indicator BR-potential:[5]

[4] Morris and Ui [11, 12] also introduced alternative BR-potentials, which are special classes of (ordinal) BR-potentials of Voorneveld [16] and the pseudo-potentials of Dubey *et al.* [1].

[5] Proposition 5 implies that a game is a BR-potential game if, and only if, it is an indicator BR-potential game by setting $\mathcal{T} = \{N\}$.

Proposition 5. *Let $\mathcal{T}$ be a partition of N. A game $\mathbf{g}^N$ is a partition $\mathcal{T}$-BR-potential game if, and only if, $\mathbf{g}^N$ is a partition $\mathcal{T}$-indicator BR-potential game.*

Proof. See Appendix. □

Now, we define the nested (resp. indicator) BR-potential games by applying the nested construction of Uno [14] to the partition (resp. indicator) BR-potentials:

Definition 6. Let $\mathcal{H} = (\mathcal{T}^k)_{k=0}^K$ be a hierarchy. A game $\mathbf{g}^N$ is a *nested $\mathcal{H}$ (resp. indicator) BR-potential game* if there exists a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K := ((f_T^k)_{T\in\mathcal{T}^k})_{k=0}^K$ of tuples such that

- $\mathbf{f}^{\mathcal{T}^0}$ is the original game $\mathbf{g}^N$: for each $i \in N$, $f_{\{i\}}^0(a) = g_i(a)$ for all $a \in A$; and
- for each $k = 1, 2, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k}$ is a partition $\mathcal{T}^k$ (resp. indicator) BR-potential of $\mathbf{f}^{\mathcal{T}^{k-1}}$.

For two person game $\mathbf{g}^{\{1,2\}}$, note that a nested BR-potential game is a BR-potential game, an indicator BR-potential game, and a nested $\mathcal{H}$-indicator BR-potential game, where $\mathcal{H} = (\{\{1\},\{2\}\},\{1,2\})$ by Proposition 5. However, with more than two players, the nested indicator BR-potential game strictly generalizes the BR-potential game, which is demonstrated in Example 14 later. Note also that the nested indicator BR-potential game is a special class of the nested BR-potential game, as shown in Example 15 below.

A nested indicator BR-potential game includes a stag hunt team game inspired by Van Huyck [15].

Example 7 (Stag hunt team game)**.** There are n players and m teams. Each player is a member of either a team. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ be a partition of N, where T_k is a set of members of team k. Each player $i \in N$ chooses an effort $e_i \in \{0, 1, \ldots, 10\} =: A_i$ with cost ce_i. For each team $k = 1, \ldots, m$, let $\underline{e_k} := \min_{i\in T_k} e_i$ be the minimal effort in team k. For $k = 1, 2, \ldots, m$, let $h_{T_k}(e)$ be the common payoff for members of team k as follows:

$$h_{T_k}(e) = \begin{cases} a\underline{e}_k & (\underline{e}_k > \max_{l\neq k} \underline{e}_l) \\ a\underline{e}_k/n(e) & (\underline{e}_k = \max_{l\neq k} \underline{e}_l) \\ 0 & (\text{otherwise}) \end{cases},$$

where a is a positive real number and $n(e)$ is the number of teams that take the maximal of the minimal effort $\max_l \underline{e}_l$ of each team. Then, for each team k and each member $i \in T_k$, i's payoff function $g_i : A \to \mathbb{R}$ is given by

$$g_i(e) = h_{T_k}(e) - ce_i$$

for all $e \in A$. It is easy to show that the game has a $\mathcal{H}$-indicator BR-potential, where $\mathcal{H} = \{\{T_1, T_2, \ldots, T_m\}, \{N\}\}$.

4 Hierarchical Adaptation in Nested Indicator Potential Games

This section provides a sufficient condition for the $\mathcal{H}$-FBRP in terms of nested indicator BR-potential games.

Theorem 8. *Let $\mathcal{H} := (\mathcal{T}^k)_{k=0}^K$ be a hierarchy. If $\mathbf{g}^N$ is a nested $\mathcal{H}$-indicator BR-potential game then it has the $\mathcal{H}$-FBRP.*

Theorem 8 implies that a nested indicator BR-potential game has the weak FBRP.

Corollary 9. *If there exists a hierarchy $\mathcal{H} := (\mathcal{T}^k)_{k=0}^K$ such that $\mathbf{g}^N$ is a nested $\mathcal{H}$-indicator BR-potential game then it has the weak FBRP.*

To prove Theorem 8, we introduce the notion of Nash response cycles. Let $\mathcal{T}$ be a partition of N. A finite sequence $(a^0, a^1, \dots, a^M)$ of action profiles is a *Nash response cycle* under $\mathcal{T}$ if $a^0 = a^M$, and, for each $m \in \{1, 2, \dots, M\}$ there exists $T(m) \in \mathcal{T}$ such that $a^m_{T(m)} \in E_{T(m)}(a^{m-1}_{-T(m)})$ and $a^m_{-T(m)} = a^{m-1}_{-T(m)}$; and there exists $m \in \{1, 2, \dots, M\}$ such that $a^{m-1}_{T(m)} \notin E_{T(m)}(a^{m-1}_{-T(m)})$. If a partition $\mathcal{T}$ is finest, i.e., $\mathcal{T} = \{i | i \in N\}$, a Nash response cycle under $\mathcal{T}$ corresponds to *a best-response cycle* defined by Voorneveld [16].

Voorneveld [16] characterized the class of BR-potential games in terms of no best-response cycles:

Proposition 10. *A finite strategic form game $\mathbf{g}^N$ is a BR-potential game if, and only if, $\mathbf{g}^N$ has no best-response cycle.*[6]

We provide a characterization of nested indicator BR-potential games in terms of no Nash response cycles by applying Proposition 10 iteratively.

Proposition 11. *Let $\mathcal{H}$ be a hierarchy. A game $\mathbf{g}^N$ is a nested $\mathcal{H}$-indicator BR-potential game if, and only if, for each $k = 1, 2, \dots, K$, each $T \in \mathcal{T}^k$, and each $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no Nash response cycle under $\mathcal{T}^{k-1}|_T$, where $\mathcal{T}^{k-1}|_T = \{T' \in \mathcal{T}^{k-1} | T' \subseteq T\}$.*

Proof. Suppose that $\mathbf{g}^N$ is a nested $\mathcal{H}$-indicator BR-potential game. We will prove by mathematical induction on K that, for each $k = 1, 2, \dots, K$, each $T \in \mathcal{T}^k$, and each $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no Nash response cycle under $\mathcal{T}^{k-1}|_T$, where $\mathcal{T}^{k-1}|_T = \{T' \in \mathcal{T}^{k-1} | T' \subseteq T\}$. If $K = 1$ then the statement is true by Propositions 5 and 10. Assume the statement holds for $K - 1$. Fix an integer h $(1 \leq h \leq K)$, $T \in \mathcal{T}^k$, and $a_{-T} \in A_{-T}$. For any $h \leq K - 1$, it is clear that the game $\mathbf{g}^N|_{a_{-T}}$ restricted by a_{-T} is a nested $(\mathcal{T}^k|_T)_{k=0}^h$-indicator BR-potential game. For each $k = 1, 2, \dots, K - 1$, the statement holds by the assumption of induction.

[6] Voorneveld [16] allowed action sets to be countable rather than finite.

Let $\mathbf{f}^{\mathcal{T}^{K-1}}$ be a $\mathcal{T}^{K-1}$-indicator BR-potential of $\mathbf{f}^{\mathcal{T}^{K-2}}$. Regarding $\mathbf{f}^{\mathcal{T}^{K-1}}$ as a game with player set $\mathcal{T}^{K-1}$, $\mathbf{f}^{\mathcal{T}^{K-1}}$ is a BR-potential game and the best-response cycle of $\mathbf{f}^{\mathcal{T}^{K-1}}$ is the Nash-response cycle under $\mathcal{T}^{K-1}$ of $\mathbf{g}^N$ since $\mathbf{g}^N$ is a nested $\mathcal{H}$-indicator BR-potential game. By Propositions 5 and 10 the statement holds.

Suppose that, for each $k = 1, 2, \ldots, K$, each $T \in \mathcal{T}^k$, and each $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no Nash response cycle under $\mathcal{T}^{k-1}|_T$. Firstly, for each $T \in \mathcal{T}^1$ and each $a_{-T} \in A_{-T}$, since no Nash response cycle of $\mathbf{g}^N|_{a_{-T}}$ under $\mathcal{T}^0|_T$ is no best-response cycle of $\mathbf{g}^N|_{a_{-T}}$, $\mathbf{g}^N|_{a_{-T}}$ has no best-response cycle. By Proposition 10, there exists a $\mathcal{T}^1$-BR-potential $\mathbf{f}^{\mathcal{T}^1}$ of $\mathbf{f}^{\mathcal{T}^0}$, and so there also exists a $\mathcal{T}^1$-indicator BR-potential $\mathbf{f}^{\mathcal{T}^1}$ of $\mathbf{f}^{\mathcal{T}^0}$ by Proposition 5.

Assume that there exists a $\mathcal{T}^{k-1}$-indicator BR-potential $\mathbf{f}^{\mathcal{T}^{k-1}}$ of $\mathbf{f}^{\mathcal{T}^{k-2}}$. Since $\mathbf{f}^{\mathcal{T}^{k-1}}$ is a $\mathcal{T}^{k-1}$-indicator BR-potential of $\mathbf{f}^{\mathcal{T}^{k-2}}$, for each $T \in \mathcal{T}^{k-1}$, we have

$$E_T(a_{-T}) = \arg\max_{a_T \in A_T} f_T(a_T, a_{-T})$$

for any $a_{-T} \in A_{-T}$. Then, for any $T^k \in \mathcal{T}^k$, we can regard the Nash response path under $\mathcal{T}^{k-1}|_{T^k}$ as a best-response path of $\mathbf{f}^{\mathcal{T}^{k-1}}$. Since $\mathbf{f}^{k-1}|_{a_{-T^k}}$ has no Nash response cycle, we can interpret that $\mathbf{f}^{k-1}|_{a_{-T^k}}$ has no best-response cycle. By Proposition 10 and Proposition 5, there exists a $\mathcal{T}^k$-indicator BR-potential $\mathbf{f}^{\mathcal{T}^k}$ of $\mathbf{f}^{\mathcal{T}^{k-1}}$. $\square$

If $\mathbf{g}^N$ has no Nash response cycle under $\mathcal{T}$ and, for each $T \in \mathcal{T}$ and each $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no best-response cycle under a hierarchy $\mathcal{H}_T$ of T then $\mathbf{g}^N$ has no best-response cycle under the hierarchy $\mathcal{H}$ generated by $(\mathcal{H}_T)_{T\in\mathcal{T}}$ and $\{N\}$:

Lemma 12. *Let $\mathcal{T}$ be a partition of N. If, for each $T \in \mathcal{T}$, there exists a hierarchy $\mathcal{H}_T$ of T such that, for each $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no best-response cycle under $\mathcal{H}_T$; and $\mathbf{g}^N$ has no Nash response cycle under $\mathcal{T}$ then $\mathbf{g}^N$ has no best-response cycle under the hierarchy $\mathcal{H}$ generated by $(\mathcal{H}_T)_{T\in\mathcal{T}}$ and $\{N\}$.*

Proof. Suppose that, for each $T \in \mathcal{T}$, there exists a hierarchy $\mathcal{H}_T$ of T such that, for each $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no best-response cycle under $\mathcal{H}_T$; and $\mathbf{g}^N$ has no Nash response cycle under $\mathcal{T}$. Fix any $\alpha^0 \in A$ and fix any $T_{(0)} \in \mathcal{T}$. Since $\mathbf{g}^N|_{\alpha^0_{-T_{(0)}}}$ has no best-response cycle under $\mathcal{H}_{T_{(0)}}$, every best-response improvement path from $\alpha^0_{T_{(0)}}$ under $\mathcal{H}_{T_{(0)}}$ connects a Nash equilibrium $\alpha^1_{T_{(0)}}$ of $\mathbf{g}^N|_{\alpha^0_{-T_{(0)}}}$.

Let $\alpha^1 := (\alpha^1_{T_{(0)}}, \alpha^0_{-T_{(0)}})$. Let

$$T^*(\alpha^1) := \{i \in N\backslash T_{(0)} | \alpha_i \notin BR_i(\alpha^1_{-i})\}.$$

If $T^*(\alpha^1) = \emptyset$ then α^1 is a Nash equilibrium of $\mathbf{g}^N$. Suppose that $T^*(\alpha^1) \neq \emptyset$. Fix any $i \in T^*(\alpha^1)$. Let $T_{(1)} \in \mathcal{T}$ be such that $i \in T$. Since $\mathbf{g}^N|_{\alpha^1_{-T_{(1)}}}$ has no best-response cycle under $\mathcal{H}_{T_{(1)}}$, every best-response improvement path from $\alpha^1_{T_{(1)}}$ under $\mathcal{H}_{T_{(1)}}$ connects a Nash equilibrium $\alpha^2_{T_{(1)}}$ of $\mathbf{g}^N|_{\alpha^1_{-T_{(1)}}}$.

We apply the above arguments to $m = 2, 3, \ldots$, iteratively. That is, let $\alpha^m := (\alpha^m_{T_{(m-1)}}, \alpha^{m-1}_{-T_{(m-1)}})$. Let

$$T^*(\alpha^m) := \{i \in N \backslash T_{(m-1)} | \alpha^m_i \notin BR_i(\alpha^m_{-i})\}.$$

If $T^*(\alpha^m) = \emptyset$ then α^m is a Nash equilibrium of $\mathbf{g}^N$. Suppose that $T^*(\alpha^m) \neq \emptyset$. Fix any $i \in T^*(\alpha^m)$. Let $T_{(m)} \in \mathcal{T}$ be such that $i \in T$. Since $\mathbf{g}^N|_{\alpha^1_{-T_{(m)}}}$ has no best-response cycle under $\mathcal{H}_{T_{(m)}}$, every best-response improvement path from $\alpha^m_{T_{(m)}}$ under $\mathcal{H}_{T_{(m)}}$ connects a Nash equilibrium $\alpha^{m+1}_{T_{(m)}}$ of $\mathbf{g}^N|_{\alpha^m_{-T_{(m)}}}$. Then, $(\alpha^0, \alpha^1, \ldots)$ is a Nash response path under $\mathcal{T}$. Since $\mathbf{g}^N$ has no Nash response cycle, every best-response improvement path under $\mathcal{H}$ from α^0 also connects a Nash equilibrium of $\mathbf{g}^N$. □

Let us prove the theorem.

Proof of Theorem 8. Suppose that $\mathbf{g}^N$ is a nested indicator BR-potential game with a hierarchy $\mathcal{H} := (\mathcal{T}^k)_{k=0}^K$. We will prove the theorem by induction on K. Let $K = 1$. Then, since $\mathbf{g}^N$ is a BR-potential game, it has no best-response cycle under $\mathcal{H}$ by Proposition 10. Assume that, for any $K \leq L - 1$, $\mathbf{g}^N$ has no best-response cycle under $\mathcal{H}$. Let $K = L$. By the assumption of induction, for any $T \in \mathcal{T}^{L-1}$ and any $a_{-T} \in A_{-T}$, $\mathbf{g}^N|_{a_{-T}}$ has no best-response cycle under a hierarchy $(\mathcal{T}^k|_T)_{k=0}^{L-1}$ of T, where $\mathcal{T}^k|_T := \{T' \in \mathcal{T}^k | T' \subset T\}$ for $k = 0, \ldots, L-1$. And, by the definition of nested indicator BR-potentials, $\mathbf{f}^{\mathcal{T}^{L-1}}$ has a partition $\{N\}$-indicator BR-potential. By Proposition 10, $\mathbf{f}^{\mathcal{T}^{L-1}}$ has no Nash response cycle under $\mathcal{T}$. By Lemma 12, $\mathbf{g}^N$ has no best-response cycle under $\mathcal{H}$. □

5 Examples

The converse of Theorem 8 does not hold: a game with the $\mathcal{H}$-FBRP may not be a nested $\mathcal{H}$-indicator BR-potential games as shown by the following example.

Example 13. Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ represented as Table 1, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix. We can show that $\mathbf{g}^{\{1,2,3\}}$ has the $\mathcal{H}$-FBRP, where $\mathcal{H} := (\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1,2,3\}\})$. However, $\mathbf{g}^{\{1,2,3\}}$ is not a nested $\mathcal{H}$-indicator BR-potential game. Indeed, since $\mathbf{g}^{\{1,2,3\}}$ has a Nash response path cycle $(a'_1, a'_2, a_3) \to (a''_1, a''_2, a_3) \to (a''_1, a''_2, a'_3) \to (a'_1, a'_2, a'_3) \to (a'_1, a'_2, a_3)$ under partition $\{\{1,2\},\{3\}\}$, $\mathbf{g}^{\{1,2,3\}}$ is not a nested $\mathcal{H}$-indicator BR-potential game by Proposition 11.

a_3	a_2	a_2'	a_2''	a_3'	a_2	a_2'	a_2''
a_1	$2,2,2$	$1,0,1$	$0,0,0$	a_1	$1,1,0$	$0,0,0$	$1,0,1$
a_1'	$0,1,1$	$0,0,1$	$0,0,0$	a_1'	$0,0,0$	$1,1,0$	$0,0,1$
a_1''	$0,0,0$	$0,0,0$	$1,1,0$	a_1''	$0,1,1$	$0,0,1$	$0,0,1$

Table 1: A game with the $\mathcal{H}$-FBRP, where $\mathcal{H} := (\{\{1\},\{2\},\{3\}\},\{\{1,2\},\{3\}\},\{\{1,2,3\}\})$, but without a nested $\mathcal{H}$-indicator BR-potential

The nested indicator BR-potential game strictly generalizes the BR-potential game introduced by Voorneveld [16]. A BR-potential game may be not a nested indicator BR-potential game, as demonstrated in the following example.

Example 14. Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 2. Note that $\mathbf{g}^{\{1,2,3\}}$ has a best-response cycle $(a_1, a_2, a_3') \to (a_1', a_2, a_3') \to (a_1', a_2', a_3') \to (a_1, a_2', a_3') \to (a_1', a_2, a_3')$. By Proposition 10, this game is not a BR-potential game. However, $\mathbf{g}^{\{1,2,3\}}$ is a nested $\mathcal{H}$-indicator BR-potential game, where $\mathcal{H} = (\{\{1\},\{2\},\{3\}\},\{\{1,2\},\{3\}\},\{\{1,2,3\}\})$. Indeed, $(f^1_{\{1,2\}}, f^1_{\{3\}})$ given in Table 2 is a $\{\{1\},\{2,3\}\}$-indicator BR-potential of $\mathbf{g}^{\{1,2,3\}}$. Regarding the $\{\{1\},\{2,3\}\}$-indicator BR-potential $(f^1_{\{1\}}, f^1_{\{2,3\}})$ as a strategic form game, we can show that f defined in Table 4 is an indicator BR-potential of $(f^1_{\{1\}}, f^1_{\{2,3\}})$. Thus $\mathbf{g}^{\{1,2,3\}}$ is a nested indicator BR-potential game.

a_3	a_2	a_2'	a_3'	a_2	a_2'
a_1	$1,1,1$	$1,0,1$	a_1	$0,1,0$	$1,0,0$
a_1'	$0,1,1$	$0,0,1$	a_1'	$1,0,0$	$0,1,0$

Table 2: $\mathbf{g}^{\{1,2,3\}}$ is not a BR-potential game but a nested indicator BR-potential game.

	a_2, a_3	a_2', a_3	a_2, a_3'	a_2', a_3'
a_1	$1,3$	$1,1$	$0,1$	$1,0$
a_1'	$0,3$	$0,2$	$1,0$	$0,1$

Table 3: A partition $\{\{1\},\{2,3\}\}$-indicator BR-potential $(f^1_{\{1\}}, f^1_{\{2,3\}})$ of $\mathbf{g}^{\{1,2,3\}}$

	a_2, a_3	a_2', a_3	a_2, a_3'	a_2', a_3'
a_1	3	1	0	1
a_1'	2	0	1	0

Table 4: A nested indicator BR-potential f of $\mathbf{g}^{\{1,2,3\}}$

A nested BR-potential game may not have the weak FBRP as shown by the following example. This implies that, for any hierarchy $\mathcal{H}$, a nested BR-potential game may not have the $\mathcal{H}$-FBRP.

Example 15. Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ represented as Table 5. $\mathbf{g}^{\{1,2,3\}}$ is a nested BR-potential game. Indeed, $\mathbf{g}^{\{1,2,3\}}$ has a partition $\{\{1\},\{2,3\}\}$-BR-potential as in

Table 6. Moreover, $\mathbf{f}^{\{\{1\},\{2,3\}\}}$ has a partition $\{\{1,2,3\}\}$-BR-potential as in Table 7. Thus, $\mathbf{g}^{\{1,2,3\}}$ is a nested BR-potential game.

However, for any hierarchy $\mathcal{H}$, $\mathbf{g}^{\{1,2,3\}}$ does not have the $\mathcal{H}$-FBRP. Indeed, pick an action profile (a_1', a_2', a_3'). Since we can show that $\mathbf{g}^{\{1,2,3\}}$ has no best-response path connecting from (a_1', a_2', a_3') to a Nash equilibrium, $\mathbf{g}^{\{1,2,3\}}$ does not have the weak FBRP. So, for any hierarchy $\mathcal{H}$, $\mathbf{g}^{\{1,2,3\}}$ does not have the $\mathcal{H}$-FBRP.

a_3	a_2	a_2'	a_2''	a_3'	a_2	a_2'	a_2''
a_1	$4,4,4$	$0,0,0$	$0,0,0$	a_1	$0,0,0$	$0,0,0$	$0,0,0$
a_1'	$3,3,3$	$0,0,0$	$0,0,0$	a_1'	$0,0,0$	$1,2,2$	$2,1,1$
a_1''	$3,3,3$	$0,0,0$	$0,0,0$	a_1''	$0,0,0$	$2,1,1$	$1,2,2$

Table 5: $\mathbf{g}^{\{1,2,3\}}$: A nested BR-potential game without an nested indicator BR-potential

	a_2,a_3	a_2',a_3	a_2'',a_3	a_2,a_3'	a_2',a_3'	a_2'',a_3'
a_1	$4,4$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
a_1'	$3,3$	$0,0$	$0,0$	$0,0$	$1,2$	$2,1$
a_1''	$3,3$	$0,0$	$0,0$	$0,0$	$2,1$	$1,2$

Table 6: A partition $\{1,\{2,3\}\}$-BR-potential $\mathbf{f}^{\{1,\{2,3\}\}}$ of $\mathbf{g}^{\{1,2,3\}}$

	a_2,a_3	a_2',a_3	a_2'',a_3	a_2,a_3'	a_2',a_3'	a_2'',a_3'
a_1	4	0	0	0	0	0
a_1'	3	0	0	0	1	2
a_1''	3	0	0	0	2	1

Table 7: A nested BR-potential of $\mathbf{g}^{\{1,2,3\}}$

A Appendix

Proof of Proposition 5 It is clear that we have the if-part, since a partition $\mathcal{T}$-indicator BR-potential of $\mathbf{g}^N$ is a partition $\mathcal{T}$-BR-potential of $\mathbf{g}^N$.

We show the only if-part. Suppose that $\mathbf{g}^N$ has a partition $\mathcal{T}$-BR-potential $\mathbf{f}^{\mathcal{T}}$. Let $c > 0$ be a sufficiently large number such that $c > \max_{T\in\mathcal{T}, a\in A} f_T(a)$. For each $T \in \mathcal{T}$, define a function $\hat{f}_T : A \to \mathbb{R}$ such that, for any $a_T \in A_T$ and any $a_{-T} \in A_{-T}$,

$$\hat{f}_T(a_T, a_{-T}) = \begin{cases} c & \text{if } a_T \in E_T(a_{-T}) \\ f(a) & \text{otherwise} \end{cases} .$$

Then, for each $T \in \mathcal{T}$, we have $E_T(a_{-T}) = \arg\max_{a_T \in A_T} \hat{f}_T(a_T, a_{-T})$ for all $a_T \in A_{-T}$. And, we can show that $\hat{\mathbf{f}}^{\mathcal{T}} := (\hat{f}_T)_{T \in \mathcal{T}}$ is a partition $\mathcal{T}$-BR-potential of $\mathbf{g}^N$. Indeed, fix any $T \in \mathcal{T}$, any $i \in T$ and any $a_{-T} \in A_{-T}$. Let

$$E_{T \setminus \{i\}}(a_{-T}) := \{a_{T \setminus \{i\}} \in A_{T \setminus \{i\}} | (a_i, a_{T \setminus \{i\}}) \in E_T(a_{-T}) \text{ for some } a_i \in A_i\}$$

. If $a_{T \setminus \{i\}} \notin E_{T \setminus \{i\}}(a_{-T})$, we have

$$\arg\max_{a_i \in A_i} \hat{f}_T(a_i, a_{-i}) = \arg\max_{a_i \in A_i} f_T(a_i, a_{-i})$$

by the construction of $\hat{f}_T$. Since $\mathbf{f}^{\mathcal{T}}$ is a partition $\mathcal{T}$-BR-potential of $\mathbf{g}^N$, we have

$$\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i}) = \arg\max_{a_i \in A_i} g_i(a_i, a_{-i}).$$

Suppose that $a_{T \setminus \{i\}} \in E_{T \setminus \{i\}}(a_{-T})$. Then, by the construction of $\hat{f}_T$, we have

$$E_T(a_{-T}) = \arg\max_{a_i \in A_i} \hat{f}(a_i, a_{T \setminus \{i\}}, a_{-T}) \times \{a_{T \setminus \{i\}}\}.$$

Since $E_T(a_{-T})$ is the set of Nash equilibria of $\mathbf{g}^N|_{a_{-T}}$, we have

$$E_T(a_{-T}) = \arg\max_{a_i \in A_i} g_i(a_i, a_{T \setminus \{i\}}, a_{-T}) \times \{a_{T \setminus \{i\}}\}.$$

Thus we have

$$\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{T \setminus \{i\}}, a_{-T}) = \arg\max_{a_i \in A_i} g_i(a_i, a_{T \setminus \{i\}}, a_{-T}).$$

Hence $\hat{\mathbf{f}}^{\mathcal{T}} = (\hat{f}_T)_{T \in \mathcal{T}}$ is a partition $\mathcal{T}$-BR-potential of $\mathbf{g}^N$.

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