On height functions on Jacobian surfaces

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Introduction

First we recall some fundamental facts on the canonical height function on an abelian variety. For an algebraic number field K, let A be an abelian variety defined over K. Take a very ample divisor D. Then we have the projective embedding $\phi_D\colon A \to \mathbf{P}^n$. Using this embedding, we define the logarithmic height h_D on A by $h \circ \phi_D$, where h is the logarithmic height on $\mathbf{P}^n$ (see [16],Chapter 3). The canonical height $\hat{h}_D$ on A attached to D is defined by $\hat{h}_D(z) = \lim_{n\to\infty} h_D(nz)/n^2$. The function $\hat{h}_D$ has a property that $\hat{h}_D(z) \geq 0$ for any $z \in A(\overline{K})$ and $\hat{h}_D(z) = 0$ if and only if z is a torsion point. If D' is any ample divisor and mD' is very ample, then we define $\hat{h}_{D'}$ by $\dfrac{1}{m}\hat{h}_{mD'}$. The height pairing $\langle\, ,\, \rangle_D$ is defined by $\langle x, y\rangle_D = \hat{h}_D(x+y) - \hat{h}_D(x) - \hat{h}_D(y)$. The regulator, which is an important factor of the Birch-Swinnerton-Dyer Conjecture, is defined via this height pairing. In general, it is very difficult to compute the canonical height directly by definition. Néron and Tate have shown that the canonical height decomposes into canonical local heights. In the case of elliptic curves, the archimedean local height is expressed using a theta function and the non-archimedean local height is expressed in a simple form (see [26],Chapter VI).

In this paper we compute canonical heights on Jacobian surfaces attached to the theta divisor and, as an example, we shall verify the Birch-Swinnerton-Dyer Conjecture numerically for certain Jacobian surface. We use Néron's formula [14], p.332, which asserts that the computation of the height pairing on Jacobian variety reduces to that of Néron's symbol. Néron's symbol is decomposed into Néron's local pairings first introduced by Néron [21],Chapter 2. Néron's local pairing is defined via Green's function at archimedean places and via intersection theory on an arithmetic surface at non-archimedean places; see [14].

The main result of this paper is the relation between the archimedean canonical local height and archimedean Néron's local pairing. In the case of elliptic curves, there exists the following relation between the canonical local height and Néron's local pairing at an archimedean place. Let $E : y^2 = 4x^3 - g_2 x - g_3$ be an elliptic curve defined over an algebraic number field K. Let v be an archimedean place of K. Let $E \simeq \mathbf{C}/\Lambda$ and $\wp(z)$ be the Weierstrass $\wp$-function relative to Λ. Let $k(z)$ be the Klein function, that is, $k(z) = \Delta(\Lambda)^{1/12} e^{-\frac{1}{2} z\eta(z)} \sigma(z)$, where $\Delta(\Lambda) = g_2^3 - 27g_3^2$, σ is the Weierstrass σ-function, and η is the quasi-period map associated to Λ (see [26],p.41 and p.465). For $z \in \mathbf{C}$, we denote by $\tilde{z}$ its image in $\mathbf{C}/\Lambda$. Then $\hat{\lambda}_v(\tilde{z}) = -\log|k(z)|$ is the archimedean canonical local height on $E - \{O\}$, where O is the origin of E. Néron's local pairing is defined via Green's function, which is, in this case, also expressed in terms of the Klein function. For any $P \in E(\bar{K})$, if we take, as an uniformizer, $\dfrac{y}{2x^2}$ at O and its translation at P, then we have

$$\langle (P) - (O), (P) - (O) \rangle_v = 2(\hat{\lambda}_v(\tilde{z}_P) + \frac{1}{12} \log|\Delta(\Lambda)|),$$

where $P = (\wp(z_P), \wp'(z_P))$.

We shall generalize the relation above for the case of hyperelliptic curves of genus 2. That is : For an algebraic number field K, let C be a hyperelliptic curve of genus 2 defined by $y^2 = f(x) = x^5 + a_1 x^4 + \cdots + a_5 \in K[x]$ and $\mathcal{B}$ be the set of finite Weierstrass points. Let J be the Jacobian variety of C and Θ be the theta divisor of J. For a divisor D of degree 0 on C, we denote its image in $J = \mathrm{Pic}^0(C)$ by $\overline{D}$. For $P \in C(\mathbf{C})$, we denote the hyperelliptic integral from ∞ to P by $u^P \in \mathbf{C}^2$, which is defined up to the period lattice Λ. For $z \in \mathbf{C}^2$, we denote its image in $J = \mathbf{C}^2/\Lambda$ by $\tilde{z}$. Let ϕ be the function as in Proposition 1.10. Let v be an archimedean place of K, $\hat{\lambda}_v$ be the canonical local height on $J - \Theta$ which is normalized as in Definition 2.1 with the fixed function ϕ as above, and $\langle\ ,\ \rangle_v$ be Néron's local pairing explicitly defined as in (2.5). Then we have (Theorem 2.18):

Main Theorem *For $P_i(x_i, y_i) \in C(K)$, $(i = 1, 2)$, let $b = P_1 - P_2$ with $\bar{b} \notin \Theta$, and $z_b = u^{P_1} - u^{P_2} \in \mathbf{C}^2$. As the base of tangent space at P_i, we take $2\,y_i \dfrac{\partial}{\partial x} = f'(x_i) \dfrac{\partial}{\partial y}$. Then we can take $\dfrac{x - x_i}{2\,y_i}$ if $P_i \notin \mathcal{B}$ and $\dfrac{y - y_i}{f'(x_i)}$ if $P_i \in \mathcal{B}$ as an uniformizer at P_i. In both cases, for an archimedean place v, if we take the uniformizer as above, the relation*

$$\langle b, b \rangle_v = 2\,\hat{\lambda}_v(\tilde{z}_b)$$

between Néron's local pairing and the canonical local height holds.

We can compute the canonical local height at archimedean places numerically. In the case of elliptic curves, one can achieve this by evaluating a rapidly convergent series, which is called Tate's series [25]. In Call and Silverman [4], they generalized

Tate's series for a class of varieties with a divisor and a morphism which satisfy certain conditions, including higher dimensional abelian varieties. Thus we can use their series to evaluate $\hat{\lambda}_v$ for archimedean places v. We shall give concrete expression of this series. In Grant [12], defining equations of Jacobian surfaces and the addition theorem are formulated by using the theory of hyperelliptic $\mathfrak{p}$ functions which goes back to an old book of Baker [1]. To construct the generalized Tate's series, we must first find appropriate domains in the Jacobian variety. We can take three domains which are obtained by partitioning the Jacobian surface by three translations of the theta divisor. Then we can construct the generalized Tate's series explicitly via hyperelliptic $\mathfrak{p}$ functions and compute the archimedean canonical local height numerically. By virtue of Theorem 2.18, we can compute Néron's local pairing at archimedean places. At non-archimedean places, we compute Néron's local pairing using intersection theory on an arithmetic surface, and hence we can compute the canonical height.

In section 1, we shall review some facts on Jacobian surfaces and hyperelliptic $\mathfrak{p}$-functions. In section 2, we shall give the explicit formula of Green's function using naturally generalized Klein function (Proposition 2.9). Using this formula, we shall prove Main Theorem. In section 3, we shall construct the generalized Tate's series in our case, using hyperelliptic $\mathfrak{p}$-functions. Finally, in section 4, we shall give some examples. Especially, we shall check the Birch-Swinnerton-Dyer Conjecture numerically.

Some algebraic computations are executed using the mathematical computing system Maple V. The Tate's series is computed using GNU g++ Ver 2.7.2 and LiDIA library 1.2. [1]

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Notation and terminology

Throughout this paper, we use the following notation. By an algebraic number field, we understand a finite algebraic extension of $\mathbf{Q}$ in $\mathbf{C}$. For an algebraic number field K, let Σ_K^∞ denote the set of infinite places of K, Σ_K^0 the set of finite places of K, and $\Sigma_K = \Sigma_K^\infty \cup \Sigma_K^0$. For $v \in \Sigma_K^0$, let K_v be the completion of K at v, let π_v be an uniformizer at v, O_v the ring of v-adic integers, $k_v = O_v/\pi_v O_v$, q_v the number of elements of k_v, and p_v the residual characteristic. As usual, for $a \in K$ and $v \in \Sigma_K$, define:

$$|a|_v = \begin{cases} |a| & \text{if } v \in \Sigma_K^\infty \text{ and } v \text{ is a real place,} \\ |a|^2 & \text{if } v \in \Sigma_K^\infty \text{ and } v \text{ is a complex place,} \\ q_v^{-\operatorname{ord}_{\pi_v}(a)} & \text{if } v \in \Sigma_K^0. \end{cases}$$

[1] This library may be available from anonymous ftp site:ftp://crypt1.cs.uni-sb.de/pub/LiDIA.

As is well known, the product formula $\prod_{v\in\Sigma_K} |a|_v = 1$ holds. We also use additive notation $v(a) = -\log |a|_v$ for every $v \in \Sigma_K$.

For any finite set S, we denote by $\#S$ the cardinality of S. For any divisor a, we denote the support of a by $\mathrm{supp}(a)$. For divisors a and a', we write $a \sim a'$ if a is linearly equivalent to a'. For a complex number, a complex vector, or a complex matrix x, we denote by $\overline{x}$ its complex conjugate. For $z \in \mathbf{C}^2$, we denote by $\tilde{z}$ its image in $\mathbf{C}^2/\Lambda$(see § 1.2).

1. Review on Jacobian surfaces

We assume that the characteristic of a ground field K is not equal to 2; moreover, except for the section 1.1, we assume that K is a subfield of $\mathbf{C}$. Let C be a hyperelliptic curve of genus 2 over K, defined by the equation

$$y^2 = f(x) := x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = \prod_{i=1}^{5} (x - \beta_i) . \tag{1.1}$$

We consider C as a projective non-singular curve and we denote by ∞ the point at infinity. The double covering $C \to \mathbf{P}^1$, $P(x, y) \mapsto x$ is branched over 5 finite points and ∞. We denote the set of finite Weierstrass points by $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$, where $B_i = (\beta_i, 0)$, $(i = 1, \ldots, 5)$. Let P^ι be the image of P under the hyperelliptic involution with respect to this covering, that is $P^\iota = (x, -y)$ when $P = (x, y)$.

1.1. Algebraic Theory.

We review the algebraic definition of the Jacobian variety $J = \mathrm{Jac}(C)$. Let $\mathrm{Div}_0(C)$ be the divisor group of degree zero and J is the abelian variety whose points represent $\mathrm{Pic}_0(C)$. For any $D \in \mathrm{Div}_0(C)$, denote by $\overline{D}$ its image in $\mathrm{Pic}_0(C)$. For m points $P_1, P_2, \ldots, P_m \in C$, put $D(P_1, P_2, \ldots, P_m) = P_1 + P_2 + \cdots + P_m - m\,\infty \in \mathrm{Div}_0(C)$.

Definition 1.1. For $m > 0$, we define

$$\begin{aligned} \mathrm{Div}_0^m(C) &= \{D(P_1, P_2, \ldots, P_m) \mid P_1, \ldots, P_m \in C\} \subset \mathrm{Div}_0(C), \\ \mathrm{Div}_0^{+,m}(C) &= \left\{ D(P_1, P_2, \ldots, P_m) \,\middle|\, \begin{array}{l} P_i \neq \infty \text{ for every } i, \\ P_i \neq P_j^\iota \text{ whenever } i \neq j \end{array} \right\} \subset \mathrm{Div}_0^m(C), \end{aligned}$$

and denote their images in $\mathrm{Pic}_0(C)$ by $\overline{\mathrm{Div}_0^m(C)}$ and $\overline{\mathrm{Div}_0^{+,m}(C)}$.

Then $J = \overline{\mathrm{Div}_0^2(C)}$ (see [19],pp.3.28–3.31). If we put $\Theta = \overline{\mathrm{Div}_0^1(C)}$, which is the theta divisor of J, then we have $J - \Theta = \overline{\mathrm{Div}_0^{+,2}(C)}$ (loc.cit.). Hence any points of J can be written as $\overline{D(P_1, P_2)}$ with $P_1 \neq P_2^\iota$, $P_i \neq \infty$, or $\overline{D(P)}$, which belongs to the theta divisor. The zero element O_J of J is $\overline{D(\infty)} = \overline{D(P, P^\iota)}$. All 2-torsion points of J are given by $\overline{D(B_i)}$, $(i = 1, \ldots, 5)$ and $\overline{D(B_i, B_j)}$, $(i, j = 1, \ldots, 5,\ i \neq j)$. We abbreviate these to $\overline{B}_i$ and $\overline{B}_{ij}$.

Reduction of any divisor of degree 0 to the form $\overline{D(P_1, P_2)}$ is explicitly given as follows.

Let P_1, P_2, P_3 be three points of C and $P_i = (x_i, y_i)$, $(i = 1, 2, 3)$. For simplicity, we assume that the points P_i are finite, distinct and $P_i \neq P_j^\iota$ for $i \neq j$. Then we can find the polynomial $V(x)$ of degree 2 satisfying the equations $V(x_1) = y_1$, $V(x_2) = y_2$, $V(x_3) = y_3$. We write V_{P_1,P_2,P_3} for this V.

Then we define a rational function $\tilde{V}$ on C by

$$\tilde{V}(x, y) = \frac{y + V(x)}{(x - x_1)\,(x - x_2)\,(x - x_3)},$$

which has poles at P_i, $(i = 1, 2, 3)$ and has a simple zero at ∞. Hence either $\tilde{V}$ has zeros of order 1 at two finite points P_4, P_5 or $\tilde{V}$ has a zero of order 2 at one point $P_4 = P_5$. That is, $V(x_4) = -y_4$ and $V(x_5) = -y_5$, or $V(x_4) = -y_4$, $V'(x_4) = -\left.\dfrac{dy}{dx}\right|_{\substack{x=x_4\\ y=y_4}}$. Since x_i, $(i = 1, \ldots, 5)$ is the solutions $f(x) - V^2(x) = 0$, we can find x_4 and x_5 and by the equations $y_4 = -V(x_4)$, $y_5 = -V(x_5)$, we can find the coordinates of P_4, P_5 which satisfy $P_1 + P_2 + P_3 \sim P_4 + P_5 + \infty$, that is $\overline{D(P_1, P_2, P_3)} = \overline{D(P_4, P_5)}$. For any divisor of degree 0, we can reduce it using the procedure above recursively. For the reduction algorithm, see Cantor [6].

1.2. **Analytic Theory.** For convenience of the reader, we review analytic theory of hyperelliptic integrals following [1].

First we take a basis $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ of the first homology group $H_1(C, \mathbf{Z})$ with intersection numbers $\gamma_1 \cdot \gamma_2 = \gamma_1' \cdot \gamma_2' = 0$, $\gamma_i \cdot \gamma_j' = \delta_{ij}$ (Kronecker's δ). We take a basis of the differentials of the first kind, $\mu_1 = \dfrac{dx}{2\,y}$, $\mu_2 = \dfrac{x\,dx}{2\,y}$, and write $\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$. We denote their periods by

$$\omega_{ij} = \int_{\gamma_j} \mu_i, \quad \omega_{ij}' = \int_{\gamma_j'} \mu_i, \quad (i, j = 1, 2).$$

It is well known that $\tau = \omega^{-1}\omega'$ belongs to the Siegel upper half space $\mathfrak{h}_2$. If we define the period lattice $\Lambda = \omega\mathbf{Z}^2 \oplus \omega'\mathbf{Z}^2$, then $J = \mathbf{C}^2/\Lambda$ is the Jacobian variety of C.

We define a hyperelliptic integral $u^{P,P_0} = \displaystyle\int_{P_0}^{P} \mu \in \mathbf{C}^2$, and $u^P = u^{P,\infty}$. Let u_i^{P,P_0} denote the i-th coordinate of u^{P,P_0}, $(i = 1, 2)$. The integral u^{P,P_0} is determined up to Λ. For any divisor $b = \sum_P m_P P$, we denote the corresponding integral $\sum_P m_P u^P$ by z_b. For any vector $z \in \mathbf{C}^2$, we denote by $\tilde{z}$ its image in $J = \mathbf{C}^2/\Lambda$.

The map $(P_1, P_2) \mapsto u^{P_1} + u^{P_2}$ becomes a surjection from the symmetric 2-product of C $S^2(C)$ to J. The image of $\{(P, \infty) \in S^2(C)\}$ under this map is the theta divisor of J. Put $\tilde{\Theta} = (\mathbf{C}^2 \to J)^*\Theta$, that is the pullback of Θ in $\mathbf{C}^2$.

Next let ζ_1, ζ_2 be the differentials of the second kind on C defined by

$$\zeta_1 = \frac{(\,3\,x^3 + 2\,a_1\,x^2 + a_2\,x)\,dx}{2\,y}, \quad \zeta_2 = \frac{x^2\,dx}{2\,y},$$

and define their periods $\eta = (\eta_{ij})$, $\eta' = (\eta'_{ij})$ by

$$\eta_{ij} = \int_{\gamma_j} \zeta_i, \quad \eta'_{ij} = \int_{\gamma'_j} \zeta_i, \quad (i, j = 1, 2).$$

We define an $\mathbf{R}$-linear map $\tilde{\eta}\colon \mathbf{C}^2 \to \mathbf{C}^2$ by

$$\tilde{\eta}(u) = \eta\, r + \eta'\, r', \quad \text{where} \quad u = \omega\, r + \omega'\, r', r, r' \in \mathbf{R}^2\,.$$

Between the periods, the following relation holds

$$\eta' = \eta\,\tau + 2\,\pi\,i\,{}^t\omega^{\,-1}\,. \tag{1.2}$$

Furthermore, we have

$$\eta\,\omega^{-1} \quad \text{is symmetric, which is equivalent to } \eta\,{}^t\eta' = \eta'\,{}^t\eta\,. \tag{1.3}$$

1.3. **σ-function.** As in [18],Chapter 2, for $\tau \in \mathfrak{h}_2$, we define a theta function on $\mathbf{C}^2$ by

$$\theta(z, \tau) = \sum_{n \in \mathbf{Z}^2} \exp\left[\pi\,i\,{}^t n\,\tau n + 2\pi i\,{}^t n\,z\right],$$

and for $a, b \in \mathbf{Q}^2$,

$$\begin{aligned}\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) &= \exp(\pi i^t a\,\tau a + 2\pi i^t a\,(z+b))\theta(z+\tau a+b)\\ &= \sum_{n\in\mathbf{Z}^2} \exp\left[\pi i^t(n+a)\,\tau(n+a) + 2\pi i^t(n+a)\,(z+b)\right].\end{aligned}$$

For $m,\, n \in \mathbf{Z}^2$, the factor of automorphy is given by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+\tau m+n,\tau) = \exp\left(2\pi i\left({}^t a\, n - {}^t b\, m - {}^t m\, z\right) - \pi i\, {}^t m\, \tau m\right)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)\,.$$

For the theta characteristic $\delta = \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix}$, with $\delta' = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ and $\delta'' = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$, the function $\theta\left[\delta\right](z,\tau)$ has a simple zero only at $\tilde{\Theta}$.

Now we define the hyperelliptic σ-function.

Definition 1.2. For $u \in \mathbf{C}^2$,

$$\sigma\left(u\right) = \exp\left(-\frac{1}{2}{}^t u\, \eta\, \omega^{-1} u\right)\theta\left[\delta\right]\left(\omega^{-1} u\right)\,.$$

Here we shall list some fundamental properties of the σ-function.

Lemma 1.3. *For $p, p' \in \mathbf{Z}^2$ and $l = \omega'\, p' + \omega\, p$, let $\alpha_l(u)$ be the automorphy factor of $\sigma(u)$, that is $\sigma(u+l) = \alpha_l(u)\,\sigma(u)$, then we have $\alpha_l(u) = \exp(L_l\, u + C_l)$, where*

$$\begin{aligned} L_l &= -{}^t l\, \eta\, \omega^{-1} - 2\,\pi\, i\, {}^t p'\, \omega^{-1}, \\ C_l &= -\frac{1}{2}\, {}^t l\, \eta\, \omega^{-1} l - \pi\, i\, {}^t p'\, \tau\, p' + 2\,\pi\, i\, {}^t \delta'\, p - 2\pi\, i\, {}^t \delta''\, p'. \end{aligned}$$

Let $c = \dfrac{\partial}{\partial u_1}\sigma(u)\Big|_{u=0}$. Then $c \neq 0$ and $\sigma(u)$ has, at $u=0$, the Taylor expansion

$$\sigma(u) = c\left(u_1 + \frac{a_3}{6}\, u_1^3 - \frac{1}{3}\, u_2^3 + (\text{terms of degree} \geq 5)\right).$$

Define a polynomial of two variable F by

$$\begin{aligned} F(x_1,x_2) = {} & x_1^2\, x_2^2\,(x_1+x_2) + 2\, a_1\, x_1^2\, x_2^2 + a_2\, x_1\, x_2\,(x_1+x_2) \\ & + 2\, a_3\, x_1\, x_2 + a_4\,(x_1+x_2) + 2\, a_5, \end{aligned} \tag{1.4}$$

and define a double integral

$$R_{Q,Q_0}^{P,P_0} = \int_{P_0}^{P}\int_{Q_0}^{Q} \frac{F(x,z) + 2\, y\, s}{4\,(x-z)^2}\,\frac{dx}{y}\,\frac{dz}{s},$$

with $s^2 = f(z)$. Then the following proposition holds:

Proposition 1.4 ([1],p35)**.** *We put $u' = u^{P_1,A_1} + u^{P_2,A_2}$, $u'' = u^{Q_1,A_1} + u^{Q_2,A_2}$, with $A_i \in \mathcal{B}$. For $P, Q \in C$, $A \in \mathcal{B}$, we have*

$$R_{P_1,Q_1}^{P,Q} + R_{P_2,Q_2}^{P,Q} = \log\frac{\sigma(u^{P,A} - u')}{\sigma(u^{P,A} - u'')}\Big/\frac{\sigma(u^{Q,A} - u')}{\sigma(u^{Q,A} - u'')}.$$

Now we define hyperelliptic $\mathfrak{p}$-functions.

Definition 1.5. For $i, j, \ldots, k = 1, 2$ and $u \in \mathbf{C}^2$, we define

$$\zeta_i(u) = \frac{\partial}{\partial u_i} \log \sigma(u) \quad \textit{and} \quad \mathfrak{p}_{ij..k}(u) = -\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \cdots \frac{\partial}{\partial u_k} \log \sigma(u) .$$

We define $\mathfrak{p}(u) = \mathfrak{p}_{11}(u)\, \mathfrak{p}_{22}(u) - \mathfrak{p}_{12}^2(u)$.

Define a polynomial ψ by

$$\begin{aligned}(1.5) \quad \psi(x_1, x_2) = {} & x_1^3\, x_2\,(3\,x_1 + x_2) + 4\,a_1\,x_1^3\,x_2 + a_2\,x_1^2\,(x_1 + 3\,x_2) \\ & + 2\,a_3\,x_1\,(x_1 + x_2) + a_4\,(3\,x_1 + x_2) + 4\,a_5,\end{aligned}$$

and let F be one as (1.4). Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points on the curve C and we put $u = u^{P_1} + u^{P_2}$. Then we have

Proposition 1.6. (i) ([1],[12]) *When* $P_1 \neq P_2$ *and* $P_1 \neq P_2^\iota$,

$$\begin{aligned} \mathfrak{p}_{11}(u) &= \frac{F(x_1, x_2) - 2\,y_1\,y_2}{(x_1 - x_2)^2}, \quad \mathfrak{p}_{12}(u) = -x_1\,x_2, \quad \mathfrak{p}_{22}(u) = x_1 + x_2, \\ (1.6) \qquad \mathfrak{p}_{111}(u) &= 2\,\frac{y_2\,\psi(x_1, x_2) - y_1\,\psi(x_2, x_1)}{(x_1 - x_2)^3}, \quad \mathfrak{p}_{112}(u) = 2\,\frac{x_2^2\,y_1 - x_1^2\,y_2}{x_1 - x_2}, \\ \mathfrak{p}_{122}(u) &= -2\,\frac{x_2\,y_1 - x_1\,y_2}{x_1 - x_2}, \quad \mathfrak{p}_{222}(u) = 2\,\frac{y_1 - y_2}{x_1 - x_2}\,. \end{aligned}$$

(ii) *When* $P_1 = P_2 = (x, y) \notin \mathcal{B}$ (*equivalently* $y \neq 0$),

$$\begin{aligned} & \mathfrak{p}_{11}(u) = 4\,x^3 + 2\,a_1\,x^2 + a_2\,x + \frac{f'^2(x) - 2\,f(x)\,f''(x)}{4\,y^2} \\ & \mathfrak{p}_{12}(u) = -x^2, \ \mathfrak{p}_{22}(u) = 2x, \\ (1.7) \quad & \mathfrak{p}_{111}(u) = -(14x^2 + 8a_1x + 2a_2)y - \frac{(2\,x^3 - f''(x))f'(x)}{2\,y} - \frac{f'^3(x)}{4\,y^3}, \\ & \mathfrak{p}_{112}(u) = \frac{x^2\,f'(x) - 4\,x\,f(x)}{y}, \ \mathfrak{p}_{122}(u) = -\frac{x\,f'(x) - 2\,f(x)}{y}, \ \mathfrak{p}_{222}(u) = \frac{f'(x)}{y}. \end{aligned}$$

Proof. As for (i), see Baker [1]. The formulae (1.7) can be deduced from (1.6) by L'hôpital's rule. □

Immediately, by the proposition above, we have

Corollary 1.7. *If $u = u^{P_1} + u^{P_2}$, then*

$$\begin{aligned} x_1 + x_2 &= \mathfrak{p}_{22}(u)\,, \quad x_1\, x_2 = -\mathfrak{p}_{12}(u), \\ y_1 + y_2 &= \mathfrak{p}_{122}(u) + \frac{1}{2}\,\mathfrak{p}_{22}(u)\,\mathfrak{p}_{222}(u), \\ y_1\, y_2 &= \frac{1}{4}\left(\mathfrak{p}_{122}^2(u) - \mathfrak{p}_{112}(u)\,\mathfrak{p}_{222}(u)\right). \end{aligned} \tag{1.8}$$

For $u, v \in \mathbf{C}^2 - \tilde{\Theta}$, define

$$q(u,v) = -c^2 \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)}. \tag{1.9}$$

Then the following formula holds [1],p.100:

$$q(u,v) = \mathfrak{p}_{11}(u) - \mathfrak{p}_{11}(v) + \mathfrak{p}_{12}(u)\,\mathfrak{p}_{22}(v) - \mathfrak{p}_{12}(v)\,\mathfrak{p}_{22}(u)\,. \tag{1.10}$$

Using the fact that, at ∞ of the curve, the coordinate functions x and y has a pole of order 2 and 5 respectively, we have:

Lemma 1.8. *Let $P(x,y)$ and $u = u^P$. Then we have*

$$\begin{aligned} &(\mathfrak{p}_{11}/\mathfrak{p}_{12})(u) = (\mathfrak{p}_{12}/\mathfrak{p}_{22})(u) = -x, \\ &(\mathfrak{p}_{111}/\mathfrak{p}_{112})(u) = (\mathfrak{p}_{112}/\mathfrak{p}_{122})(u) = (\mathfrak{p}_{122}/\mathfrak{p}_{222})(u) = -x, \\ &(\mathfrak{p}/\mathfrak{p}_{222})(u) = -y. \end{aligned} \tag{1.11}$$

Let $\sigma_i(u) = \dfrac{\partial\sigma}{\partial u_i}(u)$ for $i = 1, 2$. From this lemma, since $\dfrac{\mathfrak{p}_{12}}{\mathfrak{p}_{22}}(u) = \dfrac{\sigma_1(u)}{\sigma_2(u)} = -\dfrac{1}{u_2^2} + \cdots$ for $u \in \tilde{\Theta}$, we know that $\sigma_2(u)$ is not 0 along $\tilde{\Theta}$ except for ∞. For $u \in \mathbf{C}^2 - \tilde{\Theta}$ and $v \in \tilde{\Theta}$, we define

$$Q(u,v) = -c^2 \frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\sigma_2^2(v)}. \tag{1.12}$$

Proposition 1.9 (cf. [13],p.124)**.** *Let $P(x,y)$ and $v = u^P$, then*

$$Q(u,v) = -x^2 + \mathfrak{p}_{12}(u) + \mathfrak{p}_{22}(u)x.$$

Hence if $P_i = (x_i, y_i)$ and $u = u^{P_1} + u^{P_2}$, then

$$Q(u,v) = -(x - x_1)(x - x_2) \tag{1.13}$$

Proof. First note that $(\mathfrak{p}_{12}/\mathfrak{p}_{22})(u^P) = \sigma_1(u^P)/\sigma_2(u^P) = -x$ by (1.11). Multiplying (1.10) by $\sigma(v)^2/\sigma_2^2(v)$ and taking the limit v to u^P, then we obtain the formulae above. □

The following proposition is the heart of the duplication theorem and the definition of the canonical local height.

Proposition 1.10. *For $u \in \mathbf{C}^2$,*

$$-c^3\,\frac{\sigma(2\,u)}{\sigma^4(u)} = \mathfrak{p}_{111}(u) - \mathfrak{p}_{12}(u)\,\mathfrak{p}_{122}(u) + \mathfrak{p}_{22}(u)\,\mathfrak{p}_{112}(u)\,.$$

We denote the right-hand side by $\phi(u)$.

Proof. By the definition of $q(u,v)$, $\dfrac{q(u,v)}{\sigma(u-v)} = -c^2\,\dfrac{\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)}$. Using the Taylor expansion of $\sigma(u)$ at $u=0$, we get $\dfrac{\partial}{\partial\,v_1}\sigma(u-v)\big|_{v=u} = -c$. Thus, by L'hôpital's rule, $\lim\limits_{v\to u}\dfrac{q(u,v)}{\sigma(u-v)} = -\dfrac{1}{c}\dfrac{\partial}{\partial v_1}q(u,v)\Big|_{u=v}$. By differentiating the right-hand side of (1.10) with respect to v_1 and substituting $v=u$ into the result, we conclude the assertion. □

Corollary 1.11. *Put $\phi_{ij\cdots k}(u) = \dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j}\cdots\dfrac{\partial}{\partial u_k}\,\phi(u)$. Then we have the duplication formulae:*

$$\begin{aligned}
\mathfrak{p}_{ij}(2\,u) &= \mathfrak{p}_{ij}(u) - \frac{\phi_{ij}(u)}{4\,\phi(u)} + \frac{\phi_i(u)\phi_j(u)}{4\phi(u)^2},\\
\mathfrak{p}_{ijk}(2\,u) &= \frac{1}{2}\,\mathfrak{p}_{ijk}(u) - \frac{\phi_{ijk}(u)}{8\,\phi(u)} + \\
&\quad \frac{\phi_{ij}(u)\,\phi_k(u) + \phi_{jk}(u)\,\phi_i(u) + \phi_{ki}(u)\,\phi_j(u)}{8\,\phi^2(u)} - \frac{\phi_i(u)\,\phi_j(u)\,\phi_k(u)}{4\,\phi^3(u)}.
\end{aligned}$$

1.4. **Defining equations and arithmetic.** We review defining equations for the affine model using $\mathfrak{p}$-functions([12]). We define coordinate functions X_{ij}, X_{ijk}, and X as follows.

$$X_{ij} = \mathfrak{p}_{ij}, \quad X_{ijk} = \frac{1}{2}\,\mathfrak{p}_{ijk}, \quad \text{and} \quad X = \frac{1}{2}\,(\mathfrak{p} + a_2\,\mathfrak{p}_{12} - a_4).$$

We write X^h_0, X^h_{ij}, X^h_{ijk}, X^h for homogeneous coordinates of a points of $\mathbf{P}^8$ with $X_{ij} = X^h_{ij}/X^h_0$, $X_{ijk} = X^h_{ijk}/X^h_0$, and $X = X^h/X^h_0$. We write $[X^h_0 : X^h_{11} : X^h_{12} : X^h_{22} : X^h_{111} : X^h_{112} : X^h_{122} : X^h_{222} : X^h]$ for a set of coordinates. Then we have

Theorem 1.12 ([12])**.** *Let E_i, $(i = 1,\ldots,14)$, be the equations f_i in* [12]*,pp.103–107. The ideal $I(J-\Theta)$ is generated by $E_2,\ldots,E_7$.*

Remark 1.13. There is a misprint in [12],p.106. The equation f_{10} should be read

$$f_{10} = X^2_{112} - X_{111}\,X_{122} + \cdots\,.$$

Theorem 1.14 ([12]). *Let E_i^h be the homogenization of E_i, $(i = 1, \ldots, 14)$. Then their Jacobian matrix is of full rank at every point of J and the equations give a non-singular model in $\mathbf{P}^8$.*

The theta divisor on $J \subset \mathbf{P}^8$ is given by $X_0^h = 0$. Substituting $X_0^h = 0$ into the equations E_i^h, we have

$$\Theta \subset \left\{[0:0:0:0:X_{111}^h : X_{112}^h : X_{122}^h : X_{222}^h : X^h]\right\}.$$

The following proposition follows from Lemma 1.11:

Proposition 1.15. *For $P(x,y) \in C$, the coordinates of $\overline{D(P)} \in \Theta \subset \mathbf{P}^8$ is given by*

$$[0:0:0:0:-x^3:x^2:-x:1:-y].$$

Especially, the zero element O_J of J has the coordinates

$$[0:0:0:0:1:0:0:0:0].$$

The additive formulae are described in Grant [12] and we do not reproduce here. We can obtain the formulae of the addition of points one of which is on the Theta divisor and the other of which is not on the Theta divisor using Cantor's algorithm [6] or taking the limit of the additive formulae. As for the duplication theorem, see Corollary 1.11. Also see [5] for the computation on Kummer surfaces.

2. Archimedean local heights

2.1. **Canonical local heights.** In this section we review on the Néron-Tate local heights (the canonical local heights) on a Jacobian surface J. See [4],Section 2 for more details.

Let $\Psi_2 \colon J \to J$ be the multiplication by 2 map. The theta divisor Θ satisfies

$$\Psi_2^* \Theta \sim 4\,\Theta.$$

Now we define the canonical local height $\hat{\lambda}_v : J - \Theta \to \mathbf{R}$ for $v \in \Sigma_K$ as follows:

Definition 2.1. (1) $\hat{\lambda}_v$ is a Weil local height function corresponding the divisor Θ.
(2) Let ϕ be a function such that $\Psi_2^* \Theta = 4\,\Theta + \mathrm{div}(\phi)$, then

$$\hat{\lambda}_v(2\,z) = 4\,\hat{\lambda}_v(z) + v(\phi(z)).$$

Remark 2.2. As in [4], p.171, $\hat{\lambda}_v$ is uniquely determined if we fix ϕ, since we assume the equation of the second condition holds with $v(\phi(z))$ in spite of $v(a\phi(z))$. After this, we shall fix ϕ as in Proposition 2.3 below.

Proposition 2.3. *Let ϕ be the function defined in Proposition 1.10. Then* $\Psi_2^* \Theta = 4\,\Theta + \mathrm{div}(\phi)$.

Proof. Since $\sigma(u)$ has zero at $\tilde{\Theta}$ of order 1, we have $\Psi_2^* \Theta = 4\,\Theta + \mathrm{div}\left(\dfrac{\sigma \circ \Psi_2}{\sigma^4}\right)$. By Proposition 1.10 we have the assertion. $\square$

Now we define a modified σ-function $k(u)$, which is a natural generalization of the Klein function, by

$$k(u) = c^{-1} \exp\left(\frac{1}{2}\,{}^t u\, \tilde{\eta}(u)\right) \sigma(u). \tag{2.1}$$

Proposition 2.4. *The function $|k(u)|$ on $\mathbf{C}^2$ is periodic for Λ.*

Proof. First note that ${}^t z\ \tilde{\eta}(w) = {}^t\tilde{\eta}(w)\ z$, since this is a scalar. Let $\exp(\kappa_l(u)) = k(u+l)/k(u)$ for $l = \omega\, p + \omega'\, p'$, $p,\ p' \in \mathbf{Z}^2$ and $u = \omega r + \omega' r'$ with $r, r' \in \mathbf{R}^2$. Then, by Lemma 1.3,

$$\kappa_l(u) = \begin{pmatrix} {}^t p & {}^t p' \end{pmatrix}\ M_r \begin{pmatrix} r \\ r' \end{pmatrix} + \begin{pmatrix} {}^t p & {}^t p' \end{pmatrix}\ M_p \begin{pmatrix} p \\ p' \end{pmatrix}.$$

where

$$M_r = \begin{pmatrix} \frac{1}{2}\,({}^t\eta\ \omega - {}^t\omega\ \eta) & \frac{1}{2}\,({}^t\eta\ \omega' + {}^t\omega\ \eta' - 2\,{}^t\omega\ \eta\,\tau) \\ \frac{1}{2}\,({}^t\eta'\ \omega - {}^t\omega'\ \eta) & \frac{1}{2}\,({}^t\eta'\ \omega' + {}^t\omega'\ \eta' - 2\,{}^t\omega'\,\eta\,\tau) \end{pmatrix} - 2\,\pi\, i \begin{pmatrix} 0 & 0 \\ 1 & \tau \end{pmatrix},$$

and

$$M_p = \begin{pmatrix} 0 & \frac{1}{2}\,({}^t\omega\ \eta' - {}^t\omega\ \eta\,\tau) \\ 0 & \frac{1}{2}\,({}^t\omega'\ \eta' - {}^t\omega'\ \eta\,\tau - 2\,\pi\, i\,\tau) \end{pmatrix}.$$

By (1.3), $\eta\,\omega^{-1} = {}^t(\eta\,\omega^{-1}) = {}^t\omega^{-1}\ {}^t\eta$, thus, ${}^t\eta\ \omega = {}^t\omega\ \eta$. That is $(M_r)_{11} = 0$. By (1.2), ${}^t\eta'\ \omega = {}^t(\eta\,\tau + 2\,\pi\, i\,{}^t\omega^{-1})\ \omega = \tau\,{}^t\eta\ \omega + 2\,\pi\, i\,\mathbf{1}_2$ and ${}^t\omega'\ \eta = {}^t({}^t\eta\ \omega') = {}^t({}^t\eta\ \omega\,\tau) = \tau\,{}^t({}^t\eta\,\omega) = \tau\,{}^t\omega\ \eta$. As above, ${}^t\eta\ \omega = {}^t\omega\ \eta$, so $(M_r)_{21} = -\pi\, i\,\mathbf{1}_2$. Similarly, $(M_r)_{12} = \pi\, i\,\mathbf{1}_2$. Finally, since $\tau\,{}^t\eta\ \omega' = \tau\,{}^t(\eta\,\omega^{-1}\,\omega)\ \omega' = {}^t\omega'\ \eta\,\tau$, using (1.2), we get $(M_r)_{22} = 0$. Also by (1.2), we conclude that $M_p = \begin{pmatrix} 0 & \pi\, i\,\mathbf{1}_2 \\ 0 & 0 \end{pmatrix}$.

Since any of p, p', r, r' belongs to $\mathbf{R}^2$, $\exp(\kappa_l(u))$ is of the form $\exp(i \cdot$'real number'$)$ and $|\exp(\kappa_l(u))| = 1$. $\square$

Corollary 2.5. *For $v \in \Sigma_K^\infty$ and $u \in \mathbf{C}^2 - \tilde{\Theta}$, if we take ϕ as in Proposition 1.10, then*

$$\hat{\lambda}_v(\tilde{u}) = -\log |k(u)|_v\,. \tag{2.2}$$

Proof. By the proposition above, the right-hand side of (2.2) is well defined and clearly it is a Weil function for Θ. Furthermore
$$\frac{k(2\,u)}{k^4(u)} = c^3\frac{\sigma(2\,u)}{\sigma^4(u)} = -\phi(u),$$
thus the right-hand side of (2.2) satisfies the property (2) in the definition of $\hat{\lambda}_v$. By the uniqueness, the assertion follows. □

2.2. **Green's Function.** A meromorphic differential ϱ on the Riemann surface $C(\mathbf{C})$ is said to be of the third kind if $\mathrm{ord}_x(\varrho) \geq -1$ for all $x \in C(\mathbf{C})$. For such ϱ, we define the divisor $\mathrm{Res}(\varrho)$ by $\sum_{x\in C(\mathbf{C})} \mathrm{res}_x(\varrho)\, x$, which belongs to $\mathrm{Div}_0(C)$. Conversely, by the Riemann-Roch theorem, for any $a \in \mathrm{Div}_0(C)$, there exists a differential of the third kind ϱ such that $\mathrm{Res}(\varrho) = a$, and it is determined up to an addition of a differential of the first kind. We write ω_a for this ϱ.

Lemma 2.6. *We can choose ϱ uniquely with pure imaginary periods.*

Proof. For any differential of the third kind $\tilde{\varrho}$, we define
$$\mathbf{r} = -2\left(\mathrm{Re}\left(\int_{\gamma_1}\tilde{\varrho}\right), \mathrm{Re}\left(\int_{\gamma_2}\tilde{\varrho}\right), \mathrm{Re}\left(\int_{\gamma_1'}\tilde{\varrho}\right), \mathrm{Re}\left(\int_{\gamma_2'}\tilde{\varrho}\right)\right).$$
Then we can define complex numbers c_1, c_2 by
$$(c_1, c_2, \overline{c_1}, \overline{c_2}) = \mathbf{r}\tilde{\Omega}^{-1},$$
where $\tilde{\Omega} = \begin{pmatrix} \omega & \omega' \\ \overline{\omega} & \overline{\omega'} \end{pmatrix}$. Then $\varrho = \tilde{\varrho} + c_1\,\mu_1 + c_2\,\mu_2$ has pure imaginary periods. □

Lemma 2.7. *If $a = D(P_1)$ with $P_1(x_1, y_1) \in C$, then ω_a is explicitly given by*
$$\omega_a = \frac{y + y_1}{2\,y\,(x - x_1)}dx + \omega_h,$$
where ω_h is any differential of the first kind.

Proof. Noting that $\mathrm{div}(dx) = \sum B_i - 3\,\infty$ and $\mathrm{div}(y) = \sum B_i - 5\,\infty$, it is obvious that ω_a has simple poles at only P_1 and ∞. The function $x - x_1$ is an uniformizer at P_1 and $\mathrm{res}_{P_1}(\omega_a) = 1$. By the residue theorem, $\mathrm{res}_\infty(\omega_a) = -1$ and $\mathrm{Res}(\omega_a) = P_1 - \infty = a$. □

Definition 2.8. For any $v \in \Sigma_K^\infty$ and for each $a \in \mathrm{Div}_0(C)$, Green's function on $C(\mathbf{C}) - |a|$ attached to a is a real valued harmonic function g_a such that

(1) $g_a - m_x \log|z|_v$ is harmonic near x, where z is a local parameter at x and m_x is the order of x in a.

(2) g_a is a solution of a differential equation $\partial\bar{\partial}g_a = -2\pi i\delta_a$, where δ_a is (1,1)-current which represents the evaluation of (0,0)-forms at a.

For $a \in \mathrm{Div}_0(C)$, choose ω_a so that it has pure imaginary periods (which exists by Lemma 2.6), then the differential equation $\omega_a + \bar{\omega}_a = d\,g$ has a solution g and we can take g as g_a [14].

Now we have an explicit formula of Green's function.

Proposition 2.9. (1) *When* $a = P_1 - \infty$,

$$g_a(P) \equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_1})}{k(2\,u^P)}\right|_v .$$

(2) *When* $a = P_1 + P_2 - 2\,\infty$,

$$\begin{aligned} g_a(P) &\equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_1})}{k(2\,u^P + u^{P_2})}(x-x_2)^2\right|_v \\ &\equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_2})}{k(2\,u^P + u^{P_1})}(x-x_1)^2\right|_v \\ &\equiv \frac{1}{2}\log\left|\frac{k(2\,u^P - u^{P_1})\,k(2\,u^P - u^{P_2})}{k(2\,u^P)^2}\right|_v . \end{aligned}$$

In the both cases, the symbol $\equiv$ *means equality up to a constant.*

Proof. (1) We shall prove that the function $||\theta(z)||$ in Bost [2] is coincide with $|k(z)|$ up to a constant multiple which depends only τ. Put $z = \omega r + \omega' r'$, $r, r' \in \mathbf{R}^2$. Using (1.2), we have

$$\begin{aligned} |k(z)| &= |c^{-1}\exp(\pi i({}^t r\, r' + {}^t r'\,\tau r'))|\,|\theta[\delta](\omega^{-1}z)| \\ &= |c^{-1}|\exp(-\pi {}^t r'\ \mathrm{Im}(\tau){}^t r'\,)|\theta[\delta](\omega^{-1}z)|. \end{aligned}$$

If we put $z_0 = \omega^{-1}z = r + \tau r' = x_0 + iy_0$ and $Y = \mathrm{Im}(\tau)$, then

$$y_0 = \mathrm{Im}(z_0) = \mathrm{Im}(\tau r') = \mathrm{Im}(\tau)r' = Yr'.$$

That is $r' = Y^{-1}y_0$ and ${}^t r'\ \mathrm{Im}(\tau)r' = {}^t y_0\, Y^{-1}y_0$. On the other hand, for D a divisor of degree 1, we can deduce that $||\theta||(D) = \det(Y)^{1/4}\exp(-\pi {}^t y_0\, Y^{-1}y_0)\ |\theta[\delta](z_0)|$, where $[z] \in \mathbf{C}^2/\Lambda$ is the point corresponding to $\overline{D - \infty}$ and z_0, y_0 are as above (Note that $\Delta = \delta' + \tau\delta''$ is a 2-torsion). Thus $|k(z)|$ and $||\theta||$ coincide up to the factor $|c^{-1}|\det(Y)^{1/4}$. By virtue of Bost's result [2], which is proved in the appendix of [3] we conclude the formula.

(2) The third expression of the right hand side is obvious by (1). We can also deduce the first and second ones from Bost's result [2]. □

Remark 2.10. We can give another proof of the proposition above by directly checking the differential equation $\omega_a + \overline{\omega_a} = dg_a$ (cf. Lemma 2.7).

Remark 2.11. In the formula (2), using the equation(cf. (1.13))

$$\frac{\sigma(2\,u^P - u^{P_1})\,\sigma(2\,u^P + u^{P_1})}{\sigma^2(2\,u^P)} = M(x - x_1)^2, \text{ where } M = \frac{\sigma_2^2(u^{P_1})}{c^2},$$

we see that the first and the third expressions are equal up to a constant.

2.3. **Néron's local pairing.** We review Néron's local pairing following [14]. For any $v \in \Sigma_K$, let $\mathrm{Div}_0(C)_{/K_v}$ be the K_v-rational subgroup of $\mathrm{Div}_0(C)$ and let $Z_0(C)_{/K_v}$ be the point-wise K_v-rational subgroup. Two divisors a, $b \in \mathrm{Div}_0(C)$ are called *relatively prime* if they have disjoint support, that is $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$. For any rational function f on C and a divisor $a = \sum\limits_x a_x x \in \mathrm{Div}(C)$, write $f(a) = \prod\limits_x f(x)^{a_x}$ if $\mathrm{supp}(\mathrm{div}(f)) \cap \mathrm{supp}(a) = \emptyset$. We define a modified value of f at x as follows: Fix a tangent vector $\dfrac{\partial}{\partial t}$ at x on C and take an uniformizer z around x with $\dfrac{\partial z}{\partial t} = 1$. Then we define the modified value $f[x]$ of f at x by $f[x] = \dfrac{f}{z^m}\Big|_{z=0}$, where m is the order of f at x. For any divisor $a = \sum a_x x$, we define $f[a] = \prod f[x]^{a_x}$.

Proposition 2.12 ([14],p.328)**.** *There is a unique pairing $\langle a, b\rangle_v$ on relatively prime divisors $a \in Z_0(C)_{/K_v}$, $b \in \mathrm{Div}_0(C)_{/K_v}$ with values in $\mathbf{R}$ which satisfies the following properties:*

(i) $\langle a, b\rangle_v + \langle a, c\rangle_v = \langle a, b + c\rangle_v$.
(ii) $\langle a, b\rangle_v = \langle b, a\rangle_v$ *for any* $b \in Z_0(C)_{/K_v}$.
(iii) $\langle a, \mathrm{div}(g)\rangle_v = \log|g(a)|_v$ *for any* $g \in K(C)^*$.
(iv) *For fixed b and $x_0 \in C(K_v) - \mathrm{supp}(b)$, $C(K_v) - \mathrm{supp}(b) \ni x \mapsto \langle x - x_0, b\rangle_v \in \mathbf{R}$ is continuous.*

This pairing is called Néron's local pairing. This pairing satisfies functoriality. That is, let C' be an another curve and $\Phi \in C \times C'$ be a correspondence rational over K_v, then we have $\langle a, \Phi^* b\rangle_C = \langle \Phi_* a, b\rangle_{C'}$ for $a \in \mathrm{Div}_0(C)$, $b \in \mathrm{Div}_0(C')$ when the both sides are defined. If L_w be an extension of K_v,

$$\langle a, b\rangle_w = [L_w : K_v]\ \langle a, b\rangle_v\ . \tag{2.3}$$

If $\mathrm{supp}(a) \cap \mathrm{supp}(b) \neq \emptyset$, we modify the pairing by

$$\langle a, b\rangle_v = \log|g[a]|_v + \langle a, b'\rangle_v\,, \tag{2.4}$$

where $b = b' + \mathrm{div}(g)$ such that $\mathrm{supp}(a) \cap \mathrm{supp}(b') = \emptyset$.

For $a \in Z_0(C)_{/K_v}$ and $b \in \mathrm{Div}_0(C)_{/K_v}$ with $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$, the pairing is explicitly defined as follows.

For the archimedean place v, the pairing is explicitly given by

$$\langle a, b\rangle_v = g_a(b), \tag{2.5}$$

where g_a is Green's function attached to a (§ 2.2).

Remark 2.13. In Proposition 1.4, substituting P_1, P_2, P_3, P_3, P_5, P_5 for P, Q, P_1, P_2, Q_1, Q_2 respectively, we have

$$2\,R^{P_1,P_2}_{P_3,P_5} = \log \frac{\sigma(u^1 - 2\,u^3)}{\sigma(u^1 - 2\,u^5)} \Big/ \frac{\sigma(u^2 - 2\,u^3)}{\sigma(u^2 - 2\,u^5)}\,.$$

Using (1.6), we get $\dfrac{d}{dx}\dfrac{d}{dz}\log\sigma(u^P - u^Q) = \dfrac{\mathfrak{p}_{11}(u^P - u^Q)}{4\,y\,s}$ where $P = (x, y), Q = (z, s)$. Hence we have

$$R^{P_1,P_2}_{P_3,P_5} = \int_{P_2}^{P_1}\int_{P_5}^{P_3} \frac{d}{dx}\,\frac{d}{dz}\log\sigma(u^P - u^Q)dxdz\,.$$

Thus we get

$$\log \frac{\sigma(u^1 - 2\,u^3)}{\sigma(u^1 - 2\,u^5)} \Big/ \frac{\sigma(u^2 - 2\,u^3)}{\sigma(u^2 - 2\,u^5)} = 2\,\log \frac{\sigma(u^1 - u^3)}{\sigma(u^1 - u^5)} \Big/ \frac{\sigma(u^2 - u^3)}{\sigma(u^2 - u^5)},$$

and this implies the symmetry of the pairing.

For the non-archimedean place v, the pairing is explicitly given by

$$\langle a, b\rangle_v = -(A \cdot\ B)\log q_v,$$

where $A \cdot B$ is the intersection number of A and B (See [11],Chapter 7,20 or [26], Chapter IV, Section 7 for the definition of the intersection number). Here rational divisors $A, B \in \mathrm{Div}(\mathcal{C}) \otimes \mathbf{Q}$ are extensions of divisors a, b in a regular model $\mathcal{C}$ of C over O_v that satisfy $(A \cdot \mathcal{F}) = (B \cdot \mathcal{F}) = 0$ for any fibral irreducible divisor $\mathcal{F}$ of $\mathcal{C}/O_v$.

The following lemma is useful to compute the "correction term" (cf. [7]).

Lemma 2.14. *Let $\mathcal{C}/O_v$ be a regular arithmetic surface. Let $\sum_{i=0}^{n} m_i C_i$ be the special fiber, where C_i is an irreducible divisor, and σ be a horizontal divisor of degree 0. We assume that $m_0 = 1$. Let M be a matrix given by $M_{ij} = (C_i \cdot C_j)$, for $1 \le i, j \le n$. Define rational numbers a_i $(i = 1, \dots, n)$ by*

$$(a_1, \dots, a_n) = -((\sigma \cdot C_1), \dots, (\sigma \cdot C_n))M^{-1}.$$

Then we have $(\sigma + \sum_{i=1}^{n} a_i C_i \cdot C_j) = 0$ for any $j = 0, \dots, n$.

Finally Néron's formula is $\langle \overline{a}, \overline{b} \rangle = \sum_{v \in \Sigma_K} \langle a, b \rangle_v$; see Néron [21],pp.295–296. Here $\langle \overline{a}, \overline{b} \rangle$ is the height pairing on $J \times J$ satisfying

$$\langle \overline{a}, \overline{b} \rangle = \hat{h}(\overline{a} + \overline{b}) - \hat{h}(\overline{a}) - \hat{h}(\overline{b}),$$

where $\hat{h}$ is the canonical height $\hat{h}_\Theta$ attached to Θ(see Introduction). (We identify J and $\hat{J}$ by $J \to \hat{J}$, $a \mapsto$ (class of $(\Theta - a) - \Theta$)). If L is a finite extension of K, we have

$$\langle \alpha, \beta \rangle_L = [L : K] \ \langle \alpha, \beta \rangle_K \ .$$

2.4. **The canonical local height and Néron's local pairing.** Let P_1, P_2 be K-rational points on C. Take P_3, P_4, P_5, P_6 which satisfy $P_1 + P_3 + P_4 \sim P_2 + P_5 + P_6$. Define polynomials $G_1 = V_{P_1,P_3,P_4}$ and $G_2 = V_{P_2,P_5,P_6}$ (see § 1.1). For simplicity, we write u^* for u^{P_*}.

Let $\overline{D(P_1, P_3, P_4)} = \overline{D(P_2, P_5, P_6)} = \overline{D(P_{11}, P_{12})}$, $P_{1j} = (x_{1j}, y_{1j})$. Then G_1 is characterized by $G_1(x_1) = y_1$, $G_1(x_{11}) = -y_{11}$, $G_1(x_{12}) = -y_{12}$ and G_2 is characterized in the similar way.

First we can prove the following lemma by direct computation.

Lemma 2.15. *Let G_1, G_2 as above. Then we have the following relation.*

$$q(u^1 - u^2, u^{11} + u^{12}) = \frac{(y_2 + G_1(x_2))(y_1 + G_2(x_1))}{(x_1 - x_2)^2}$$

Remark 2.16. As the referee notes, the following formula holds:

$$q(u^1 + u^2, u^3 + u^4) = \frac{\det \begin{pmatrix} y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \\ y_4 & x_4^2 & x_4 & 1 \end{pmatrix} \det \begin{pmatrix} y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ -y_3 & x_3^2 & x_3 & 1 \\ -y_4 & x_4^2 & x_4 & 1 \end{pmatrix}}{(x_1 - x_2)^2 (x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)^2}.$$

We can prove this formula by considering of zeros and poles, or by direct computation. On the other hand, by Cramer's rule, we have

$$y - G_1(x) = \frac{\det \begin{pmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_{11}^2 & x_{11} & 1 & -y_{11} \\ x_{12}^2 & x_{12} & 1 & -y_{12} \\ x^2 & x & 1 & y \end{pmatrix}}{(x_1 - x_{11})(x_1 - x_{12})(x_{11} - x_{12})}.$$

The lemma above is immediately deduced from these equations.

Proposition 2.17. *Let P_3, P_4, P_5, P_6 and G_1, G_2 as above. Then we have*

$$q(u^1-u^2,u^3+u^4-u^2)(=q(u^1-u^2,u^5+u^6-u^1))$$
$$=\frac{(x_1-x_3)(x_1-x_4)(x_2-x_5)(x_2-x_6)(y_2+G_1(x_2))(y_1+G_2(x_1))}{(x_2-x_3)(x_2-x_4)(x_1-x_5)(x_1-x_6)(x_1-x_2)^2}.$$

Proof. As the referee notes, we have the following equation:

$$q(u^1-u^2,u^3+u^4-u^2)=q(u^1-u^2,u^{11}+u^{12})\frac{Q(u^3+u^4,u^1)Q(u^5+u^6,u^2)}{Q(u^3+u^4,u^2)Q(u^5+u^6,u^1)}.$$

This is checked by the definition of $q(u,v)$ and $Q(u,v)$ ((1.9) and (1.12)). Putting (1.13) and Lemma 2.15 together, the assertion follows. $\square$

Theorem 2.18. *For $P_i(x_i,y_i)\in C(K)$, $i=1,2$, let $b=P_1-P_2$ with $\bar{b}\notin\Theta$, and $z_b=u^1-u^2\in\mathbf{C}^2$. As the base of tangent space at P_i, we take $2\,y_i\,\dfrac{\partial}{\partial x}=f'(x_i)\,\dfrac{\partial}{\partial y}$. Then we can take $\dfrac{x-x_i}{2\,y_i}$ if $P_i\notin\mathcal{B}$ and $\dfrac{y-y_i}{f'(x_i)}$ if $P_i\in\mathcal{B}$ as an uniformizer at P_i. In both cases, for an archimedean place v, if we take the uniformizer as above, the relation*

$$\langle b,b\rangle_v=2\,\hat{\lambda}_v(\tilde{z}_b) \tag{2.6}$$

between Néron's local pairing and the canonical local height holds.

Proof. For any divisor $b'\in\mathrm{Div}_0(C)$ which is linearly equivalent to b, let $G=G_{b,b'}$ be a rational function such that $b=b'+\mathrm{div}(G_{b,b'})$. Then we have

$$\langle b,b\rangle_v=\langle b,b'\rangle_v+\log|G[b]|_v\,,\quad \langle b,b'\rangle_v=g_b(b'),$$

where $G[b]$ means the modified value of G at b (see (2.4)). As in the proof of Proposition 2.17, we can take b' in the form $b'=P_5+P_6-P_3-P_4$ and $P_1+P_3+P_4\sim P_{11}+P_{12}+\infty$.

If we define

$$\tilde{G}_1(P)=\frac{y+G_1(x)}{(x-x_1)\,(x-x_3)\,(x-x_4)},$$
$$\tilde{G}_2(P)=\frac{y+G_2(x)}{(x-x_2)\,(x-x_5)\,(x-x_6)},$$

then,

$$\mathrm{div}(\tilde{G}_1)=P_{11}+P_{12}+\infty-P_1-P_3-P_4,$$
$$\mathrm{div}(\tilde{G}_2)=P_{11}+P_{12}+\infty-P_2-P_5-P_6\,.$$

Thus if we define $G=\tilde{G}_2/\tilde{G}_1$, then $\mathrm{div}(G)=b-b'$, that is we can take G as $G_{b,b'}$.

When both P_i, $(i = 1, 2)$ are not Weierstrass points, we may take the local parameter at P_i as mentioned in the theorem, and we have

$$G[b] = \frac{(x_1 - x_3)(x_1 - x_4)(x_2 - x_5)(x_2 - x_6)(y_2 + G_1(x_2))(y_1 + G_2(x_1))}{-(x_2 - x_3)\,(x_2 - x_4)\,(x_1 - x_5)\,(x_1 - x_6)\,(x_1 - x_2)^2}\,.$$

By Proposition 2.9, we have

$$\begin{aligned}\langle b, b'\rangle_v &= g_b(P_5) + g_b(P_6) - g_b(P_3) - g_b(P_4)\\ &= \frac{1}{2}\,\log\left|\frac{k(2\,u^5 - u^1)\,k(2\,u^6 - u^1)\,k(2\,u^3 - u^2)\,k(2\,u^4 - u^2)}{k(2\,u^5 - u^2)\,k(2\,u^6 - u^2)\,k(2\,u^3 - u^1)\,k(2\,u^4 - u^1)}\right|_v\,.\end{aligned}$$

Now we may assume

$$u^1 + u^3 + u^4 = u^2 + u^5 + u^6. \tag{2.7}$$

Put $V_{i,j} = {}^t u^{\,i}\tilde{\eta}(u^j)$. Then the formula in the $\log|\quad|$ is of the form $\exp(E)\,\Sigma$, where

$$\begin{aligned}E = &-(V_{1,5} - V_{2,5} + V_{1,6} - V_{2,6}) + (V_{1,3} + V_{1,4} - V_{2,3} - V_{2,3})\\ &- (V_{5,1} + V_{6,1} - V_{5,2} - V_{6,2}) + (V_{3,1} + V_{4,1} - V_{3,2} - V_{4,2}),\end{aligned}$$

and

$$\Sigma = \frac{\sigma(2\,u^5 - u^1)\,\sigma(2\,u^6 - u^1)\,\sigma(2\,u^3 - u^2)\,\sigma(2\,u^4 - u^2)}{\sigma(2\,u^5 - u^2)\,\sigma(2\,u^6 - u^2)\,\sigma(2\,u^3 - u^1)\,\sigma(2\,u^4 - u^1)}.$$

By the assumption (2.7), we have

$$E = -2\,{}^t z_b\;\tilde{\eta}(z_b).$$

On the other hand, by (1.10), we have

$$\Sigma = \Sigma_1\,\times\,\frac{\sigma^2(u^5 + u^6 - u^1)\,\sigma^2(u^3 + u^4 - u^2)}{\sigma^2(u^5 + u^6 - u^2)\,\sigma^2(u^3 + u^4 - u^1)},$$

where

$$\Sigma_1 = \frac{q(u^5 + u^6 - u^1, u^5 - u^6)\,q(u^3 + u^4 - u^2, u^3 - u^4)}{q(u^5 + u^6 - u^2, u^5 - u^6)\,q(u^3 + u^4 - u^1, u^3 - u^4)}\,.$$

Using (1.10) again, we have

$$\begin{aligned}&\sigma^2(u^5 + u^6 - u^2)\sigma^2(u^3 + u^4 - u^1)\\ &\qquad\qquad = \frac{1}{c^4}q^2(u^3 + u^4 - u^2, u^1 - u^2)\sigma^4(u^3 + u^4 - u^2)\sigma^4(z_b),\end{aligned}$$

and by the assumption (2.7), we get $\Sigma = \Sigma_1\,\Sigma_2\,(c^{-1}\sigma(z_b))^{-4}$, where

$$\Sigma_2 = \frac{1}{q^2(u^3 + u^4 - u^2, u^1 - u^2)}\,.$$

Here we prove the following lemma.

Lemma 2.19. *Let* Σ_1 *be as above, then* $\Sigma_1 = 1$ *.*

Proof. Let $i,\ j,\ k \in \{1,\ldots,6\}$ be distinct indices. We put $v^+ = u^i + u^j$, $v^- = u^i - u^j$, and $u = v^+ - u^k$. By (1.10), $q(u, v^+) = 0$ and, by (1.6), $\mathfrak{p}_{12}(v^-) = \mathfrak{p}_{12}(v^+)$ and $\mathfrak{p}_{22}(v^-) = \mathfrak{p}_{22}(v^+)$. Hence we have

$$\begin{aligned} q(u, v^-) &= \mathfrak{p}_{11}(u) - \mathfrak{p}_{11}(v^-) + \mathfrak{p}_{12}(u)\,\mathfrak{p}_{22}(v^-) - \mathfrak{p}_{12}(v^-)\,\mathfrak{p}_{22}(u) - q(u, v^+) \\ &= \mathfrak{p}_{11}(v^+) - \mathfrak{p}_{11}(v^-) = \frac{-4\,y_i\,y_j}{(x_i - x_j)^2}, \end{aligned}$$

and this does not depend on the index k, thereby completing the proof. □

Finally, by Proposition 2.17, we have

$$\langle b, b\rangle_v = \frac{1}{2}\,\log\left|\Sigma_2\,k^{-4}(z_b)\right|_v + \frac{1}{2}\,\log|G[b]|_v^2 = -2\,\log|k(z_b)|_v\,.$$

As for the case where $P_i \in \mathcal{B}$, we can prove the equation similarly, noticing that

$$\lim_{P\to P_i} \frac{x - x_i}{y - y_i} = \frac{2\,y_i}{f'(x_i)}.$$

In any case, by Corollary 2.5, we have $\langle b, b\rangle_v = 2\,\hat{\lambda}_v(\tilde{z}_b)$. This completes the proof. □

By Proposition 2.17 and the proof of Theorem 2.18, we have

Corollary 2.20. *Let* $b = P_1 - P_2 \in Z_0(C)_{K_v}$, $b' = P_5 + P_6 - P_3 - P_4 \in \mathrm{Div}_0(C)_{K_v}$ *and* u^i *be as above. Then we have*

$$\langle b, b\rangle_v = \langle b, b'\rangle_v + \log|q(u^1 - u^2, u^3 + u^4 - u^2)|_v.$$

3. Tate's series

In this section, we shall give concrete expression of Tate's series for the canonical local height.

3.1. **Generalities.** We review Tate's series [4]. In general let V be a non-singular projective variety, Ψ be a morphism $V \to V$, and Θ be a divisor $\Theta \in \mathrm{Div}(V)\otimes\mathbf{R}$, with $\Psi^*\,\Theta = \alpha\,\Theta + \mathrm{div}(\phi)$, for some real number $\alpha > 1$ and a function ϕ. Let $t_1, \ldots, t_r$, $t_i \in K(V)^* \otimes \mathbf{R}$ be functions with $\mathrm{div}(t_i) = \Theta - D_i$ satisfying $\bigcap_i \mathrm{supp}(D_i) = \emptyset$. For each $i = 1, \ldots, r$, we define functions $w_i = \phi \cdot t_i^\alpha$, $z_i = \dfrac{\phi\,\cdot\,t_i^\alpha}{t_i \circ \Psi}$, and for $i, j = 1, \ldots, r$, we define $s_{ij} = \dfrac{z_j\,w_i}{w_j}$. For any ample divisor D, we define a distance function λ_D as in [4],pp.191–192. Then we have

Theorem 3.1 ([4]). *Let $P \in V(\overline{K_v}) - \mathrm{supp}(\Theta)$ be given. Define a sequence of indices $i_0, i_1, \ldots, i_n, \ldots,$ by*

$$\lambda_{\mathrm{supp}(D_{i_n})}(\Psi^n P) = \min_{1 \le i \le r} \lambda_{\mathrm{supp}(D_i)}(\Psi^n P) .$$

Define a sequence of real numbers c_n as

$$c_n = -v(s_{i_n\, i_{n+1}}(\Psi^n P)), \quad n = 0, 1, 2, \ldots,$$

which is bounded independently of n and P. Then

$$\hat{\lambda}_\Theta(P) = v(t_{i_0}(P)) + \sum_{n=0}^{N-1} \alpha^{-n-1} c_n + O(\alpha^{-N}),$$

where the constant of $O(\alpha^{-N})$ is independent of both P and N.

3.2. **The case of Jacobian surfaces.** Now we apply the above to the case of Jacobian surfaces. That is $V = J$, Θ is the theta divisor, $\Psi = \Psi_2$, that is the multiplication by 2 map, and $\alpha = 4$.

For $P \in J$, we denote by T_P the translation map $J \to J$, $D \mapsto D + P$.

Proposition 3.2. *Let $D_1 = T^*_{\overline{B_1}}\Theta$, $D_2 = T^*_{\overline{B_2}}\Theta$, and $D_3 = T^*_{\overline{B_{13}}}\Theta$. Then D_i is irreducible and $\bigcap D_i = \emptyset$.*

Proof. The first assertion is obvious since Θ is irreducible. Any point in D_1 can be written $\overline{D(P, B_1)}$. If this point belongs to D_2, then, for some $Q \in C$, $P + B_1 \sim Q + B_2$. If $P \ne B_1 (= B_1^\iota)$ and $P \ne \infty$, by the uniqueness, $P = B_2$ and $Q = B_1$, hence the point is $\overline{D(P, B_1)} = \overline{B_{12}}$. If $P = B_1$, then $Q = B_2$ and $\overline{D(P, B_1)} = O_J$. The case $P = \infty$ does not occur. Thus $D_1 \cap D_2 = \{O_J, \overline{B_{12}}\}$, hence we have to prove that both O_J and $\overline{B_{12}}$ do not belong to D_3. If $O_J \in D_3$, that is for some $P \in C$, $B_1 + B_3 \sim P + \infty$. Since $\overline{B_{13}} \notin \Theta$, this case does not occur. If $\overline{B_{12}} \in D_3$, then $B_2 \sim B_3$, which leads to contradiction. $\square$

Proposition 3.3. *Let t_i be the elements of $K(J)^* \otimes \mathbf{R}$ corresponding to the divisors D_i of Proposition 3.2 (see § 3.1). We can take t_i as follows*:

$$\begin{aligned} t_1 &= \left(\mathfrak{p}_{12} + \beta_1 \mathfrak{p}_{22} - \beta_1^2\right)^{-1/2}, \\ t_2 &= \left(\mathfrak{p}_{12} + \beta_2 \mathfrak{p}_{22} - \beta_2^2\right)^{-1/2}, \\ t_3 &= \left(\mathfrak{p}_{11} + (\beta_1 + \beta_3) \mathfrak{p}_{12} + \beta_1 \beta_3 \mathfrak{p}_{22} + A_{13}\right)^{-1/2}, \end{aligned}$$

where $A_{13} = (\beta_1 + \beta_3)(\beta_1^2 + \beta_1 \beta_3 + \beta_3^2) + a_1 (\beta_1 + \beta_3)^2 + a_2 (\beta_1 + \beta_3) + a_3$.

Proof. First, as for t_1, t_2, we can show that the function $\mathfrak{p}_{12}(u)+\beta_i\,\mathfrak{p}_{22}(u)-\beta_i^2$ vanishes only when $u = u^{B_i} + u^P$, since if we put $u = u^{P_1} + u^{P_2}$, then this function is equal to $(x_1-\beta_i)\,(x_2-\beta_i)$. The function has poles at Θ of order 2, thus $\mathrm{div}(t_i^2) = 2\,T^*_{B_i}\,\Theta-2\,\Theta$, that is, $\mathrm{div}(t_i) = D_i - \Theta$.

To prove the formula for t_3, we use the following lemma.

Lemma 3.4. *We fix $v_0 = u^{P_1} + u^{P_2} \notin \tilde{\Theta}$, $P_1, P_2 \in C$. For $u \in \mathbf{C}^2 - \tilde{\Theta}$, define a rational function q_{P_1,P_2} on J by*

$$q_{P_1,P_2}(\tilde{u}) = \mathfrak{p}_{11}(u) + \mathfrak{p}_{22}(v_0)\,\mathfrak{p}_{12}(u) - \mathfrak{p}_{12}(v_0)\,\mathfrak{p}_{22}(u) - \mathfrak{p}_{11}(v_0)\,.$$

Then

$$\mathrm{div}(q_{P_1,P_2}) = T^*_{P_1,P_2}\,\Theta + T^*_{P_1^\iota,P_2^\iota}\,\Theta - 2\,\Theta.$$

Proof. By (1.10), the function q_{P_1,P_2} vanishes at $T^*_{P_1,P_2}\,\Theta$ and $T^*_{P_1^\iota,P_2^\iota}\,\Theta$, has poles at Θ of order 2 and has no poles at elsewhere. Thus the lemma follows. □

Proof of Proposition 3.3. By Lemma 3.4, we can take $q^{1/2}_{B_1,B_3}$ as t_3. Using the fact that $f(\beta_i) = 0$, we have $F(\beta_1, \beta_3) + A_{13}\,(\beta_1 - \beta_3)^2 = 0$, and from (1.6), the assertion follows. □

Finally, for $u \in \mathbf{C}^2$ and $\tilde{u} \in J$, as a function measuring the distance of $\tilde{u}$ and Θ, we take

$$\lambda_\Theta(\tilde{u}) = \max\left(\log|\mathfrak{p}_{ij}(u)|\,, \log|\mathfrak{p}_{ijk}(u)|\,, \log|\mathfrak{p}(u)|\right)\,.$$

If $\mathfrak{p}_I(u) = 0$ for some index I, we regard the value $\log|\mathfrak{p}_I(u)|$ as $-\infty$ and may ignore it.

4. Examples

In this section, we give some examples. Throughout this section, we denote by N_T the number of terms of the summation of Tate's series of Theorem 3.1. For the archimedean place $v = v_\infty$ of $\mathbf{Q}$, we write $\hat{\lambda}_\infty$ for $\hat{\lambda}_v$. For the symbols I_{a-b-c}, I^*_{a-b-c} etc., see [17], also [20].

Example 4.1. Let $C\colon y^2 = f(x) = x^5 - x + \dfrac{1}{4}$. The curve C has the model over $\mathbf{Z}$, $\mathcal{C}$: $y^2+y = x^5-x$. This arithmetic surface has singular fiber at $p = 139(= p_1), 449(= p_2)$, but it is regular at any point on the surface. In fact, we can prove the singular fibers $\mathcal{C}_{p_1}$ and $\mathcal{C}_{p_2}$ are both of genus 1 with one normal singularity and they are of type I_{1-0-0}.

Let $\mathcal{C}_\eta$ be the generic fiber of $\mathcal{C}$ and $\alpha : C \to \mathcal{C}_\eta$ be an isomorphism $(x, y) \mapsto (x, y - \frac{1}{2})$. Take points $P_1(1, \frac{1}{2})$ and $P_2(-1, \frac{1}{2})$ on C and put $b = P_1 - P_2$. Then for $N_T \geq 50$,

$$\hat{\lambda}_\infty(z_b) = 0.3479557596566240490280900180 47\ldots.$$

Take points $P_3(0, \frac{1}{2})$ and $P_4(-1, -\frac{1}{2})$ on C. Let P_5, P_6, with $\alpha(P_5) = (x_5, y_5)$ and $\alpha(P_6) = (x_6, y_6)$ be the points which satisfy $b \sim b'$ for $b' = P_5 + P_6 - P_3 - P_4$. Then, by the addition theorem,

$$x_5 + x_6 = 93/11^2, \quad x_5\, x_6 = -68/11^2,$$
$$y_5 + y_6 = 3 \cdot 61 \cdot 1031/11^5, \quad y_5\, y_6 = 2^2 \cdot 3^2 \cdot 17 \cdot 73/11^5.$$

We write $\tilde{P}$ for the section corresponding to the point $\alpha(P) \in \mathcal{C}_\eta$. Then $\tilde{P}_i$ and $\tilde{P}_j$ do not intersect for $i = 1, 2$ and $j = 3, 4$. Also $\tilde{P}_1$ intersects neither $\tilde{P}_5$ nor $\tilde{P}_6$. One of $\tilde{P}_5$ and $\tilde{P}_6$ intersects $\tilde{P}_2$ with multiplicity 1 on the fiber $\mathcal{C}_{73}$. Thus we have $\langle b, b' \rangle_v = \log 73$ if $p_v = 73$ and $\langle b, b' \rangle_v = 0$ for other finite places v. Since $q(u^1 - u^2, u^3 + u^4 - u^2) = -73/4$, by Corollary 2.20, we have

$$\begin{aligned} \langle \bar{b}, \bar{b} \rangle &= 2\, \hat{\lambda}_\infty(z_b) + 2\, \log 2 \\ &= 2.0822058804331387168906442790105599508865\ldots. \end{aligned}$$

Remark 4.1. In the example above, if we take $P_3 = (2, \dfrac{11}{2})$, $P_4 = (-1, -\dfrac{1}{2})$, then $\tilde{G}[b] = -31/4$. For the place v with $p_v = 31$, $\langle b, b' \rangle_v = \log 31$, and for the other places v, $\langle b, b' \rangle_v = 0$. Hence we obtain the same result for $\langle \bar{b}, \bar{b} \rangle$ and the global height is surely independent of P_3, P_4. In this way, we can check the computation of Néron's symbol.

Example 4.2. Let $N = 23$ and $X_0(N)$ be the modular curve. It has the canonical model [10],p.416:

$$y^2 = f(x) = x^6 - 14\, x^5 + 57\, x^4 - 106\, x^3 + 90\, x^2 - 16\, x - 19.$$

Let χ be the quadratic character corresponding to the quadratic field $\mathbf{Q}(\sqrt{-7})$, let $X_0(N)_\chi$ be the twisted modular curve which is given by

$$-7\, y^2 = f(x) = x^6 - 14\, x^5 + 57\, x^4 - 106\, x^3 + 90\, x^2 - 16\, x - 19, \tag{4.1}$$

and denote this by C. Let J be the Jacobian variety of C. We want to verify the Birch-Swinnerton-Dyer Conjecture for J.

Now we recall the Birch-Swinnerton-Dyer Conjecture. Let A be an abelian variety defined over $\mathbf{Q}$, let A' be the dual abelian variety of A, let V_∞ be the volume of real periods $\mathrm{Vol}(A(\mathbf{R}))$, let S be the finite set of bad primes, let V_S be $\mathrm{Vol}(\prod_{p \in S} A(\mathbf{Q}_p))$, let Ш be the Tate-Shafarevich group of A, let $A(\mathbf{Q})_{tors}$ be the torsion part of the Mordell-Weil group of A, and let r be the Mordell-Weil rank of A, which conjecturally equals the order of the Hasse-Weil zeta function $L(s, A)$ at $s = 1$. Let α_i, $1 \leq i \leq r$ be a system of generators of $A(\mathbf{Q}) \otimes \mathbf{Q}$ and $R = \det(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq r}$ be the regulator of A. Then the conjecture is as follows [15],p.51,Conjecture 2.8.2:

$$\lim_{s \to 1} (s-1)^{-r} L(s, A) = \frac{R\, V_\infty\, V_S\, \#\text{Ш}}{\#A(\mathbf{Q})_{tors} \#A'(\mathbf{Q})_{tors}}. \tag{4.2}$$

Since we do not have methods to compute the order of Ш, we want to check

$$\frac{\lim_{s \to 1} (s-1)^{-r} L(s, A) \#A(\mathbf{Q})_{tors} \#A'(\mathbf{Q})_{tors}}{R\, V_\infty\, V_S} \in \mathbf{Q}. \tag{4.3}$$

Let $S_2(N)$ be the space of cusp forms of weight 2 with respect to $\Gamma_0(N)$. The space $S_2(23)$ is 2-dimensional. Let $g \in S_2(23)$ be the one of the eigen cusp forms which has the Fourier expansion $g(q) = a_1 + a_2 q + \cdots$, with $a_1 = 1$, $a_2 = \dfrac{-1+\sqrt{5}}{2}$ (cf. [9]). It is well known that the coefficients a_n belong to $K = \mathbf{Q}(\sqrt{5})$ and g, g^σ are basis of $S_2(23)$ where σ is the generator of $\mathrm{Gal}(K/\mathbf{Q})$. Let g_χ be a cusp form given by $\sum_{n \geq 1} \chi(n) a_n\, q^n$, which belongs to $S_2(23 \cdot 7^2)$. Then the Hasse-Weil ζ-function $L(s, J)$ equals $L(s, g_\chi) L(s, g_\chi^\sigma)$. Since the signs of the functional equations are -1, both of $L(s, g_\chi)$ and $L(s, g_\chi^\sigma)$ have odd analytic rank(analytic rank means the order at $s = 1$). In fact, they are of analytic rank 1, that is the first derivatives of them do not vanish at $s = 1$. We check this by computing the special value of the derivatives of the L-functions using the following proposition.

Proposition 4.2 ([8],p.31, Prop.2.13.1)**.** *For* $g = \sum_{n=1}^{\infty} a_n q^n \in S_2(N)$,

$$L'(g, 1) = 2 \sum_{n=1}^{\infty} \frac{a_n}{n} G_1\left(\frac{2\pi n}{\sqrt{N}}\right) \quad \textit{where} \quad G_1(x) = \int_1^\infty e^{-xy} \frac{dy}{y}.$$

By this method, we have

$$\begin{aligned} L'(g_\chi, 1) &= 3.32367015912761142112490902457175944194175 \\ &\quad 48256170127399799836304033108 \cdots, \\ L'(g_\chi^\sigma, 1) &= 1.22357337805505770149941672608138385308754 \\ &\quad 69109100787909011075184313338 \cdots. \end{aligned}$$

Thus, by the conjecture, the Mordell-Weil rank of J should be 2. On the other hand, we have four rational points of C; $P_1(1, 1)$, $P_2(3, 5)$, and their images of the hyperelliptic involution: P_1^ι, P_2^ι. We define $b_1 = P_1 - P_2$, $b_2 = P_1 - P_2^\iota$, and $b_3 = P_1 - P_1^\iota$, $b_i \in Z_0(C)_{\mathbf{Q}}$. Put $\alpha_i = \overline{b_i} \in J(\mathbf{Q})$, $(i = 1, 2, 3)$, then we have $\alpha_3 = \alpha_1 + \alpha_2$. If $R' := \det\left(\langle \alpha_i, \alpha_j \rangle_{1 \le i,j \le 2}\right)$ is not 0, then α_1, α_2 are independent and R' is the regulator up to a multiple of an integer.

Next we compute the archimedean local height. We have the factorization

$$f(x) = (x^3 - 3\,x^2 + 2\,x + 1)\,(x^3 - 11\,x^2 + 22\,x - 19).$$

Take the real root x_0 of $x^3 - 3\,x^2 + 2\,x + 1$, then we have an isomorphism over $\mathbf{Q}(x_0)$

$$(x, y) \mapsto \left(\frac{f'(x_0)}{x - x_0}, \frac{f'(x_0)^2\,y}{(x - x_0)^3}\right),$$

the image of which is the curve given by $y^2 = x^5 + \cdots \in \mathbf{Q}(x_0)[x]$. Using this equation, we compute the Tate's series, and the results are as follows($N_T \ge 150$):

$$\begin{aligned}
\hat\lambda_\infty(\alpha_1) &= 8.741710830248329676715455717979070912007\\
&\qquad 70880444567048579023157642390338444909942\ldots,\\
\hat\lambda_\infty(\alpha_2) &= 8.582439336006556673512123509303683952583\\
&\qquad 74396997136609666242902462638802297360634\ldots,\\
\hat\lambda_\infty(\alpha_3) &= 8.756154336458371625892976983995127033076\\
&\qquad 13410174036934238900919308262429139876024\ldots.
\end{aligned}$$

Take points $P_3(5 - \sqrt{13}, 10 - 3\sqrt{13})$, $P_4(5 + \sqrt{13}, 10 + 3\sqrt{13})$ on C. Then we obtain points P_j, $(j = 5, \ldots, 10)$ on C such that b_i is linearly equivalent to $b_i' = P_{2i+3} + P_{2i+4} - P_3 - P_4$. Let $G_i = G_{b_i, b_i'}$ be functions satisfying $\mathrm{div}(G_i) = b_i - b_i'$. We have, by the addition theorem,

$$\begin{aligned}
P_5 &= \left(3\sqrt{-1}, 2 + 21\sqrt{-1}\right),\\
P_6 &= \left(-3\sqrt{-1}, 2 - 21\sqrt{-1}\right),\\
P_7 &= \left(\frac{1015 + 3\sqrt{32009}}{256}, \frac{-70569557 - 419001\sqrt{32009}}{4194304}\right),\\
P_8 &= \left(\frac{1015 - 3\sqrt{32009}}{256}, \frac{-70569557 + 419001\sqrt{32009}}{4194304}\right),\\
P_9 &= \left(\frac{437 + \sqrt{147206}}{107}, \frac{27721445 + 67635\sqrt{147206}}{1225043}\right),
\end{aligned}$$

$$P_{10} = \left(\frac{437 - \sqrt{147206}}{107}, \frac{27721445 - 67635\sqrt{147206}}{1225043} \right).$$

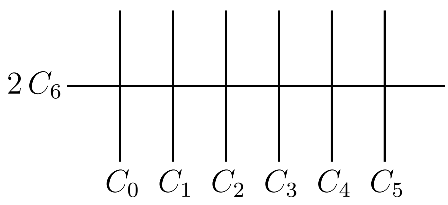
All of these points are not same as P_1 and P_2 . Thus we compute the real archimedean Néron's local pairing $r_i := \langle b_i, b_i' \rangle_\infty = 2\hat{\lambda}_\infty(\alpha_i) - \log|G_i[b_i]|$. We have

$$\begin{aligned} r_1 = -0.&37796820384747932395665162140649285119065 \\ &18770297380695556416800300082746053918 0\ldots, \\ r_2 = -0.&19953526207200136297766114857666473088636 \\ &16660026622625415043913611017785201995 15\ldots, \\ r_3 = -2.&31351754217484082343669039649918041621037 \\ &00816323168711809843820904667959580572 2\ldots. \end{aligned}$$

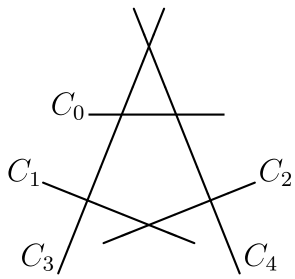
Next we consider Néron's local pairings at non-archimedean places. At $p = 2$, the model (4.1) over $\mathbf{Z}$ is not normal, thus we must blow up by $y = (x^3 + x^2 + 1) + 2Y$. Then we have

$$\langle b_1, b_1' \rangle_2 = -\log 2, \quad \langle b_2, b_2' \rangle_2 = 0, \quad \langle b_3, b_3' \rangle_2 = -\log 2.$$

Let $k = k_i$ be a biquadratic field which is generated by the coordinates of the points of the support of b_i'. Following [17], we have: At $p = 7$, the fiber of the minimal regular model is of type I^*_{0-0-0}, and if $p = 7$ is ramified in k, $C \times O_k$ has good fiber over the primes lying over 7; at $p = 23$, if p is unramified in the field k, then the fiber of the minimal regular model is of type I_{1-2-3}, and if ramified, ramification index is 2 and the type is I_{2-4-6}. In our case, both of primes 7 and 23 are unramified in k. We figure the fibres at each prime:



$p = 7$



$p = 23$

The fibres C_i are (-2)-curves, except for C_6 at $p = 7$ and C_3, C_4 at $p = 23$, all of which are (-3)-curves. At $p = 7$, we can decide which fiber the section hits by looking up the x coordinates. At $p = 23$, the sections corresponding to the P_i do not hit C_0, C_1, C_2, and we can decide which fiber they hit, by checking whether $(x + 2)(x + 5)(x + 9)$ equals $4y \mod 23$ or $-4y \mod 23$.

Remark 4.3. At $\mathfrak{P}|23$, if the reduction of x-coordinate is -2, then the section hits C_0. If the reduction is -5, putting $\xi = \dfrac{x+5}{y} \mod \mathfrak{P}$, and $\eta = \dfrac{23}{y} \mod \mathfrak{P}$, then; if $11\xi - 5\eta = 4$, hit C_1, else hit C_2, when C_3 is the fiber $(x+2)(x+5)(x+9) = 4y$. This can be proved by chasing the procedures of blowing-ups.

Thus, using Lemma 2.14, we have

$$\langle b_1, b_1' \rangle_7 = \langle b_2, b_2' \rangle_7 = \langle b_3, b_3' \rangle_7 = 0$$

$$\langle b_1, b_1' \rangle_{23} = -\frac{6}{11} \log 23, \quad \langle b_2, b_2' \rangle_{23} = 0, \quad \langle b_3, b_3' \rangle_{23} = -\frac{6}{11} \log 23.$$

It is easy to compute the intersection numbers at other primes. Summing up over all finite places, we have

$$\sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_1, b_1' \rangle_v = -\log 2 + 3 \log 3 - \frac{6}{11} \log 23,$$

$$\sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_2, b_2' \rangle_v = -3 \log 3 + \log 887,$$

$$\sum_{v \in \Sigma_{\mathbf{Q}}^0} \langle b_3, b_3' \rangle_v = -\log 2 + \log 3 + \log 179 - \frac{6}{11} \log 23.$$

Put $h_i = \langle \alpha_i, \alpha_i \rangle$ for $i = 1, 2, 3$. Summing up the above with the archimedean part, we have

$$\begin{aligned} h_1 = 0.&51445190927191370037180496873370981188956 \\ &67630743809904375965600347210307253958 5\ldots, \end{aligned}$$

$$\begin{aligned} h_1 &= 0.514451909271913700371804968733709811889 5 \\ &\quad 66763074380990437596560034721030725395 85\ldots, \\ h_2 &= 3.292472854233249053239651793468486911979 3 \\ &\quad 90261850156761513516226780837504069143 66\ldots, \\ h_3 &= 1.569063799449087814279080149817373013177 8 \\ &\quad 00861007450837869343190024866324282415 00\ldots. \end{aligned}$$

Since $\langle \alpha_1, \alpha_2 \rangle = (h_3 - h_1 - h_2)/2$, we have the regulator up to a multiple of an integer

$$\begin{aligned} R' &= 0.4418135224747459009837796585512486028911 \\ &\quad 9027232784016701161394069098323593670992\ldots. \end{aligned}$$

Next we compute the real periods using the method in Cremona [8]. We compute imaginary periods of $\Gamma_0(23)$, and multiply it by $\sqrt{-7}$ to get the real period of J up to a multiple of a rational number.

Let $(c : d)$ be the M-symbol [8],p.8. Let $\gamma_1^- = \frac{1}{2}(3 : 1) - \frac{1}{2}(-3 : 1)$, and $\gamma_2^- = \frac{1}{2}(2 : 1) - \frac{1}{2}(3 : 1) + (4 : 1) - \frac{1}{2}(-3 : 1)$, then these are basis of the anti-holomorphic homology space $H^-(N)$. Define V'_∞ be $\sqrt{7}^2 \left\langle \gamma_1^-, g \right\rangle \left\langle \gamma_1^-, g^\sigma \right\rangle$. Then this is the real period of J up to a multiple of a rational number. Computing by direct method(see [8],p.25,Prop.2.10.1),

$$V'_\infty = 10.2506719848116009699526519174413057653616\\ 563192656135793896884674121642463714 2414\ldots.$$

The number of connected components of Néron model of $J/\mathbf{Z}_7$ is 16 and that of $J/\mathbf{Z}_{23}$ is 11(cf. [17]). Hence we have $V_S = 11 \cdot 16$. Let $\tilde{J}_p$ be the reduction of J at a prime p. By the formula [5], p.80, (8.2.5), we have $\#\tilde{J}_3(\mathbf{F}_3) = 21$ and $\#\tilde{J}_{11}(\mathbf{F}_{11}) = 221$. Since C and J has good reduction at $p = 3$ and 11, and $J(\mathbf{Q})_{tors}$ is injectively mapped to $\tilde{J}_3(\mathbf{F}_3)$ and $\tilde{J}_{11}(\mathbf{F}_{11})$. Thus we have $\#J(\mathbf{Q})_{tors} = 1$. If we put

$$T := \frac{\frac{1}{2}L'(g_\chi, 1)L'(g_\chi^\sigma, 1)\#J(\mathbf{Q})_{tors}^2}{R'\,V'_\infty\,V_S},$$

then

$$392 \cdot T = 0.9999999999\cdots \text{('9' repeats at least 50 times)}.$$

Thus we can guess $T = \frac{1}{392}$ and is a rational number as desired.

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