Degeneration of Algebraic Curves and Representation of Braid Group

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ABSTRACT

In this short note we give examples of degenerations of algebraic curves which are cyclic branched coverings of degree 3 over the complex projective line and show the results of calculations of the monodromies around singular loci of them. As an application, we construct symplectic representation of the Artin braid group.

Key Words: algebraic curve, degeneration, monodromy, braid group

1 Introduction

In this short note, we study compact Riemann surfaces which are cyclic branched coverings of degree 3 over the complex projective line $\mathbf{P}^1(\mathbb{C})$. Here we discuss concrete examples of degenerations of such Riemann surfaces. The purpose of this note is to calculate monodromies around singular loci explicitly for these examples. And , as an application, we get a symplectic representation of Artin braid group B_n of n strings. (For the definition of B_n, refer, for example, [B].)

First, we recall cyclic branched coverings of degree 3 over the complex projective line $\mathbf{P}^1(\mathbb{C})$ and equivalence problem of cyclic coverings which is due to M. Namba.

Let $P_1, \cdots, P_6 \in \mathbf{P}^1(\mathbb{C})$ be six distinct points of the complex projective line and $B = \{P_1, \cdots, P_6\}$. We study here cyclic branched coverings over $\mathbf{P}^1(\mathbb{C})$ which branches at B. It is known that the fundamental group of the complement is presentated as follows (Cf. [M1], [M2], [N2] and [N3]) :

$$\pi_1\left(\mathbf{P}^1(\mathbb{C}) \setminus B\right) \cong < \gamma_1, \cdots, \gamma_6 | \gamma_6 \cdots \gamma_1 = 1 > .$$

Then we define a group homomorphism

$$\Phi : \pi_1(\mathbf{P}^1(\mathbb{C}) \setminus B) \to S_3,$$

where S_3 is the symmetric group of 3 letters, such as :

$$\Phi(\gamma_j) = (1\,2\,3)\ (j = 1, \cdots, 6).$$

Corresponding to kernel of Φ, there exists a cyclic covering of degree 3

$$\pi : X \to \mathbf{P}^1(\mathbb{C}).$$

From the Riemann-Hurwitz formula, the genus of X is 4 and the first homology group $H_1(X; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\oplus 8}$.

Let $f(x)$ be a polynomial and n be a positive integer. Let C be the Compact Riemann surface defined by

$$C : y^n = f(x).$$

The precise meaning of this is that C is a non-singular model of the closure in $\mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$ of the affine curve : $y^n = f(x)$. Let

$$\pi : (x, y) \mapsto x$$

be the projection mapping. Through the mapping π, C is considered as a cyclic covering of degree n over $\mathbf{P}^1(\mathbb{C})$ which branches at the zero points of $f(x) = 0$ or ∞.

For such cyclic coverings, M. Namba gave an answer for the equivalence problem as follows ([N1]) :

Theorem 1.1 (Namba). *Let $s \geq 3$ be an integer. Define algebraic curves C_1 and C_2 asa follows:*

$$\begin{aligned} C_1 &: y^3 = (x-\alpha_1)\cdots(x-\alpha_s) \\ C_2 &: y^3 = (x-\beta_1)\cdots(x-\beta_s), \end{aligned}$$

where α_j and β_k are complex numbers satisfying $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$, $\beta_k \neq \beta_{k'}$ $(k \neq k')$.
Then C_1 is biholomorphic to C_2 if and only if there is an automorphism $T \in Aut\left(\mathbf{P}^1(\mathbb{C})\right)$ such that

$$T(\{\alpha_1, \cdots, \alpha_s, \infty\}) = \{\beta_1, \cdots, \beta_s, \infty\}.$$

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2 Examples of degeneration over isolated points

In this section we give examples of families of compact Riemann surfaces which degenerate over some points and show the result of calculation of the monodromies. Let $\triangle$ be a domain of $\mathbb{C}$.

Example 1. Let $\mathfrak{X} = \{X_t\}_{t\in\triangle}$ be the families of cyclic coverings of degree 3 over $\mathbf{P}^1(\mathbb{C})$ defined as :

$$X_t = \{y^3 = x^5 - t\}.$$

Let $f_1 : \mathfrak{X} \to \triangle$ be projection mapping defined as $(x, y, t) \mapsto t$.

For general $t \in \triangle$, X_t are nonsingular curves of genus 4. For $t = 0$, X_0 is a singular fiber. We take here $t_0 \in \triangle \setminus \{0\}$ and fix it. For the convenience, we assume $t_0 \in \mathbb{R}_{>0}$. It is natural that the fundamental group $\pi_1(\triangle \setminus \{0\}, t_0)$ acts on $H_1(X_{t_0}; \mathbf{Z})$. Let

$$M_1 : \pi_1(\triangle \setminus \{0\}, t_0) \to Aut(H_1(X_{t_0}; \mathbf{Z}))$$

be the monodromy representation of f_1.

Let $\delta_0 \in \pi_1(\triangle \setminus \{0\}, t_0)$ be a homtopy class of closed path which starts from t_0, rounds around 0 once in the counter-clockwise direction and goes back to t_0.

For δ_0, it is possible to compute $M_1(\delta_0)$. We choose suitable symplectic basis of $H_1(X_{t_0}; \mathbf{Z})$ and use the the Reidemeister-Schreier method, then we have by direct calculations :

$$M_1(\delta_0) = \begin{pmatrix} 0 & 1 & -2 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & -1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -5 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let n be a positive integer and E_n be the indentity matrix. And let

$$J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

In this calculations, considering the intersecton numbers of the basis, we have chosen suitable basis of $H_1(X_{t_0}; \mathbf{Z})$ so that $M_1(\delta_0) \in S_p(4, \mathbf{Z})$. That is

$$J_4 = {}^tM_1(\delta_0)\, J_4\, M_1(\delta_0).$$

So , as the result, we have a group homomorphism

$$M_1 : \pi_1(\triangle \setminus \{0\}, t_0) \to S_p(4, \mathbf{Z}).$$

Remark1. From Theorem1.1, the moduli of $\mathfrak{X} = \{X_t\}_{t\in\triangle}$ does not change along δ_0 and $\#M_1(\delta_0)$, the order of the matrix, $M_1(\delta_0)$ is finite. In fact the order is 15 .

Next we consider the following:

Example 2. Let $\mathfrak{Y} = \{Y_t\}_{t\in\triangle}$ be the families of cyclic coverings of degree 3 over $\mathbf{P}^1(\mathbb{C})$ defined as :

$$Y_t = \{y^3 = x(x + 1)(x + 2)(x + 3)(x + t)\}.$$

Let $f_2 : \mathfrak{Y} \to \triangle$ be projection mapping defined as $(x, y, t) \mapsto t$.

For general $t \in \triangle$, Y_t are curves of genus 4. For $t = 0, -1, -2, -3, \infty$, $Y_0, Y_{-1}, Y_{-2}, Y_{-3}, Y_\infty$ are singular fibers. We take here $t_0 \in \triangle \setminus \{0, -1, -2, -3\}$ and fix it. For the convenience, we assume $t_0 \in \mathbb{R}_{>0}$, as above. It is natural that the fundamental group $\pi_1(\triangle \setminus \{0, -1, -2, -3\}, t_0)$ acts on $H_1(Y_{t_0}; \mathbf{Z})$. Let

$$M_2 : \pi_1(\triangle \setminus \{0, -1, -2, -3\}, t_0) \to Aut(H_1(Y_{t_0}; \mathbf{Z}))$$

be the monodromy representation of f_2. Let $\eta_0 \in \pi_1(\triangle \setminus \{0, -1, -2, -3\}, t_0)$ be a homtopy class of closed path which starts from t_0, rounds around 0 once in the counterclockwise direction and goes back to t_0. In the simimilar way above (Example 1), it is possible to compute $M_2(\eta_0)$. Direct calculations show :

$$M_2(\eta_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & -1 & -2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$M_2(\eta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 2 & -2 \\ 0 & -4 & 4 & 1 & 0 & -1 & 1 & -1 \\ 0 & 4 & -3 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 6 & 4 & 0 & -3 & -2 & -3 \\ 0 & 2 & -1 & -2 & 0 & 0 & 1 & -1 \\ 0 & -2 & 0 & -1 & 0 & 1 & 2 & -1 \end{pmatrix}$$

$$M_2(\eta_2) = \begin{pmatrix} 0 & -1 & -2 & -1 & 1 & 1 & 2 & 2 \\ -1 & 0 & -2 & -1 & 1 & 1 & 2 & 2 \\ -2 & -2 & 3 & 1 & -1 & -1 & 1 & -1 \\ 2 & 2 & -2 & 0 & 1 & 1 & -1 & 1 \\ -1 & -1 & 4 & 2 & -1 & -2 & -1 & 1 \\ -1 & -1 & 4 & 2 & -2 & -1 & -1 & 1 \\ -2 & -2 & -4 & -2 & 2 & 2 & 5 & 4 \\ -1 & -1 & -2 & -1 & 1 & 1 & 2 & -1 \end{pmatrix}$$

$$M_2(\eta_3) = \begin{pmatrix} 0 & 1 & -4 & -2 & 1 & 2 & 1 & -1 \\ -2 & 0 & -2 & -1 & -1 & 1 & 2 & -2 \\ -1 & -2 & 3 & 1 & -2 & -1 & 1 & -1 \\ 1 & 2 & -2 & 0 & 2 & 1 & -1 & 1 \\ -1 & -2 & 2 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 4 & 2 & -1 & -1 & -1 & 1 \\ -4 & -2 & -4 & -2 & 2 & 2 & 5 & 4 \\ -2 & -1 & -2 & -1 & -1 & 1 & 2 & -1 \end{pmatrix}$$

$$M_2(\eta_\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this calculations, considering the intersecton numbers of the basis, we have chosen suitable basis of $H_1(Y_{t_0};\mathbf{Z})$ so that $M_2(\eta_j) \in S_p(4,\mathbf{Z})$ for $(j = 0,1,2,3,\infty)$. That is

$$J_4 = {}^tM_2(\eta_j)\, J_4\, M_2(\eta_j).$$

So , as the result, we have a group homomorphism

$$M_2 : \pi_1(\Delta \setminus \{0,-1,-2,-3\}, t_0) \to S_p(4,\mathbf{Z}).$$

Remark2. From Theorem1.1, the moduli of $\mathfrak{Y} = \{Y_t\}_{t\in\Delta}$ change along η_j and $\#M_2(\eta_j) = \infty$.

3 Examples of degeneration along discriminant locus

In this section we consider examples of families of compact Riemann surfaces which degenerate along the discriminant locus and then give a symplectic representation of the Artin braid group.

Example 3. Let $\mathbf{a} = (a_1,a_2,a_3,a_4,a_5) \in \mathbb{C}^5$. Next we consider the families of cyclic 3-gonal curves $\mathfrak{Z} = \{Z_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{C}^5}$ defined as :

$$Z_{\mathbf{a}} = \{y^3 = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5\}.$$

Let $f_3 : \mathfrak{Z} \to \mathbb{C}^5$ be projection mapping defined as $(x,y,\mathbf{a}) \mapsto \mathbf{a} \in \mathbb{C}^5$.

Let $\Delta(f_3) \subset \mathbb{C}^5$ be the discriminant locus of f_3. It is known that ([D]):

$$\pi_1\left(\mathbb{C}^5 \setminus \Delta(f_3)\right) \cong B_5,$$

where B_5 is the Artin braid group. Through the above isomorphism, we identify $\pi_1\left(\mathbb{C}^5 \setminus \Delta(f_3)\right)$ with B_5. B_5 is presented as follows:

$$< \sigma_1,\cdots,\sigma_4 | \sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1} (k = 1,2,3) > .$$

Under the situation, it is also natural that $\pi_1\left(\mathbb{C}^5 \setminus \Delta(f_3)\right)$ acts on $H_1(Z_{\mathbf{a}_0};\mathbf{Z})$. Let

$$M_3 : \pi_1(\Delta \setminus \{0\}, t_0) \to Aut\left(H_1(Z_{\mathbf{a}_0};\mathbf{Z})\right)$$

be the monodromy representation of f_3. For each $\sigma_k \in \pi_1\left(\mathbb{C}^5 \setminus \Delta(f_3)\right)$, direct caluculatios show:

$$M_3(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_3(\sigma_2) = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & -0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_3(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_3(\sigma_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this calculations, considering the intersecton numbers of the basis, we have chosen suitable basis of $H_1(Z_{a_0}; \mathbf{Z})$ so that $M_3(\sigma_k) \in S_p(4, \mathbf{Z})$ for $(k = 1, 2, 3, 4)$. That is

$$J_4 = {}^t M_3(\sigma_k)\, J_4\, M_3(\sigma_k).$$

So , as the result, we have a symplectic representaion of the braid group B_5

$$M_3 : B_5 \to S_p(4, \mathbf{Z}).$$

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