# An Application of Theory of Strongly Branched Coverings

# AN APPLICATION OF THEORY OF STRONGLY BRANCHED COVERINGS

Takanori MATSUNO*

**ABSTRACT**

R. D. M. Accola [A] developed a theory of strongly branchecd coverings of compact Riemann surfaces and among other applications of the theory he constructed a Riemann surface admitting only the identity automorphism. In this short note, applying the theory of strongly branched coverings, we construct a Riemann surface whose automorphism group is a finite simple group.

**Key Words** : Riemann surface, automorphism group, finte simple group, branched covering

# 1 Introduction

Let $G$ be any finite simple group. The main purpose of this short note is to give a method to construct a compact Riemann surface whose automorphism group is isomorphic to $G$. This is an application of a theory of strongly branched coverings of compact Riemann surfaces due to R. D. M. Accola ([A]).

Though Greenberg's theorem is known, our method is very simple and easy ( Cf. [G], [M-N] ).

Let $\pi : C_1 \to C_2$ be a holomorphic mapping of compact Riemann surfaces of degree $d_\pi$ and total ramification $r_\pi$. We denote by $g_i = g(C_i)$ for $i = 1, 2$ the genus of $C_i$. From the Riemann-Hurwitz formula, we have

$$2g_1 - 2 = d_\pi(2g_2 - 2) + r_\pi.$$

**Definition 1.1.** The mapping $\pi$ will be called *strongly branched* if

$$r_\pi > 2d_\pi(d_\pi - 1)(g_2 + 1).$$

Notice that if $d_\pi = 2$ and $g_2 = 0$, and $r_\pi > 4$, we are in the hyperelliptic case.

For a non-constant meromorphic function $f : C_i \to \mathbf{P}^1(\mathbb{C})$, we denote by $o(f)$ the order of $f$. The function field on $C_i$ will be denoted by $M_i$. If $\pi : C_1 \to C_2$ is a holomorphic mapping of compact Riemann surfaces of degree $d_\pi$, then $M_2(\subset M_1)$ will be the subfield of index $d_\pi$ obtained by lifting functions from $C_2$ to $C_1$.

Let $f : C_1 \to \mathbf{P}^1(\mathbb{C})$ be a meromorphic function on $C_1$ of order $o(f)$. Let $\pi : C_1 \to C_2$ be a branched covering of degree $d_\pi$, not necessarily strongly branched. We denote by $B(\subset C_2)$ the branch locus of $\pi$. For a point $Q \in C_2 \setminus B$, let $\pi^{-1}(Q) = \{P_1, \cdots, P_{d_\pi}\}$ be the inverse image of $Q$.

(Received August 20, 2009)
* Dept. of Industrial Systems Engineering : Natural Science

Define $\Delta_\pi(f)$ as follows,

$$\Delta_\pi(f)(Q) = \prod_{i<j} (f(P_i) - f(P_j))^2 .$$

$\Delta_\pi(f)$ is a well-defined meromorphic function on $C_2 \setminus B$ and , from Riemann's extension theorem, $\Delta_\pi(f)$ extends to a meromorphic function on $C_2$. The order of $\Delta_\pi(f)$ is at most $2(d_\pi - 1)o(f)$ while the number of zeros of $\Delta_\pi(f)$ is at least $r_\pi$. Thus if $r_\pi > 2(d_\pi - 1)o(f)$, $\Delta_\pi(f) \equiv 0$. From the unicity theorerm, we have the following lemma.

**Lemma 1.1.** *If $r_\pi > 2(d_\pi - 1)o(f)$, then $\pi : C_1 \to C_2$ admits a factorization such that $\mu : C_1 \to C_3$ and $\nu : C_3 \to C_2$, where $\pi = \nu \circ \mu$ and there is a meromorphic function $\lambda : C_3 \to \mathbf{P}^1(\mathbb{C})$ on $C_3$ so that $f = \lambda \circ \mu$.* (See Figure1.)

$$\begin{tikzcd}
C_1 \arrow[dd, "\pi"'] \arrow[dr, "\mu"] \arrow[drr, "f"] & & \\
& C_3 \arrow[dl, "\nu"] \arrow[r, "\lambda"'] & \mathbf{P}^1(\mathbb{C}) \\
C_2 & &
\end{tikzcd}$$

**Figure 1.**

**Lemma 1.2.** *Let $C_1$ be a compact Riemann surface of genus $g_1$. Let $f_1$ and $f_2$ be meromorphic functions on $C_1$ of order $o_1$ and $o_2$, respectively. If $f_1$ and $f_2$ generate $M_1 = \mathbb{C}(f_1, f_2)$, the full function field on $C_1$, then*

$$g_1 \leqq (o_1 - 1)(o_2 - 1).$$

*Proof.* Consider $f_1 : C_1 \to \mathbf{P}^1(\mathbb{C})$ as a branched covering. From the Riemann-Hurwitz formula,

$$2g_1 - 2 = -2o_1 + r_1,$$

where, $r_1$ is a total ramification of $f_1$.

Assume $r_1 > 2(o_1 - 1)o_2$. Then $\triangle_{f_1}(f_2) \equiv 0$. So from Lemma1.1, there is a factorization $\mu : C_1 \to C_2$ and $\nu : C_2 \to \mathbf{P}^1(\mathbb{C})$ where $f_1 = \nu \circ \mu$ . The degree of $\mu$, $d_\mu$, is strictly greater than 1 and there is a meromorphic function $\lambda : C_2 \to \mathbf{P}^1(\mathbb{C})$ so that $f_2 = \lambda \circ \mu$. The index $[M_1 : M_2] = d_\mu > 1$ and $M_2 \subsetneqq M_1$.

$$\begin{tikzcd}
& C_1 \arrow[d, "\mu"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\
& C_2 \arrow[dl, "\nu"] \arrow[dr, "\lambda"'] & \\
\mathbf{P}^1(\mathbb{C}) & & \mathbf{P}^1(\mathbb{C})
\end{tikzcd}$$

**Figure 2.**

From the diagram above (Figure 2), we have :

$$\begin{aligned} M_1 &= \mathbb{C}(f_1, f_2) \\ &= \mathbb{C}(\nu \circ \mu, \lambda \circ \mu) \\ &= \mu^* \mathbb{C}(\nu, \lambda) \\ &\subset M_2 \subsetneqq M_1. \end{aligned}$$

This is a contradiction. So we have:

$$r_1 \leqq 2(o_1 - 1)o_2.$$

Then

$$\begin{aligned} 2g_1 - 2 &\leqq -2o_1 + 2(o_1 - 1)o_2 \\ g_1 &\leqq (o_1 - 1)(o_2 - 1). \quad q.e.d. \end{aligned}$$

## 2 Preliminaries

In this section we recall a theory of strongly branched coverings of compact Riemann surfaces developed by R. D. M. Accola [A] .

Applying the Riemann-Hurwitz formula to the definition of strongly branched coverings, we have the following criterions.

**Lemma 2.1.** *If $\pi : C_1 \to C_2$ is a holomorphic mapping of compact Riemann surfaces then the following conditions are equivalent to $\pi$ being strongly branched:*

(1) $g_1 > d_\pi^2 g_2 + (d_\pi - 1)^2$,

(2) $d_\pi \cdot r_\pi > (d_\pi - 1)(2g_1 - 2 + 4d_\pi)$.

**Definition 2.1.** $M_2$ will be called strongly branched subfield of $M_1$, if $\pi : C_1 \to C_2$ is a strongly branched covering.

We need the following inequality to prove the lemma after the next.

**Lemma 2.2.** *Let $n$, $m$ be positive integers. Then the following inequality holds :*

$$n^2(m-1)^2 + (n-1)^2 \leqq (nm-1)^2.$$

*Proof.*

$$\begin{aligned} & (nm-1)^2 - n^2(m-1)^2 - (n-1)^2 \\ = & -2mn + 2n^2m - n^2 - n^2 + 2n \\ = & 2n(n-1)(m-1) \geqq 0. \quad q.e.d. \end{aligned}$$

**Lemma 2.3.** *Let $\pi : C_1 \to C_2$ , $\mu : C_1 \to C_3$, $\nu : C_3 \to C_2$be coverings such that $\pi = \nu \circ \mu$. If $\pi$ is strongly branched then $\mu$ or $\nu$ is.*

*Proof.* Since $\pi$ is strongly branched, from Lemma2.1,

$$g_1 > d_\pi^2 g_2 + (d_\pi - 1)^2.$$

Assume that neither $\mu$ nor $\nu$ is strongly branched. Then from Lemma2.1,

$$\begin{cases} g_1 & \leqq d_\mu^2 g_3 + (d_\mu - 1)^2 \\ g_3 & \leqq d_\nu^2 g_2 + (d_\nu - 1)^2. \end{cases}$$

Eliminating $g_3$, we have:

$$\begin{aligned} g_1 &\leqq d_\mu^2 \left\{ d_\nu^2 g_2 + (d_\nu - 1)^2 \right\} + (d_\mu - 1)^2 \\ &= d_\mu^2 d_\nu^2 g_2 + d_\mu^2 (d_\nu - 1)^2 + (d_\mu - 1)^2. \end{aligned}$$

From Lemma2.2,

$$\begin{aligned} g_1 &\leqq d_\mu^2 d_\nu^2 g_2 + (d_\mu d_\nu - 1)^2 \\ &= d_\pi^2 g_2 + (d_\pi - 1)^2. \end{aligned}$$

This is a contradiction. *q.e.d.*

**Definition 2.2.** A strongly branched subfield $M_2$ will be called a maximal strongly branched subfield of $M_1$, if whenever $M_2 \subset M_3 \subset M_1$ and $M_2 \neq M_3$ then $M_3$ is not a strongly branched subfield. The corresponding definition will also holds for coverings.

**Lemma 2.4.** *Let $\pi : C_1 \to C_2$ , be a maximal strongly branched covering of degree $d_\pi$. Suppose $f_1 : C_1 \to \mathbf{P}^1(\mathbb{C})$ is a meromorphic function on $C_1$ such that $2(d_\pi - 1)o(f_1) < r_\pi$. Then there is a meromorphic function $f_2$ on $C_2$ so that $f_1 = f_2 \circ \pi$ (i.e.$f_1 \in M_2$).*

*Proof.* Because of the condition $2(d_\pi - 1)o(f_1) < r_\pi$, from Lemma1.1, there exists a factorization $\mu : C_1 \to C_3$, $\nu : C_2 \to C_3$ where $\pi = \nu \circ \mu$,
and the degree of $\mu$, $d_\mu$, is stirictly greater than 1, and there is a meromorphic function $f_3$ on $C_3$ so that $f_1 = f_3 \circ \mu$. Choose $\nu : C_2 \to C_3$ as $d_\nu$ is minimum. Here we assume $d_\nu > 1$. Since $\pi$ is maximal, $\mu$ is

not strongly branched. From Lemma, $\nu$ is strongly branched. So we have :

$$r_\mu \leqq 2d_\mu(d_\mu - 1)(g_3 + 1)$$

and

$$r_\nu > 2d_\nu(d_\nu - 1)(g_2 + 1).$$

From the Riemann-Hurwitz formula

$$g_3 - 2 = d_\nu(2g_2 - 2) + r_\nu$$

and direct calculation shows that

$$r_\mu < \frac{d_\mu \cdot d_\nu(d_\mu - 1)}{d_\nu - 1} \cdot r_\nu.$$

It is easy to see that $r_\pi = r_\mu + d_\mu \cdot r_\nu$ from the Riemann-Hurwitz formula. Then we have:

$$\begin{aligned} r_\pi &< \frac{d_\mu \cdot d_\nu(d_\mu - 1)}{d_\nu - 1} \cdot r_\nu + d_\mu \cdot r_\nu \\ &= \frac{d_\mu \cdot d_\nu(d_\mu - 1) + (d_\nu - 1) \cdot d_\mu}{d_\nu - 1} \cdot r_\nu \\ &= \frac{d_\mu(d_\pi - 1)}{d_\nu - 1} \cdot r_\nu. \end{aligned}$$

So it follows that:

$$\begin{aligned} 2(d_\nu - 1) \cdot o(f_3) &= \frac{2(d_\nu - 1)}{d_\mu} \cdot o(f_1) \\ &< \frac{2(d_\nu - 1)}{d_\mu} \cdot \frac{r_\pi}{2(d_\pi - 1)} \\ &= \frac{d_\nu - 1}{d_\mu(d_\pi - 1)} \cdot r_\pi \\ &< r_\nu. \end{aligned}$$

Then there must exist a factorization of $\nu$ and this contradicts the minimality of $d_\nu$. So $d_\nu = 1$ and $\nu : C_3 \to C_2$ is a biholomorphic mappinng. Put $f_2 = f_3 \circ \nu^{-1} : C_2 \to \mathbf{P}^1(\mathbb{C})$. Then

$$\begin{aligned} f_2 \circ \pi &= (f_3 \circ \nu^{-1}) \circ (\nu \circ \mu) \\ &= f_3 \circ (\nu^{-1} \circ \nu) \circ \mu \\ &= f_3 \circ \mu \\ &= f_1. \end{aligned}$$

So $f_2$ is a required function. *q.e.d.*

**Lemma 2.5.** *If a maximal strongly branched subfield of $M_1$ exists, then it is unique.*

*Proof.* Suppose that there exist two maximal storngly branched coverings $\mu : C_1 \to C_2$ and $\nu : C_1 \to C_3$. We denote by $d_\mu$ and by $d_\nu$ the degree of $\mu$ and of $\nu$, respectively. We may assume that

$$d_\mu(g_2 + 1) \geqq d_\nu(g_3 + 1).$$

Let $f_1$ and $f_2$ be meromorphic functtions on $C_3$ of order $o(f_1)$ and $o(f_2)$, respectively and generate the full function field of $C_3$.

From Lemma1.2, both $o(f_1)$ and $o(f_2)$ are no greater than $g_3 + 1$. Then the order of $f_1 \circ \nu$ is no greater than $d_\nu(g_3 + 1)$. So, by the assumption, the order of $f_1 \circ \nu$ is no greater than $d_\mu(g_2 + 1)$. From Lemma2.4, $f_1 \circ \nu \in M_2$. By the same argument, $f_2 \circ \nu \in M_2$ also holds. Because $f_1 \circ \nu$ and $f_2 \circ \nu$ generate $M_3$, it follows that $M_3 \subset M_2 \subset M_1$. Since $M_3$ is maximal, we have $M_3 = M_2$. *q.e.d.*

We denote by $A(C)$ the full group of holomorphic automorphisms. Let $f$ be a meromorphic function on $C$ and take any automorphism $\sigma \in A(C)$. Then $f \circ \sigma$ is a meromorphic funciton on $C$ again. So, in this manner, $A(C)$ acts on the function field on $C$. Let $\pi : C_1 \to C_2$ be a maximal storngly branched covering. Since a maximal stongly branched subfield is unique, $A(C_1)$ acts on $M_1$ leaving $M_2$ invariant. Let $N$ be the subgroup of $A(C_1)$ which leaves the functions of $M_2$ poitwise fixed. Let $f_2 \in M_2$ and $\tau \in N$. For any $\sigma \in A(C_1)$,

$$\begin{aligned} & f_2 \circ (\sigma^{-1} \circ \tau \circ \sigma) \\ = \; & \{(f_2 \circ \sigma^{-1}) \circ \tau\} \circ \sigma \\ = \; & (f_2 \circ \sigma^{-1}) \circ \sigma \\ = \; & f_2 \circ (\sigma^{-1} \circ \sigma) \\ = \; & f_2. \end{aligned}$$

Then $\sigma^{-1} \circ \tau \circ \sigma \in N$. So $N$ is a normal subgroup of $A(C_1)$. $C_2$ is biholomorphic to the quotient space $C_1/N$ and $N$ is the covering transformation group of $\pi : C_1 \to C_2$. Naturally there is an exact sequence of group homomorphisms :

$$\{1\} \to N \to A(C_1) \overset{\alpha}{\to} A(C_1/N).$$

So the quotient group $A(C_1)/N$ is isomorphic to a finite subgroup of $A(C_1/N)$.

Then we have the following commutative diagram (Figure 3):

$$\begin{tikzcd} C_1 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & C_1 \arrow[d, "\pi"] \\ C_2 \arrow[r, "\alpha(\sigma)"] & C_2 \end{tikzcd}$$

**Figure 3.**

**Definition 2.3.** ([N1]) For a branched covering $\pi : C_1 \to C_2$, if the covering transformation group acts transitively on every fiber of $\pi$, then $\pi$ is said to be Galois.

Then we have the following theorem :

**Theorem 2.6.** *If $\pi : C_1 \to C_2$ is a Galois and strongly branched covering whose covering transformation group $G$ ($\subset A(C_1)$) is a simple group , then $\pi$ is a maximal strongly branched covering.*

*Proof.* If $\pi : C_1 \to C_2$ is not maximal, then there exists a maximal branched covering $\mu : C_1 \to C_3$ and a morphism $\nu : C_3 \to C_2$ such that $\pi = \nu \circ \mu$. Since $\mu$ is maximal, there is a normal subgroup $N$ ($\subset A(C_1)$) such that $C_3$ is biholomorphic to the quotient space $C_1/N$ and $N \subset G$. It is trivial that $N$ is a normal subgroup not only of $A(C_1)$ but also of $G$. This contradicts the assumption that $G$ is simple. *q.e.d.*

# 3 Main theorem

In [A], as an application of the theory of strongly branched coverings, compact Riemann surfaces which admit only the identity automorphism are constructed. In this section modifying the idea of the proof of the above result, we shall prove the following main theorem :

**Theorem 3.1.** *Let $G$ be any finite simple group. Then there is a compact Riemann surface whose automorphism group is isomorphic to $G$.*

*Proof.* Let $d$ be the order of $G$. We denote $G$ as $\{x_1, x_2, \cdots, x_{d-1}, x_d = e\}$, where $e$ is a unit element of $G$. Let $s$ be an integer such as $s > 4d - 3$. Choose a set of $s$ points $B = \{P_1, \ldots, P_s\}$ on the complex projective line $\mathbf{P}^1(\mathbb{C})$ such that no projective transformation except the identity leaves $B$ setwise fixed. Since $s > 5$, this choice is possible. It is known that the fundamental group of the complement of $B$ is presentated as follows ( Cf. [M1], [N1]) :

$$\pi_1\left(\mathbf{P}^1(\mathbb{C}) \setminus B\right) \cong <\gamma_1, \cdots, \gamma_s | \gamma_s \cdot \gamma_{s-1} \cdots \gamma_1 = 1 > .$$

Then we define a group homomorphism $\Phi$ such as :

$$\Phi(\gamma_j) = \begin{cases} x_j & (1 \leq j \leq d-1) \\ x_1 & (d \leq j \leq s-1) \\ x_1^{-1} \cdot \cdots \cdot x_{d-1}^{-1} \cdot x_1^{-(s-d)} & (j = s). \end{cases}$$

$\Phi$ is well-defined and is surjective . There is an exact sequence of group homomorphisms as follows :

$$\{1\} \to Ker\,(\Phi) \to \pi_1\left(\mathbf{P}^1(\mathbb{C}) \setminus B\right) \xrightarrow{\Phi} G \to \{1\}.$$

Corresponding to the kernel of $\Phi$, $Ker(\Phi)$, there exists a finite Galois covering $\pi : C \to \mathbf{P}^1(\mathbb{C})$ which branches at $B$. The covering transformation group of $\pi$ is isomorphic to $G$ (Cf. [M2], [N1], [N2]). Since $s > 4d - 3$,

$$r_\pi \geqq \frac{d}{2}(s-1) > 2d(d-1).$$

So $\pi : C \to \mathbf{P}^1(\mathbb{C})$ is a strongly branched covering. From Theorem2.6, since $G$ is simple, $\pi$ is a maximal strongly branched covering. So $G$ is a normal subgroup of $A(C)$. Since $\pi$ is Galois, the quotient space $C/G$ is biholomorphic to the base space $\mathbf{P}^1(\mathbb{C})$. Then we have an injection:

$$A(C)/G \hookrightarrow A\left(\mathbf{P}^1(\mathbb{C})\right).$$

Because of our choice of set of branching points $B$, the quotient group $A(C)/G$ must be isomorphic to a unit group $\{1\}$. Then the full group of holomorphic automorphism of $C$, $A(C)$, must be equal to $G$. The proof is completed.

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