The Kepler Motion and the Harmonic Oscillator on Spaces of Constant Curvature

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ABSTRACT

Two dynamical systems such as the Kepler motion and the harmonic oscillator on constant curvature spaces are defined from the viewpoints of dynamical symmetry. These systems in Euclidean space are well known as celebrated examples for both classical and quantum mechanics. Our systems under consideration are regarded as the generalization of the usual systems. One property common to our systems is that any "bounded" orbit is closed. The period of these closed orbits is calculated analytically and the dependency on the constant curvature K of our configuration spaces is clarified.

Key Words: Kepler motion, harmonic oscillator, constant curvature space, Bertrand's theorem, period

1. Introduction.

In usual central potential dynamical systems, Kepler motion and the harmonic oscillator are well known as typical examples which admit dynamical symmetry. One property of these systems is that any bounded orbit is closed. Of central potential dynamical systems, Bertrand showed that any bounded orbit is closed for only these systems, the Kepler motion and the harmonic oscillator. This theorem is known as Bertrand's theorem. The other property of these systems is that proper quadratic first integrals are admitted. Namely, Runge- Lenz vector is conserved for the Kepler motion and conserved symmetric tensor for the harmonic oscillator.

In previous papers [1,2], central potential dynamical systems on constant curvature spaces have been defined and investigated from the viewpoints of dynamical symmetry. The Kepler motion and the harmonic oscillator are generalized to the ones on constant curvature spaces. These systems are derived from the existence of the proper quadratic first integrals [1]. In [2], the periodicity of all bounded orbits leads the Kepler motion and the harmonic oscillator on constant curvature spaces which have already been found in [1]. Thus, these researches have shown the reasonable results.

In this short note, our attention is paid to periods of these systems. And we will show how the periods depend on the constant curvature K of our configuration space.

2. Preliminaries.

In this section, we give some preliminaries for later use. At first, let us assume that the configuration space is an N-dimensional constant curvature space ($N \geq 2$). The fundamental form is given by

$$ds^2 = h(r)\sum_{i=1}^{N}(dx^i)^2 , \tag{1}$$

where

$$h(r) = \left(1 + \left(Kr^2/4\right)\right)^{-2}, r = \sqrt{\sum_{i=1}^{N}(x^i)^2} . \tag{2}$$

The constant parameter K stands for the curvature. This metric is known as a conformally flat metric and the coordinates system is similar to the Cartesian in a flat space.

Here we consider the motion of the unit particle in our configuration space. Therefore the kinetic energy T takes the following form

$$T = (1/2)h(r)\sum_{i=1}^{N}(dx^i/dt)^2 . \tag{3}$$

And the potential function U is assumed to be central. This means that U is a function which depends only on the variable r.

$$U = U(r). \tag{4}$$

Thus we have dynamical systems which are considered as central potential dynamical systems on constant curvature spaces.

In [2], our N-degree-of freedom systems are shown to be reduced to two-degree-of freedom systems by making use of angular momentums, which are known as linear first integrals.

Therefore we may consider the dynamical systems defined by

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$$T = (1/2)h(r)\left((dx^1/dt)^2 + (dx^2/dt)^2\right). \quad (5)$$
$$U = U(r), \quad r = \sqrt{(x^1)^2 + (x^2)^2}$$

Here we adopt a well-known polar coordinates system (r, φ) given by

$$x^1 = r\cos(\varphi), \qquad x^2 = r\sin(\varphi). \quad (6)$$

So our dynamical system takes the following form

$$T = (1/2)h(r)\left((dr/dt)^2 + r^2(d\varphi/dt)^2\right) \quad (7)$$
$$U = U(r), \quad r = \sqrt{(x^1)^2 + (x^2)^2}.$$

In [1], potential function $U(r)$ have been investigated in order that the systems (3) and (4) admit proper quadratic first integrals. Here 'proper' quadratic first integrals mean the constants of motion which are quadratic in generalized velocity dx^i/dt and are not expressed with linear first integrals except the total energy.

As a result, with constant parameter a

$$U_K(r) = ar^{-1}(1 - Kr^2/4) \quad , \quad (8)$$

and

$$U_H(r) = ar^2(1 - Kr^2/4)^{-2} \quad , \quad (9)$$

have been found to be generalized potential functions in the Kepler motion and the harmonic oscillator on constant curvature spaces, respectively. Of course in addition to the form of potential functions, quadratic first integrals for (8) and (9) are shown explicitly.

In the next paper concerning central potential systems on constant curvature spaces [2], the orbits have been studied. Of the dynamical system (7), a necessary and sufficient condition for any 'bounded' orbit to be closed is that the potential function $U(r)$ takes the form $U_K(r)$ or $U_H(r)$ given by (8) and (9), respectively.

3. Main Results.

In this section, we calculate the periods of the closed orbits for $U_K(r)$ and $U_H(r)$.

Energy and angular momentum conservation law lead

$$2E = h(r)((dr/dt)^2 + r^2(d\varphi/dt)^2) + 2U(r), \quad (10)$$

$$L = h(r)r^2(d\varphi/dt) \quad , \quad (11)$$

where E and L are the total energy and the areal velocity respectively and they are constants of motion.

Let us introduce the variable u by

$$u = r^{-1}(1 - Kr^2/4). \quad (12)$$

Then (10) and (11) are transformed into the forms

$$(du/dt)^2 = 2(E - V(u))(u^2 + K)^2 - L^2(u^2 + K)^3, \quad (13)$$

$$d\varphi/dt = L(u^2 + K) \quad , \quad (14)$$

where $V(u)$ is a central potential function of u through the transformation (12). The generalized Kepler motion potential (8) and the harmonic oscillator potential (9) become

$$V_K(u) = U_K(r) = ar^{-1}(1 - Kr^2/4) = au \quad , \quad (15)$$

and

$$V_H(u) = U_H(r) = ar^2(1 - Kr^2/4)^{-2} = au^{-2} \quad , (16)$$

respectively.

Since the case of $K = 0$ is well known, we are restricted to the case of $K \neq 0$

CASE 1. The harmonic oscillator in the case of $K > 0$.

We will make a brief review of orbit calculation shown in [2]. By taking into account of (13) and (14), we can show the periodicity of the orbit. We denote by $u_1, u_2 \quad (u_1 \leq u_2)$ the values of u such that $du/dt = 0$ then we have

$$2(E - au_i^{-2}) - L^2(u_i^2 + K) = 0 \qquad (i = 1,2) \quad . \quad (17)$$

When u varies from u_1 to u_2 the increase of φ becomes

$$\Delta\varphi = \int_{u_1}^{u_2}(d\varphi/du)du = \int_{u_1}^{u_2}(d\varphi/dt)(du/dt)du$$
$$= \int_{u_1}^{u_2}\left(2(E - au^{-2})L^{-2} - (u^2 + K)\right)^{-1/2}du. \quad (18)$$

Making use of (17), the integral (18) can be easily calculated to be

$$|\Delta\varphi| = \pi/2. \quad (19)$$

Thus we have the periodicity of the bounded orbit for the harmonic oscillator on constant curvature spaces. It is to be noted that some assumptions are needed for the orbit to be bounded. As our case is $K>0$, we assume $a>0$ and $E \geq KL^2/2+\sqrt{2a}L$. These facts have been shown in the paper [2].

We focus our attention to the period of the closed orbit and the dependency of the period on the constant parameter K is considered.

With a slightly modification of the orbit calculation, we can get the period of the closed orbit. From (19) the period T_H is given by

$$T_H = 4\int_{u_1}^{u_2} 1/(du/dt)du$$

$$= 4\int_{u_1}^{u_2} u du / \left((u^2+K)\left((2E-KL^2)u^2 - L^2u^4 - 2a\right)^{1/2} \right). \quad (20)$$

We put $w=u^2$ and also $w_i = u_i^2, (i=1,2)$. Then the period becomes

$$T_H = 2\int_{w_1}^{w_2} dw / \left((w+K)\left((2E-KL^2)w - L^2w^2 - 2a\right)^{1/2} \right). \quad (21)$$

From (17) we have

$$(2E-KL^2)w_i - L^2 w_i^2 - 2a = 0 \quad (i=1,2) \ . \quad (22)$$

Then we get

$$w_1 + w_2 = (2E-KL^2)/L^2 \ , \quad (23)$$

$$w_1 w_2 = 2a/L^2 \ , \quad (24)$$

and

$$\left((2E-KL^2)w - L^2w^2 - 2a\right) = L^2(w_2-w)(w-w_1). \quad (25)$$

Therefore (21) is transformed into

$$T_H = (2/L)\int_{w_1}^{w_2} dw / \left((w+K)\left((w_2-w)(w-w_1)\right)^{1/2} \right). \quad (26)$$

Further we take the variable θ defined below

$$w = (w_1+w_2)/2 + (w_2-w_1)\sin(\theta)/2 \ . \quad (27)$$

Then the period T_H becomes

$$T_H = (2/L)\int_{-\pi/2}^{\pi/2} d\theta / \left(K + (w_2+w_1)/2 + (w_2-w_1)\sin(\theta)/2 \right). \quad (28)$$

Here we put

$$A = (w_2-w_1)/2, \quad B = K + (w_2+w_1)/2, \quad (29)$$

then we obtain, after a calculation,

$$T_H = (2/L)\int_{-\pi/2}^{\pi/2} d\theta / \left(A\sin(\theta) + B \right) = (2\pi/L)/\sqrt{B^2-A^2} \ . \quad (30)$$

Finally, from (23),(24),(29) and (30) we have gotten

$$T_H = (\sqrt{2}\pi)/\sqrt{EK+a} \quad (31)$$

The other cases can be discussed in the same way as in Case 1. The results are as follows.

CASE 2. The harmonic oscillator in the case of $K<0$.

If $a>0$ and $a/(-K) \geq E \geq KL^2/2+\sqrt{2a}L$ are valid, we get

$$T_H = (\sqrt{2}\pi)/\sqrt{EK+a} \ . \quad (32)$$

It should be noted that the period (31) holds in all cases including the case for $K=0$.

CASE 3. The Kepler motion in the case of $K>0$.

If $a<0$ and $E \geq (KL^4-a^2)/(2L^2)$ hold good, we get the following period

$$T_K = \left(\pi|a|\right) / \left(\sqrt{(E^2+a^2K)\left(-E+\sqrt{E^2+a^2K}\right)} \right). \quad (33)$$

CASE 4. The Kepler motion in the case of $K<0$.

If $a<0$ and $a\sqrt{-K} \geq E \geq (KL^4-a^2)/(2L^2)$ hold good, after a long calculation, we get

$$T_K = \left(\pi|a|\right) / \left(\sqrt{(E^2+a^2K)\left(-E+\sqrt{E^2+a^2K}\right)} \right), \quad (34)$$

It is also pointed out that the expression (33) for $K>0$ is same as the one for $K \leq 0$.

4. Concluding Remarks and Further Discussions.

We have shown the periods for the Kepler motion and the harmonic oscillator with the total energy E and constant parameter K , which stands for the curvature of the configuration space under consideration. Of course, our results are easily seen to exactly coincide with the ones for $K = 0$. This means that our results are said to be a generalization of the usual Kepler motion and harmonic oscillator.

As for the results, total energy E depends on K through kinetic energy and potential function. If we assume that E is fixed as a constant E_0, it should be concluded that periods for our generalized dynamical systems are monotone decreasing with respect to K.

As another generalization of the Kepler motion and the harmonic oscillator, multifold Kepler systems have been found, which can be characterized for any bounded orbit to be closed [3,4]. The dependency of the periods on the parameters in the multifold Kepler systems will be studied in the future.

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