# Extension problem and duality of conditional entropy associated with a commutative hypergroup

# Extension problem and duality of conditional entropy associated with a commutative hypergroup

Satoe YAMANAKA

Osaka Prefecture University

2012

## Contents

## 1. Introduction

The notion of a hypergroup is one of generalizations of the concept of the measure algebra on a locally compact group. The axiomatic setting of a hypergroup was set up by C. Dunkl [D], R. Jewett [J] and R. Spector [S] around 1975. A hypergroup is suitable for describing a random walk on symmetric graphs. Some models of a hypergroup are association schemes, a hypergroup coming from double cosets of a group by a compact subgroup, the (conjugacy) class hypergroup coming from conjugacy classes of a compact group, and the character hypergroup coming from irreducible representations of a compact group.

One of the important problems for a hypergroup is to determine the structures of hypergroups. N. Wildberger analyzed finite hypergroups in 1995 ([W1]) and determined the structures of hypergroups of order three in 2002 ([W2]). However the structures of hypergroups of low order, for examples four and five, has not been determined.

It is important to solve an extension problem in order to determine the structures of hypergroups. Here we introduce an extension problem in the category of hypergroups. Let $H$ and $L$ be locally compact hypergroups. A locally compact hypergroup $K$ is called an extension hypergroup of $L$ by $H$ if the sequence:

$$1 \longrightarrow H \xrightarrow{\iota} K \xrightarrow{\varphi} L \longrightarrow 1$$

is exact. An extension problem is to determine all structures of extension hypergroups $K$ of $L$ by $H$ when $L$ and $H$ are given.

In the present thesis, the author reports to solve some extension problems and to discover the structures of hypergroups of low order.

We investigate certain extension problems.

First, in the category of finite commutative hypergroups, we considered the extension problem of the case that $H$ is a finite abelian group and $L$ is the Golden hypergroup, and we have succeeded in solving it. Moreover, we characterize splitting extension hypergroups. When N. Wildberger determined all structures of hypergroups of order three, he pointed out that the Golden hypergroup was in an interesting position among strong hypergroups of order three. This is a motivation that we consider an extension problem of the Golden hypergroup.

Secondly, in the category of locally compact commutative hypergroups, we considered the extension problem of the case that $H$ is a locally compact abelian group and $L$ is a hypergroup of order two, and we solved it. As a result, when we set a locally compact abelian group $H$ with the one dimensional torus $\mathbb{T}$, it turns out that the extension hypergroups agree with the hypergroups on two tori $\mathbb{T} \cup \mathbb{T}$ determined by M. Voit [V].

Thirdly, in the category of locally compact commutative hypergroups, we solved the extension problem of the case that $H$ is a locally compact abelian group and $L$ is the Golden hypergroup. As a result, when we set a locally compact abelian group $H$ with the one dimensional torus $\mathbb{T}$, we determine a part of the structures of hypergroups on three tori $\mathbb{T} \cup \mathbb{T} \cup \mathbb{T}$. This result is a generalization of the result by M. Voit [V].

As a next step, we considered the duality of extension problems. For a finite commutative signed hypergroup $K$, we denote the set of all characters of $K$ by $\hat{K}$. Then, $\hat{K}$ becomes a signed commutative hypergroup with the product as functions on $K$. For a finite commutative hypergroup $K$, $\hat{K}$ is not necessarily to be a hypergroup. In the category of finite commutative signed hypergroups, the duality of a hypergroup holds, i.e. $\hat{\hat{K}} \cong K$. The duality of an extension means that the sequence:

$$1 \longrightarrow \hat{L} \xrightarrow{\hat{\iota}} \hat{K} \xrightarrow{\hat{\varphi}} \hat{H} \longrightarrow 1$$

is exact for the exact sequence:

$$1 \longrightarrow H \xrightarrow{\iota} K \xrightarrow{\varphi} L \longrightarrow 1.$$

This duality always holds in the category of finite commutative signed hypergroups. Therefore we need to consider extension problems in the category of a signed hypergroups.

Through our research, we noticed that a signed action of a hypergroup played an essential role to determine extension hypergroups. Hence we introduced a signed action of a signed hypergroup on a finite set referring to the definition of actions of a hypergroup by Sunder and Wildberger [SW].

We determined all irreducible signed action of a hypergroup of order two. Applying these actions, one knows that the structures of extensions of a hypergroup of order two by a hypergroup of order two can be obtained easily ([KSTY]). This is our developed method for solving extension problems for hypergroups.

Moreover we introduce the notion of entropy of an irreducible signed action of a signed hypergroup. We show that this entropy is the complete invariant for two dimensional irreducible signed actions of a signed hypergroup of order two.

Let $K$ be a commutative signed hypergroup and $H$ a signed subhypergroup of $K$. We give the conditional entropy $\mathcal{H}_\phi(K|H)$ associated with a canonical state $\phi$ of the measure algebra $M^b(K)$ of $K$. Moreover for the quotient hypergroup $L$ of $K$ by $H$, we introduce the conditional entropy $\mathcal{H}(K|L)$

associated with the normalized Haar measure of $K$. For these entropy, we show the dual relation:

$$\mathcal{H}_{\phi}(K|H) = \mathcal{H}(\hat{K}|\hat{H}), \quad \mathcal{H}(K|L) = \mathcal{H}_{\hat{\phi}}(\hat{K}|\hat{L})$$

where $\hat{\phi}$ be the canonical state of the measure algebra $M^b(\hat{K})$.

Applying these entropy to extension problems, we have determined the equivalence classes of extension hypergroups of a hypergroup of order two by a hypergroup of order two. This is a new approach for considering the extension problems for hypergroups.

Moreover for a generalized orbital hypergroup $K^E$ of a finie commutative hypergroup $K$, we also introduce two kinds of conditional entropy $\mathcal{H}(K|K^E)$ and $\mathcal{H}_{\phi}(K|K^E)$, and show the dual relation:

$$\mathcal{H}_{\phi}(K|K^E) = \mathcal{H}(\hat{K}|\widehat{K^E}), \quad \mathcal{H}(K|K^E) = \mathcal{H}_{\hat{\phi}}(\hat{K}|\widehat{K^E}).$$

The present thesis is organized as follows.

In Chapter 2, we describe fundamental notions for hypergroups.

In Chapter 3, we study three extension problems: the extension of the Golden hypergroup by finite abelian groups, the extension of hypergroups of order two by locally compact abelian groups and the extension of the Golden hypergroup by locally compact abelian groups.

In Chapter 4, we introduce a notion of irreducible signed actions of a signed hypergroup and apply it to certain extension problems.

In Chapter 5, we introduce two kinds of conditional entropy. One is the conditional entropy associated with the normalized Haar measure of a finite commutative signed hypergroup $K$ and the other is the conditional entropy associated with the canonical state of the measure algebra $M^b(K)$ of $K$. Moreover, the dual relation of these entropy is discussed.

## 2. Preliminaries

2.1. **Definitions of hypergroups.** We recall some notions and facts on locally compact hypergroups from Bloom-Heyer's book [BH]. Let $K$ be a locally compact Hausdorff space, i.e. each point has a compact neighborhood and any two points can be separated by the compact neighborhoods.

Let $C_c(K)$ be the set of all continuous functions with compact supports on $K$.

Let $\mu$ be a Radon measure, i.e., $\mu$ is a continuous linear mapping from $C_c(K)$ to $\mathbb{C}$. Let $M(K)$ be the set of all Radon measures on $K$. Then $M(K)$ become a linear space. We denote the norm $||\cdot||$ on $M(K)$ by

$$||\mu|| = \sup\{|\mu(f)| : f \in C_c(K), ||f||_\infty \leq 1\} \in [0, \infty]$$

where $||f||_\infty = \max\{|f(c)| : c \in K\}$ is the uniform norm. Let $M^b(K), M^b_+(K)$ and $M^1(K)$ be the set of all bounded Radon measures, all bounded positive Radon measures and all probability measures on $K$ respectively i.e.

$$M^b(K) = \{\mu \in M(K) : ||\mu|| < \infty\},$$
$$M^b_+(K) = \{\mu \in M^b(K) : \mu(f) \geq 0 \text{ for } f \geq 0\}$$
$$M^1(K) = \{\mu \in M^b_+(K) : \mu(K) = 1\}$$

where $f \geq 0$ ($f \in C_c(K)$).

For $\mu \in M^b(K)$, the support of $\mu$ is define by

$$\text{supp}(\mu) = \cap\{F \subset K : F \text{ is closed, } |\mu|(F^c) = 0\}.$$

We can make $M^b(K)$ a topological vector space with weak topology obtained from $\sigma(M(X), C_c(X))$.

For $c \in K$, we write the Dirac measure at $c$ by $\varepsilon_c \in M^b_+(K)$ i.e.

$$\varepsilon_c(f) = f(c) \text{ for } f \in C_c(K).$$

**Proposition 2.1.** *Let $\Psi(c) = \varepsilon_c$ for $c \in K$. The mapping $\Psi$ is a homeomorphism from $K$ to $\{\varepsilon_c : c \in K\}$.*

*Proof.* Put $\Psi(c) = \varepsilon_c$. When $c_j \to c$, we have

$$\varepsilon_{c_j}(f) = f(c_j) \to f(c) = \varepsilon_c(f),$$

because $f \in C_c(K)$ is continuous. Hence we get $\varepsilon_{c_j} \to \varepsilon_c$. □

Let $\mathcal{C}(K)$ is the family of all non-empty compact subsets of $K$. For open subsets $U$ and $V$ of $K$, we denote

$$\mathcal{C}_U(V) = \{C \in \mathcal{C}(K) : C \cap U \neq \emptyset, C \subset V\}.$$

Then, the set $\{\mathcal{C}_U(V) : U, V \subset K,\ U \text{ and } V \text{ are open.}\}$ gives a topology in $\mathcal{C}(K)$. This topology is called *Michael topology.*

**Definition** (locally compact hypergroups)**.** Let $K$ be a non-empty locally compact Hausdorff space. The quaternary $K = (K, M^b(K), *, ^-)$ will be called a *hypergroup* if the following conditions are satisfied.

(1) The vector space $M^b(K)$ is a Banach algebra by the binary product $*$ respect to the norm $||\cdot||$. The product $*$ called the *convolution.*

(2) For $x, y \in K$, $\varepsilon_x * \varepsilon_y \in M^1(K)$ and $\mathrm{supp}(\varepsilon_x * \varepsilon_y)$ is compact.

(3) The mapping $K \times K \ni (x, y) \mapsto \varepsilon_x * \varepsilon_y \in M^1(K)$ is continuous by weak topology on $M^b(K)$.

(4) $K \times K \ni (x, y) \mapsto \mathrm{supp}(\varepsilon_x * \varepsilon_y) \in \mathcal{C}(K)$ is continuous by Michael topology.

(5) For any $x \in K$, there exists the element $e \in K$ such that $\varepsilon_x * \varepsilon_e = \varepsilon_e * \varepsilon_x = \varepsilon_x$.

(6) There exists a homeomorphism $K \ni x \to x^- \in K$ such that $(x^-)^- = x$ and $(\varepsilon_x * \varepsilon_y)^- = \varepsilon_{y^-} * \varepsilon_{x^-}$ for all $x, y \in K$, called the *involution* where $\mu^-$ is the image of $\mu$ under the involution. Moreover, $e \in \mathrm{supp}(\varepsilon_x * \varepsilon_y)$ if and only if $x = y^-$.

We note that the involution is weakly continuous.

When $\varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x$ for any $x, y \in K$, we call $K$ *commutative.* When $x^- = x$ for any $x \in K$, we call $K$ *hermitian.*

If a hypergroup $K$ is hermitian, then $K$ is commutative because

$$\varepsilon_x * \varepsilon_y = (\varepsilon_x * \varepsilon_y)^- = \varepsilon_{y^-} * \varepsilon_{x^-} = \varepsilon_y * \varepsilon_x.$$

Using the convolution $*$ for point measures of $K$, we define the convolution $*$ on $M^b(K)$ i.e.

$$\mu * \nu = \int_K \int_K \varepsilon_x * \varepsilon_y d\mu(x) d\nu(y).$$

Let $K_1$ and $K_2$ be hypergroups. We call a mapping $\varphi$ *(hypergroup) homomorphism* from $K_1$ to $K_2$ if $\varphi$ is a mapping from $K_1$ to $K_2$ and the mapping $\tilde{\varphi}$ from $M^b(K_1)$ to $M^b(K_2)$ defined $\tilde{\varphi}(\varepsilon_x) := \varepsilon_{\varphi(x)}$ for $x \in K_1$ satisfies

$$\varphi(\mu * \nu) = \varphi(\mu) * \varphi(\nu), \ \ \varphi(\mu^-) = \varphi(\mu)^-$$

for any $\mu$ and $\nu \in M^b(K_1)$.

Moreover, if a homomorphism $\varphi$ from $K_1$ to $K_2$ is bijection, then $\varphi$ is called *isomorphism.*

**Lemma 2.2.** *The homomorphism $\varphi$ maps a point measure of a hypergroup $K_1$ to some point measure of a hypergroup $K_2$. Especially, the unit $e_{K_1}$ is mapped to the unit $e_{K_2}$.*

*Proof.* By the simple calculation, we have

$$\begin{aligned}\varphi(\mu*\nu)(f) &= \int_{K_2} f(t')d\varphi(\mu*\nu)(t') = \int_{K_1} f(\varphi(t))d(\mu*\nu)(t)\\ &= \int_{K_1}\int_{K_1}\int_{K_1} f(\varphi(t))d(\varepsilon_x*\varepsilon_y)(t)d\mu(x)d\nu(y)\\ &= \int_{K_1}\int_{K_1}\int_{K_2} f(t')d\varphi(\varepsilon_x*\varepsilon_y)(t')d\mu(x)d\nu(y)\end{aligned}$$

and

$$\begin{aligned}(\varphi(\mu)*\varphi(\nu))(f) &= \int_{K_2}\int_{K_2}\int_{K_2} f(t')d(\varepsilon_{x'}*\varepsilon_{y'})(t')d\varphi(\mu)(x')d\varphi(\nu)(y')\\ &= \int_{K_1}\int_{K_1}\int_{K_2} f(t')d(\varepsilon_{\varphi(x)}*\varepsilon_{\varphi(y)})(t')d\mu(x)d\nu(y).\end{aligned}$$

Hence we have $\varphi(\varepsilon_x*\varepsilon_y)=\varepsilon_{\varphi(x)}*\varepsilon_{\varphi(y)}$.

For the involution, we can calculate that

$$\begin{aligned}\varphi(\mu^-)(f) &= \int_{K_2} f(t')d\varphi(\mu^-)(t') = \int_{K_1} f(\varphi(t))d\mu^-(t)\\ &= \int_{K_1} f(\varphi(t)^-)d\mu(t),\end{aligned}$$

and

$$\begin{aligned}\varphi(\mu)^-(f) &= \int_{K_2} f(t')d\varphi(\mu)^-(t') = \int_{K_2} f(t'^-)d\varphi(\mu)(t')\\ &= \int_{K_1} f(\varphi(t^-))d\mu(t).\end{aligned}$$

Hence $\varepsilon_{\varphi(t)^-}=\varepsilon_{\varphi(t^-)}$ i.e. $\varphi(\varepsilon_t)^-=\varepsilon_{\varphi(t^-)}$ because we know $f(\varphi(t)^-)=\varepsilon_{\varphi(t)^-}(f)$ and $f(\varphi(t)^-)=\varepsilon_{\varphi(t^-)}(f)$.

Moreover, for unit $e_{K_1}$ of $K_1$, $\varphi(\varepsilon_{e_{K_1}})=\varphi(\varepsilon_{e_{K_1}}^-*\varepsilon_{e_{K_1}})=\varphi(\varepsilon_{e_{K_1}})^-*\varphi(\varepsilon_{e_{K_1}})$. Since there exists $k\in K_2$ such that $\varphi(\varepsilon_{e_{K_1}})=\varepsilon_k$, we have

$$\mathrm{supp}(\varepsilon_k^-*\varepsilon_k)\ni e_{K_2}$$

by the axiom of a hypergroup. Therefore, we have $\varphi(e_{K_1})=e_{K_2}$ because the element $k$ such that $\varepsilon_k=\varepsilon_k^-*\varepsilon_k$ is the unit $e_{K_2}$. □

**Example 2.3.** Let $G$ be a locally compact group with unit $e$ and $H$ be a compact group. A continuous affine action of $H$ on $G$ is a continuous mapping $(x,s)\to x^s$ from $G\times H$ to $G$ satisfying that $x^e=x, (x^s)^t=x^{st}$ and there exists $c\in G$ and $\varphi\in\mathrm{Aut}(G)$ such that $x^s=c\varphi(x)$. We denote the normalized Haar measure of $H$ by $\omega_H$.

Then, we have the hypergroup $G^H$ with the quotient topology whose convolution structure is given by

$$\varepsilon_{x^H} * \varepsilon_{y^H} = \int_H \int_H \varepsilon_{(x^s y^t)^H} d\omega_H(s) d\omega_H(t).$$

Next we introduce the finite case conforming to the axiom of locally compact hypergroups referring to Wildberger [W1].

Let $K$ be a finite set. The sets $M^b(K), M^1_{\mathbb{R}}(K), M^1(K)$ of all measures, all probability measures and all non-negative probability measures on $K$ are described as follows respectively.

$$M^b(K) = \left\{ \sum_{c \in K} a_c \varepsilon_c : a_c \in \mathbb{C} \right\},$$

$$M^1_{\mathbb{R}}(K) = \left\{ \sum_{c \in K} a_c \varepsilon_c : a_c \in \mathbb{R}, \sum_{c \in K} a_c = 1 \right\},$$

$$M^1(K) = \left\{ \sum_{c \in K} a_c \varepsilon_c : a_c \geq 0, \sum_{c \in K} a_c = 1 \right\}$$

where $\varepsilon_c$ is the Dirac measure on $c \in K$. The support of the element $\mu = \sum_{c \in K} a_c \varepsilon_c$ is

$$\mathrm{supp}(\mu) = \{c \in K : a_c \neq 0\}.$$

**Definition** (generalized (finite) hypergroup)**.** Let $K = \{c_0, c_1, \cdots, c_n\}$ be a finite set. The quaternary $(K, M^b(K), *, ^-)$ is called a *generalized (finite) hypergroup* if $K$ satisfies the following conditions.

(1) The triple $(M^b(K), *, ^-)$ is a $*$-algebra with unit $\varepsilon_{c_0}$.

(2) $K^- = K$.

(3) The structure constant $n_{ij}^k \in \mathbb{C}$ is defined as follows.

$$\varepsilon_{c_i} * \varepsilon_{c_j} = \sum_{k=0}^{n} n_{ij}^k \varepsilon_{c_k}.$$

The constant $n_{ij}^k$ satisfies the following conditions.

$$c_i^- = c_j \text{ if and only if } n_{ij}^0 > 0 \text{ and}$$
$$c_i^- \neq c_j \text{ if and only if } n_{ij}^0 = 0.$$

We denote $(K, M^b(K), *, ^-)$ by $K$ simply and we say that the order of $K$ is $n+1$. For any $i, j$, if $\varepsilon_{c_i} * \varepsilon_{c_j}$ belongs to $M^1_{\mathbb{R}}(K)$ then $K$ is called a *signed hypergroup* and if $\varepsilon_{c_i} * \varepsilon_{c_j}$ belongs to $M^1(K)$ then $K$ is called a *hypergroup*.

In this paper, $c_{i^-}$ means $c_i^-$. The weight $w(c_i)$ of $c_i \in K$ is defined by

$$w(c_i) := (n_{i^- i}^0)^{-1}.$$

The total weight $w(K)$ of $K$ is

$$w(K) = \sum_{i=0}^{n} w(c_i).$$

We note that a generalized hypergroup $K$ become a group if and only if $w(c_i) = 1$ for all $i$.

**Example 2.4.** Consider the symmetric random walk on the edge of a regular triangle. Fix a vertex $x_0$ as the origin. A vertex $x$ is said to have the distance $i$ from the origin $x_0$ if there exists a minimal $i$-step path of edges which connects $x_0$ and $x$. Let $\varepsilon_{c_i}$ be the random walk which comes from a movement from a vertex to another vertex having the distance $i$. We denote the walk $c_i$ after the walk $c_j$ by $\varepsilon_{c_i} * \varepsilon_{c_j}$. Then, using the probability, we can write that

$$\varepsilon_{c_1} * \varepsilon_{c_1} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_1}.$$

Hence we have the hypergroup $K = \{c_0, c_1\}$ of order two with above structure. If we consider the symmetric random walk on the edge of a regular pentagon, then we have the *Golden hypergroup* $\mathbb{G} = \{c_0, c_1, c_2\}$ which has the following structures:

$$\varepsilon_{c_1} * \varepsilon_{c_1} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_2},$$

$$\varepsilon_{c_2} * \varepsilon_{c_2} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_1},$$

$$\varepsilon_{c_1} * \varepsilon_{c_2} = \frac{1}{2}\varepsilon_{c_1} + \frac{1}{2}\varepsilon_{c_2}.$$

**Example 2.5.** By the definition of finite signed hypergroup, we have all hypergroups $\mathbb{Z}_q(2) = \{c_0, c_1\}$ of order two with a parameter $q$ $(q > 0)$ and the following structure.

$$\varepsilon_{c_1} * \varepsilon_{c_1} = q\varepsilon_{c_0} + (1-q)\varepsilon_{c_1}.$$

We note that if the parameter $q$ satisfies $0 < q \leq 1$ then the signed hypergroup $\mathbb{Z}_q(2)$ becomes a hypergroup, and if the parameter $q$ equals to 1 then we have $\mathbb{Z}_q(2) = \mathbb{Z}_2$.

2.2. **Harmonic analysis of a finite commutative signed hypergroup.** We generalized the some results of Sunder-Wildberger's work [SW] and Wildberger's work [W1] in the category of finite signed hypergroups.

Hereafter, let $K$ be a finite signed hypergroup.

**Lemma 2.6.** *We define the constant* $n_{ij}^k \in \mathbb{R}$ *by* $\varepsilon_{c_i} * \varepsilon_{c_j} = \sum_{c_k \in K} n_{ij}^k \varepsilon_{c_k}$ *for* $c_i, c_j \in K$*. Then, we have*

(1) $n_{ij}^k = n_{j^- i^-}^{k^-}$,

(2) $\dfrac{n_{ij}^k}{w(c_k)} = \dfrac{n_{i^- k}^j}{w(c_j)}$,

(3) $\dfrac{n_{ij}^k}{w(c_k^-)} = \dfrac{n_{kj^-}^i}{w(c_i^-)}$.

*Proof.* (1) For $c_i, c_j \in K$, we calculate

$$(\varepsilon_{c_j^-} * \varepsilon_{c_i^-})^- = \left(\sum_{c_k \in K} n_{j^- i^-}^k \varepsilon_{c_k}\right)^- = \sum_{c_k \in K} n_{j^- i^-}^k \varepsilon_{c_k}^- = \sum_{c_k \in K} n_{j^- i^-}^{k^-} \varepsilon_{c_k}.$$

Since $\varepsilon_{c_i} * \varepsilon_{c_j} = (\varepsilon_{c_j^-} * \varepsilon_{c_i^-})^-$, we have $n_{ij}^k = n_{j^- i^-}^{k^-}$.

(2) By simple calculation,

$$\begin{aligned}(\varepsilon_{c_k}^- * \varepsilon_{c_i}) * \varepsilon_{c_j} &= \left(\sum_l n_{k^- i}^l \varepsilon_{c_l}\right) * \varepsilon_{c_j} = n_{k^- i}^{j^-} \varepsilon_{c_j^-} * \varepsilon_{c_j} + \sum_{l \neq j^-} n_{k^- i}^l \varepsilon_{c_l} * \varepsilon_{c_j} \\ &= n_{k^- i}^{j^-} n_{j^- j}^0 \varepsilon_{c_0} + \cdots .\end{aligned}$$

In the similar way, we have $\varepsilon_{c_k}^- * (\varepsilon_{c_i} * \varepsilon_{c_j}) = n_{ij}^k n_{k^- k}^0 \varepsilon_{c_0} + \cdots$ . Comparing the coefficient of the unit $\varepsilon_{c_0}$, we get $n_{ij}^k n_{k^- k}^0 = n_{k^- i}^{j^-} n_{j^- j}^0$.

(3) In the similar calculation of (2), we have

$$(\varepsilon_{c_i} * \varepsilon_{c_j}) * \varepsilon_{c_k}^- = n_{ij}^k n_{kk^-}^0 \varepsilon_{c_0} + \cdots$$

and

$$\varepsilon_{c_i} * (\varepsilon_{c_j} * \varepsilon_{c_k}^-) = n_{jk^-}^{i^-} n_{ii^-}^0 \varepsilon_{c_0} + \cdots .$$

Since $n_{jk}^{i^-} = n_{kj^-}^i$ by (1), we have $n_{ij}^k n_{kk^-}^0 = n_{kj^-}^i n_{ii^-}^0$. □

We call $e_K \in M^1(K)$ *the normalized left Haar measure* if $\mu * e_K = e_K$ for any $\mu \in M_{\mathbb{R}}^1(K)$.

**Lemma 2.7.** *The normalized left Haar measure* $e_K$ *of* $K$ *is uniquely given by*

$$e_K = \sum_{c \in K} \frac{w(c)}{w(K)} \varepsilon_c.$$

*Proof.* Suppose that the measure $e_K \in M^1(K)$ is a normalized Haar measure. Put $e_K = \sum_{c_i \in K} a_i \varepsilon_{c_i}$. For any $c_j^- \in K$,

$$\varepsilon_{c_j^-} * e_K = \sum_{c_i \in K} a_i \sum_{c_k \in K} n_{j-i}^k \varepsilon_{c_k}.$$

Here, the coefficient of the unit $\varepsilon_{c_0}$ of above measure is $a_j n_{j-j}^0$. On the other hand, $\varepsilon_{c_j^-} * e_K = e_K$ by the supposition. Comparing the coefficients of the unit, we get $a_j = a_0 (n_{j-j}^0)^{-1} = a_0 w(c_j)$. Hence we have

$$e_K = \sum_{c_j \in K} a_0 w(c_j) \varepsilon_{c_j}.$$

Since $e_K$ is a probability measure, $\sum_{c_j \in K} a_0 w(c_j) = a_0 \sum_{c_j \in K} w(c_j) = a_0 w(K) = 1$. Therefore $a_0 = \frac{1}{w(K)}$.

Conversely, we suppose that $e_K = \sum_{c_j \in K} \frac{w(c_j)}{w(K)} \varepsilon_{c_j}$. For any $c_i \in K$, we have

$$\varepsilon_{c_i} * e_K = \sum_{c_j \in K} \frac{w(c_j)}{w(K)} \sum_{c_l \in K} n_{ij}^l \varepsilon_{c_l}.$$

For any $c_k^- \in K$,

$$\varepsilon_{c_k^-} * (\varepsilon_{c_i} * e_K) = \sum_{c_j \in K} \frac{w(c_j)}{w(K)} \sum_{c_l \in K} n_{ij}^l \varepsilon_{c_k^-} * \varepsilon_{c_l} = \sum_{c_j \in K} \frac{w(c_j)}{w(K)} \sum_{c_l \in K} n_{ij}^l \sum_{c_p \in K} n_{k-l}^p \varepsilon_{c_p}.$$

Here, the coefficient of the unit $\varepsilon_{c_0}$ of above measure is

$$\sum_{c_j \in K} \frac{w(c_j)}{w(K)} n_{ij}^k n_{k-k}^0 = \sum_{c_j \in K} \frac{w(c_j)}{w(K)} \frac{n_{ij}^k}{w(c_k)} = \sum_{c_j \in K} \frac{w(c_j)}{w(K)} \frac{n_{i-k}^j}{w(c_j)} = \frac{1}{w(K)}$$

by Lemma 2.6 (2). On the other hand, when we put $\varepsilon_{c_i} * e_K = \sum_{c_j \in K} b_j \varepsilon_{c_j} \in M_{\mathbb{R}}^1(K)$,

$$\varepsilon_{c_k}^- * (\varepsilon_{c_i} * e_K) = \sum_{c_j \in K} b_j \sum_{c_l \in K} n_{k-j}^l \varepsilon_{c_l}.$$

The coefficient of the unit $\varepsilon_{c_0}$ of above measure is $b_k n_{k-k}^0 = \frac{b_k}{w(k)}$. Comparing the coefficients of the unit, we have $b_k = \frac{w(c_k)}{w(K)}$ i.e. $\varepsilon_{c_i} * e_K = \sum_{c_k \in K} \frac{w(c_k)}{w(K)} \varepsilon_{c_l} = e_K$. □

**Proposition 2.8.** *For* $c_i \in K$,

$$w(c_i^-) = w(c_i).$$

*Proof.* Since the left normalized Haar measure $e_K$ satisfies the condition $\mu * e_K = e_K$ for any $\mu \in M^1(K)$, it is obvious that $e_K$ is a projection.

Using Lemma 2.7, for $c_j \in K$, we have

$$e_K * \varepsilon_{c_j} = \sum_{i,k} \frac{w(c_i)}{w(K)} n_{ij}^k \varepsilon_{c_k} = \sum_{i,k} \frac{w(c_i)}{w(K)} \frac{n_{ij}^k}{w(c_k)} w(c_k) \varepsilon_{c_k}.$$

Since we can calculate that

$$\frac{n_{ij}^k}{w(c_k)} = \frac{n_{i^-k}^j}{w(c_j)} = \frac{n_{k^-i}^{j^-}}{w(c_j^-)}\frac{w(c_j^-)}{w(c_j)} = \frac{n_{kj^-}^i}{w(c_i)}\frac{w(c_j^-)}{w(c_j)}$$

by Lemma 2.6, we have

$$\begin{aligned} e_K * \varepsilon_{c_j} &= \sum_{i,k} \frac{w(c_i)}{w(K)} \left( \frac{n_{kj^-}^i}{w(c_i)} \frac{w(c_j^-)}{w(c_j)} \right) w(c_k) \varepsilon_{c_k} \\ &= \frac{w(c_j^-)}{w(c_j)} \sum_k \frac{w(c_k)}{w(K)} \left( \sum_i n_{kj^-}^i \right) \varepsilon_{c_k} = \frac{w(c_j^-)}{w(c_j)} e_K. \end{aligned}$$

Here we have known that $e_K = e_K * e_K = e_K * (\varepsilon_{c_j} * e_K) = (e_K * \varepsilon_{c_j}) * e_K = \frac{w(c_j)}{w(c_j^-)} e_K * e_K$. Since $e_K \neq 0$, we have $\frac{w(c_j^-)}{w(c_j)} = 1$, namely, $w(c_j^-) = w(c_j)$.

□

**Corollary 2.9.** *The normalized left Haar measure $e_K$ is an orthogonal projection of $M^b(K)$ and the normalized right Haar measure.*

*Proof.* For any $c_j \in K$, it is obvious that $e_K * \varepsilon_{c_j} = e_K$ by the proof of Proposition 2.8 and the normalized right Haar measure is unique. Since $e_K^- = (e_K * \varepsilon_{c_j})^- = \varepsilon_{c_j}^- * e_K^-$, we have $e_K^- = e_K$ because of the uniqueness of the normalized Haar measure. □

Let $K$ be a finite signed hypergroup. We define a linear mapping $\phi$ from $M^b(K)$ to $\mathbb{C}$ by

$$\phi(\mu) = a_0$$

for any $\mu = \sum_{c_k \in K} a_k \varepsilon_{c_k}$. Obviously,

$$\phi(\varepsilon_{c_i}^- * \varepsilon_{c_i}) = \phi\left( \frac{1}{w(c_i)} \varepsilon_{c_0} + \cdots \right) = \frac{1}{w(c_i)} > 0 \text{ and } \phi(c_0) = 1.$$

When $\phi(\mu^- * \mu) = 0$, we have $\mu = 0$ because

$$\phi(\mu^- * \mu) = \phi\left( \sum_{c_k, c_l \in K} \overline{a_k} a_l \varepsilon_{c_k}^- * \varepsilon_{c_l} \right) = \sum_{c_k \in K} |a_k|^2 \frac{1}{w(c_k)}.$$

Hence $\phi$ is a faithful positive state of $M^b(K)$. We call $\phi$ the *canonical* state. We define the inner product $(\cdot|\cdot)$ of $M^b(K)$ by

$$(\mu|\nu) = \phi(\nu^- * \mu).$$

**Proposition 2.10.** (1) $(\varepsilon_{c_i}|\varepsilon_{c_j}) = \frac{1}{w(c_i)}\delta_{i,j}$ *where $\delta_{i,j}$ is Kronecker's delta.*

(2) $(\varepsilon_{c_k} * \varepsilon_{c_i}|\varepsilon_{c_j}) = (\varepsilon_{c_i}|\varepsilon_{c_k}^- * \varepsilon_{c_j})$.

*Proof.* (1) It is easy to see that $\phi(\varepsilon_{c_j}^- * \varepsilon_{c_i}) = 0$ for $i \neq j$ by the axiom of a hypergroup.

(2) By the definition, we have

$$(\varepsilon_{c_k} * \varepsilon_{c_i}|\varepsilon_{c_j}) = \phi(\varepsilon_{c_j}^- * (\varepsilon_{c_k} * \varepsilon_{c_i})) = \phi((\varepsilon_{c_k}^- * \varepsilon_{c_j})^- * \varepsilon_{c_i}) = (\varepsilon_{c_i}|\varepsilon_{c_k}^- * \varepsilon_{c_j}).$$

□

**Corollary 2.11.** *For* $\mu = \sum_{c_k \in K} a_k \varepsilon_{c_k} \in M^b(K)$, *we have*

$$a_k = (\mu|c_k)w(c_k).$$

**Proposition 2.12.** $M^b(K)$ *is a* $C^*$*-algebra.*

*Proof.* For $\mu \in M^b(K)$, we denote $||\mu||_2 := (\mu|\mu)^{\frac{1}{2}}$. Then $M^b(K)$ becomes a finite dimensional Hilbert space. We denote a Hilbert space $M^b(K)$ by $H$.

Let $\mathcal{L}(H)$ be a set of all linear mapping from $M^b(K)$ to $M^b(K)$. For $\mu \in M^b(K)$ and $x \in H$, we put $\pi(\mu)x = \mu * x$. We know that $\pi$ is a $*$-isomorphism from $M^b(K)$ into $\mathcal{L}(H)$. Since $\pi(M^b(K))$ is a $*$-subalgebra of $C^*$-algebra $\mathcal{L}(H)$, $\pi(M^b(K))$ is a $C^*$-algebra with the norm $||\cdot||$ by $||\pi(\mu)|| = \sup_{x \in H, ||x||_2 \leq 1} ||\pi(\mu)x||_2$. Hence $M^b(K)$ becomes a $C^*$-algebra with the same norm of $\pi(M^b(K))$. □

We call a complex valued function $\chi$ on a finite commutative signed hypergroup $K$ a *character* of $K$ if $\chi$ satisfies

$$\chi(c_0) = 1 \text{ and } \chi(c_i)\chi(c_j) = \sum_{c_k \in K} n_{ij}^k \chi(c_k)$$

where $\varepsilon_{c_i} * \varepsilon_{c_j} = \sum_{c_k \in K} n_{ij}^k c_k$. There exists the character $\chi$ such that $\chi(c_i) = 1$ for all $c_i \in K$; we write it by $\chi_0$. Let $\hat{K}$ be the set of all character of $K$

We can expand $\chi$ on $K$ into $M^b(K)$ by

$$\chi(a_i \varepsilon_{c_i} + a_j \varepsilon_{c_j}) := a_i \chi(c_i) + a_j \chi(c_j)$$

for $a_i, a_j \in \mathbb{C}$ and $c_i, c_j \in K$.

**Proposition 2.13.** *Let* $e_K$ *be the normalized Haar measure of* $K$*. For any* $j$,

$$\chi_j(e_K) = \delta_{0,j}.$$

*Proof.* For any $c_i \in K$,

$$\chi_j(\varepsilon_{c_j})\chi_j(e_K) = \chi_j(\varepsilon_{c_j} * e_K) = \chi_j(e_K).$$

Hence we get $\chi_j(\varepsilon_{c_i}) = 1$ or $\chi_j(e_K) = 0$. When $\chi_j(c) = 1$ for all $c \in K$ namely $\chi_j = \chi_0$, we have

$$\chi_0(e_K) = \frac{1}{w(K)} \sum_{c_k \in K} w(c_k)\chi_0(\varepsilon_{c_k}) = \frac{1}{w(K)} \sum_{c_k \in K} w(c_k) = 1.$$

Since for $\chi_j$ $(j \neq 0)$, there exists $c_i$ such that $\chi_j(c_i) \neq 1$, we get $\chi_j(e_K) = 0$. □

**Proposition 2.14.** *When* $K = \{c_0, c_1, \cdots, c_n\}$, *we have* $\hat{K} = \{\chi_0, \chi_1, \cdots, \chi_n\}$.

*Proof.* For a character $\tilde{\chi}$ on $M^b(K)$, the restriction $\chi$ of $\tilde{\chi}$ to $K$ is a character of $K$. Conversely, for a character $\chi$ of $K$, the value of character $\tilde{\chi}$ of $\mu = \sum_{c \in K} a_c \varepsilon_c \in M^b(K)$ is given by $\tilde{\chi}(\mu) = \sum_{c \in K} a_c \varepsilon_{\chi(c)}$. Hence we see a one-to-one correspondence between $\hat{K}$ and $\widehat{M^b(K)}$. For $\tilde{\chi_i} \in \widehat{M^b(K)}$, we can take the minimal projection $e_j$ on $M^b(K)$ such that $\tilde{\chi_i}(e_j) = \delta_{i,j}$ and $\sum_{j=0}^{n} e_j = 1$. Since the numbers of minimal projections on $M^b(K)$ is $n+1$, we have $\widehat{M^b(K)} = \{\tilde{\chi_0}, \tilde{\chi_1}, \cdots, \tilde{\chi_n}\}$. Therefore we know that the order of $\hat{K}$ is $n+1$. □

Hereafter, Let $\{e_j\}_j$ be the minimal projections of $M^b(K)$ such that

$$\chi_i(e_j) = \delta_{i,j}, \ e_j * e_j = e_j, \ e_j^- = e_j.$$

**Proposition 2.15.**

$$\varepsilon_{c_i} * e_j = \chi_j(c_i) e_j.$$

*Proof.* By the fact that $M^b(K) \cong \sum_j \mathbb{C} e_j$, we can write $\varepsilon_{c_i} = \sum_k a_k e_k$. Then we have

$$\varepsilon_{c_i} * e_j = \sum_k a_k e_k * e_j = a_j e_j$$

from the property of projections.

On the other hands, $\chi_j(c_i) = \chi_j\left(\sum_k a_k e_k\right) = \sum_k a_k \chi_j(e_k) = a_j$, so we get

$$\chi_j(c_i) e_j = a_j e_j = \varepsilon_{c_i} * e_j.$$

□

**Proposition 2.16.**

$$\chi_i(c_j^-) = \overline{\chi_i(c_j)}.$$

*Proof.* For $M^b(K) \ni \mu = \sum_k a_k e_k$, it is easy to see that $\mu^- = \left(\sum_k a_k e_k\right)^- = \sum_k \overline{a_k} e_k^- = \sum_k \overline{a_k} e_k$ and $\chi_i(\mu) = \chi_i\left(\sum_k a_k e_k\right) = \sum_k a_k \chi_i(e_k) = a_i$. Hence we have

$$\chi_i(\mu^-) = \chi_i\left(\sum_k \overline{a_k} e_k\right) = \overline{a_i} = \overline{\chi_i(\mu)}.$$

This conclusion holds if we restrict $\chi_i$ on $K$.

□

Let $A(\hat{K})$ be the $*$-algebra generated by $\hat{K}$ with following product and involution:

$$(\chi_i\chi_j)(c) = \chi_i(c)\chi_j(c) \text{ and } \chi_i^-(c) = \overline{\chi_i(c)}$$

for $\chi_i, \chi_j \in A(\hat{K})$ and $c \in K$. Then any complex valued function on $K$ belongs to $A(\hat{K})$.

For $\chi_i, \chi_j \in \hat{K}$, we put

$$(\chi_i|\chi_j) := \frac{1}{w(K)} \sum_{c_k \in K} \chi_i(c_k)\overline{\chi_j(c_k)}w(c_k).$$

Then we define the inner product of $A(\hat{K})$ as follows:
For $a = \sum_{\chi_i \in \hat{K}} \alpha_i\chi_i, b = \sum_{\chi_j \in \hat{K}} \beta_j\chi_j \in A(\hat{K})$,

$$(a|b) := \sum_{\chi_i,\chi_j \in \hat{K}} \alpha_i\overline{\beta_j}(\chi_i|\chi_j).$$

**Proposition 2.17.** *$\hat{K}$ is a finite commutative signed hypergroup with unit $\chi_0$.*

*Proof.* By the definition, we know that

$$(\chi_i|\chi_i) = \frac{1}{w(K)} \sum_{c_k \in K} |\chi_i(c_k)|^2 w(c_k) > 0.$$

For $\chi_j \in \hat{K}$ $(i \neq j)$, since $\chi_i\chi_j^-$ belongs to $A(\hat{K})$, we can write $\chi_i\chi_j^- = \sum_{\chi_k \in \hat{K}} \alpha_k\chi_k$. For the normalized Haar measure $e_K$ of $K$, we have

$$\chi_i\chi_j^-(e_K) = \sum_{\chi_k \in \hat{K}} \alpha_k\chi_k(e_K) = \alpha_0$$

by Proposition 2.13. On the other hands, we have

$$\chi_i\chi_j^-(e_K) = \chi_i(e_K)\chi_j^-(e_K) = 0$$

because $i \neq j$. Hence we get $\alpha_0 = 0$, namely, $\mathrm{supp}(\chi_i\chi_j^-) \not\ni \chi_0$. We also get $(\chi_i|\chi_j) = 0$ because

$$\begin{aligned}\chi_i\chi_j^-(e_K) &= \frac{1}{w(K)} \sum_{c_k \in K} w(c_k)\chi_i\chi_j^-(c_k) = \frac{1}{w(K)} \sum_{c_k \in K} w(c_k)\chi(c_k)\overline{\chi_j(c_k)} \\ &= (\chi_i|\chi_j).\end{aligned}$$

Therefore $\{\chi_i\}_i$ are orthogonal basis of $A(\hat{K})$, so we can write

$$\chi_i\chi_j = \sum_{\chi_k \in \hat{K}} m_{ij}^k \chi_k$$

where $m_{ij}^k \in \mathbb{C}$.

We note that $\overline{\chi_i\chi_j(c_l)} = \sum_k \overline{m_{ij}^k}\overline{\chi_k(c_l)}$ and $\chi_i\chi_j(c_l^-) = \sum_k m_{ij}^k\chi_k(c_l^-)$. Hence we have $\overline{m_{ij}^k} = m_{ij}^k$ i.e. $m_{ij}^k \in \mathbb{R}$ because of Proposition 2.16.

Since $\chi_i(c_0)\chi_j(c_0) = \sum_k m_{ij}^k \chi_k(c_0)$ and $\chi(c_0) = 1$ for all $\chi \in \hat{K}$, we get $\sum_k m_{ij}^k = 1$.

□

We identify $\chi \in A(\hat{K})$ with $\varepsilon_\chi \in M^b(\hat{K})$.

**Corollary 2.18.**

$$(\chi_i|\chi_i) = \frac{1}{w(\chi_i)}.$$

**Proposition 2.19.**

$$\hat{\hat{K}} \cong K.$$

*Proof.* Since we already know that $M^b(K)$ is a commutative $C^*$-algebra by Proposition 2.12, we can see that the set $M^b(\hat{\hat{K}})$ generated by all character of $\hat{K}$ is isomorphic to $M^b(K)$ by Gelfand representation. □

We call $\hat{K}$ the *dual* signed hypergroup of a finite commutative signed hypergroup $K$.

For a commutative hypergroup $K$, when the dual signed hypergroup $\hat{K}$ satisfies the hypergroup conditions, we call that $K$ is strong. For a commutative signed hypergroup $K$, when the dual signed hypergroup $\hat{K}$ satisfies the dual relation $\hat{K} \cong K$, we call that $K$ is self-dual.

**Proposition 2.20.**

$$e_j = \frac{w(\chi_j)}{w(K)} \sum_i w(c_i)\overline{\chi_j(c_i)}\varepsilon_{c_i}.$$

*Proof.* Put $e_j = \sum_k a_k \varepsilon_{c_k}$ for $a_k \in \mathbb{C}$. For any $c_i \in K$, we have

$$(e_j|c_i) = \sum_k a_k(c_k|c_i) = \sum_k a_k \phi(\varepsilon_{c_i}^- * \varepsilon_{c_k}) = a_i \cdot \frac{1}{w(c_i)}.$$

On the other hands, we have

$$(e_j|c_i) = \phi(\varepsilon_{c_i}^- * e_j) = \phi(\chi_j(c_i^-)e_j) = \overline{\chi_j(c_i)}a_0$$

by Proposition 2.15 and Proposition 2.16. Hence, we have $a_i = \overline{\chi_j(c_i)}w(c_i)a_0$. Then we have

$$\begin{aligned} \chi_j(e_j) &= \chi_j\left(\sum_i \overline{\chi_j(c_i)}w(c_i)a_0\varepsilon_{c_i}\right) = a_0 \sum_i \overline{\chi_j(c_i)}\chi_j(c_i)w(c_i) \\ &= a_0 w(K)(\chi_j|\chi_j) = a_0 \cdot \frac{w(K)}{w(\chi_j)} \end{aligned}$$

by Corollary 2.18. Since $\chi_j(e_j) = 1$, we get $a_0 = \dfrac{w(\chi_j)}{w(K)}$.

□

Next, we introduce some methods of making new hypergroups from the materials of given hypergroups.

(1) Direct product hypergroup $H \times L$.

Let $H$ and $L$ be locally compact commutative signed hypergroups with unit $h_0 \in H$ and $l_0 \in L$ respectively. The *direct product hypergroup* $H \times L = \{(h, l) : h \in H, l \in L\}$ is defined as follows.

The point measure $\varepsilon_{(h,l)}$ of an element $(h, l) \in H \times L$ is identified with $\varepsilon_h \otimes \varepsilon_l \in M^b(H) \otimes M^b(L)$. The convolution $\cdot$ on $H \times L$ is calculated as follows.

$$\varepsilon_{(h,l)} \cdot \varepsilon_{(h',l')} := (\varepsilon_h * \varepsilon_{h'}) \otimes (\varepsilon_l * \varepsilon_{l'}).$$

Then we immediately know that the unit is $(h_0, l_0)$ and involution $^-$ is given by $(h, l)^- := (h^-, l^-)$.

For $\chi \in \hat{H}$ and $\tau \in \hat{L}$, we define the *double character* $(\chi, \tau)$ by $(\chi, \tau)(h, l) := \chi(h)\tau(l)$. Then it is obvious that $(\chi, \tau)$ is a character of $H \times L$ namely $\widehat{H \times L} = \hat{H} \times \hat{L}$.

(2) Let $H$ be a compact commutative signed hypergroup and $L$ be a finite commutative signed hypergroup. We denote $L \setminus \{\text{unit of } L\}$ by $L_0$. The a hypergroup *join* $H \vee L := H \cup L_0$ of $H$ by $L$ is defined as follows.

(a) $\varepsilon_h * \varepsilon_l = \varepsilon_l$ for $h \in H$ and $l \in L_0$.

(b) $\varepsilon_{l_i^-} * \varepsilon_{l_i} = \dfrac{1}{w(l)} e_H + \sum_{k \neq 0} n_{i^- i}^k \varepsilon_{l_k}$ for $l_i \in L_0$ where $e_H$ is the normalized Haar measure of $H$.

(3) Let $H$ be a finite signed hypergroup and $G$ be a finite abelian group. Let $\alpha$ be a homomorphism from $G$ to $\mathrm{Aut}(H)$, called *(group) action* of $G$ on $H$. We denote an $\alpha$-orbit by $C_i$ and $\varepsilon_{C_i} := \dfrac{1}{|C_i|} \sum_{c \in C_i} \varepsilon_c$. Then the set $K = \{C_0, C_1, \cdots, C_n\}$ of all orbits by $\alpha$ become a commutative signed hypergroup, called *orbital hypergroup* of $H$ by $G$ and denoted by $H^\alpha$.

Especially, when $H$ is a group and an action $\alpha$ is the adjoint action of $H$, $K$ is called the *(conjugacy) class hypergroup* and denoted by $K(H)$.

**Example 2.21.** Let $S_3 = \{e, h, h^2, g, hg, h^2 g\}$ be the symmetric group of order three where $h^3 = e, g^2 = e$ and $gh = h^2 g$.

The classes are as follows:

$$C_0 = \{e\},\ C_1 = \{h, h^2\},\ C_2 = \{g, hg, h^2 g\}.$$

Let $c_i = C_i/|C_i|$. The set $K(S_3)$ of class hypergroup of $S_3$ is $K(S_3) = \{c_0, c_1, c_2\}$ and the structure constants are seen to be

$$\varepsilon_{c_1} * \varepsilon_{c_1} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_1}, \quad \varepsilon_{c_2} * \varepsilon_{c_2} = \frac{1}{3}\varepsilon_{c_0} + \frac{2}{3}\varepsilon_{c_1},$$
$$\varepsilon_{c_1} * \varepsilon_{c_2} = \varepsilon_{c_2}.$$

(4) Let $K$ be a locally compact signed hypergroup. Let $N$ be a subalgebra of $M^b(K)$ with unit of $M^b(K)$. For a state $\phi$ of $M^b(K)$, there exists the unique conditional expectation $E$ from $M^b(K)$ onto $N$ such that $\phi \circ E = \phi$ namely $E$ satisfies following conditions.
   (a) $E$ is a linear mapping from $M^b(K)$ to $N$.
   (b) $E(\varepsilon_a * \varepsilon_x * \varepsilon_b) = \varepsilon_a * E(\varepsilon_x) * \varepsilon_b$ for $a, b \in N$ and $x \in M^b(K)$.
   (c) $\phi \circ E = \phi$.

   If for a locally compact signed hypergroup $K'$, there exists the isomorphism $\Psi$ from $M^b(K')$ onto $N$ and for any $x \in K$ there exists $c' \in K'$ such that $E(\varepsilon_x) = \Psi(\varepsilon_{c'})$, then $K$ is called the *generalized orbital hypergroup* of $K$ by the conditional expectation $E$ and denote by $K^E$.

   **Remark.** Any orbital hypergroup is a generalized orbital hypergroup.

(5) Let $H$ be a finite group and $\hat{H}$ be a set of all irreducible representation of $H$. For $\hat{H} \ni \pi_i, \pi_j$, the tensor product of $\pi_i$ and $\pi_j$ is given by

$$\pi_i \otimes \pi_j := \sum_k \oplus M_{ij}^k \pi_k$$

   where $M_{ij}^k$ is the multiplicity of $\pi_k$. Remarking the dimension, we can see that $(\dim\pi_i)(\dim\pi_j) = \sum_k M_{ij}^k \dim\pi_k$. We denote the normalized character of $\pi_i$ by $\chi_i$ namely

$$\chi_i(h) := \frac{\mathrm{tr}(\pi_i(h))}{\dim\pi_i}.$$

   If we put $m_{ij}^k = \dfrac{M_{ij}^k \dim\pi_k}{(\dim\pi_i)(\dim\pi_j)}$, then we have

$$\chi_i\chi_j = \sum_k m_{ij}^k \chi_k, \quad \sum_k m_{ij}^k = 1.$$

   The hypergroup is called a *character hypergroup* and denote by $K(\hat{H})$.

**Example 2.22.** Let $\hat{S}_3 = \{x_0, x_1, \pi\}$ be the set of all irreducible representations of the symmetric group $S_3$ of order three where $\dim\chi_i = 1$ and $\dim\pi = 2$. We denote the normalized character of $\pi$ by $\chi_2$.

The set $K(\hat{S_3})$ of character hypergroup of $S_3$ is $K(\hat{S_3}) = \{\chi_0, \chi_1, \chi_2\}$ and the structure is determined by

$$\varepsilon_{\chi_1} * \varepsilon_{\chi_1} = \varepsilon_{\chi_0}, \quad \varepsilon_{\chi_2} * \varepsilon_{\chi_2} = \frac{1}{4}\varepsilon_{\chi_0} + \frac{1}{4}\varepsilon_{\chi_1} + \frac{1}{2}\varepsilon_{\chi_2},$$
$$\varepsilon_{\chi_1} * \varepsilon_{\chi_2} = \varepsilon_{\chi_2}.$$

**Remark.** For a finite group $H$, we have

$$\widehat{K(H)} \cong K(\hat{H}).$$

## 3. Extension problem of some hypergroups

Let $K$ be a locally compact commutative hypergroup and $H \subset K$ be a subhypergroup. It is well-known that the quotient $K/H$ is also a commutative hypergroup. In order to describe this situation, we often use the form of short exact sequence:

$$1 \longrightarrow H \xrightarrow{\iota} K \xrightarrow{\varphi} L \longrightarrow 1$$

where $L = K/H$ and $\varphi$ is the quotient mapping. Then, the hypergroup $K$ is called an *extension* hypergroup of $L$ by $H$.

**Problem.** For given locally compact commutative hypergroups $H$ and $L$, find all commutative extension hypergroups $K$ of $L$ by $H$.

In this Chapter, we consider three extension problems.

### 3.1. **Extensions of the Golden hypergroup by finite abelian groups.**

3.1.1. *The structures of extension hypergroups.* Let $L = \{\ell_0, \ell_1, \ell_2\}$ be the Golden hypergroup $\mathbb{G}$ where $\ell_0$ is the unit of $L$. The hypergroup structure of $L$ is determined by

$$\delta_{\ell_1} \circ \delta_{\ell_1} = \frac{1}{2}\delta_{\ell_0} + \frac{1}{2}\delta_{\ell_2}, \quad \ell_1^- = \ell_1,$$

$$\delta_{\ell_2} \circ \delta_{\ell_2} = \frac{1}{2}\ell_0 + \frac{1}{2}\delta_{\ell_1}, \quad \ell_2^- = \ell_2,$$

$$\delta_{\ell_1} \circ \delta_{\ell_2} = \frac{1}{2}\delta_{\ell_1} + \frac{1}{2}\delta_{\ell_2}$$

where $\delta_{\ell_i}$ is the Dirac measure at $\ell_i \in L$. Let $H = \{h_0, h_1, \cdots, h_n\}$ be a finite abelian group where $h_0$ is the unit of $H$.

We investigate the structure of extensions $K$ of $L$ by $H$. Let $\varphi$ be a homomorphism from $K$ onto $L$ such that Ker $\varphi = H$, where $H$ is assumed to be a subhypergroup of $K$. Then $K$ is written as the disjoint union of $H = \varphi^{-1}(\ell_0)$, $S := \varphi^{-1}(\ell_1)$ and $T := \varphi^{-1}(\ell_2)$. Let $H(\ell_1)$ and $H(\ell_2)$ denote the stability group of $H$ at $s_0 \in S$ and $t_0 \in T$ respectively, i.e.

$$H(\ell_1) = \{h \in H : \varepsilon_h * \varepsilon_{s_0} = \varepsilon_{s_0}\},$$

$$H(\ell_2) = \{h \in H : \varepsilon_h * \varepsilon_{t_0} = \varepsilon_{t_0}\}.$$

We note that $H(\ell_1)$ does not depend on the choice of $s_0 \in S$ but only on $S$ and $H(\ell_2)$ also depends only on $T$.

**Proposition 3.1.** *For each $s \in S$ and $t \in T$, there exist $h$ and $k \in H$ such that $\varepsilon_s = \varepsilon_h * \varepsilon_{s_0}$ and $\varepsilon_t = \varepsilon_k * \varepsilon_{t_0}$.*

*Proof.* If $s \in \operatorname{supp}(\varepsilon_h * \varepsilon_{s_0})$ for $h \in H$, then $\operatorname{supp}(\varepsilon_h^- * \varepsilon_s)$ is contained in $\operatorname{supp}(\varepsilon_h^- * \varepsilon_h * \varepsilon_{s_0})$. Since $H$ is a group, we have $\varepsilon_h^- * \varepsilon_h = \varepsilon_{h_0}$ so that

$$\operatorname{supp}(\varepsilon_h^- * \varepsilon_h * \varepsilon_{s_0}) = \operatorname{supp}(\varepsilon_{h_0} * \varepsilon_{s_0}) = \operatorname{supp}(\varepsilon_{s_0}) = \{s_0\}.$$

Hence we see that $\varepsilon_h^- * \varepsilon_s = \varepsilon_{s_0}$, namely $\varepsilon_s = \varepsilon_h * \varepsilon_{s_0}$. By the fact that

$$S = H * \varepsilon_{s_0} = \bigcup_{h \in H} \operatorname{supp}(\varepsilon_h * \varepsilon_{s_0}),$$

we get the desired conclusion. In a similar way, we have the same conclusion for $t \in T$. □

Let $e_{H_0}$ denote the normalized Haar measure of a subgroup $H_0$ of $H$. The next Lemma is useful for our arguments hereafter.

**Lemma 3.2.** *For a subgroup $H_0$ of $H$, if $c \in M^1(H)$, supp(c) $\subset H_0$ and $e_{H_0} * c = c$, then we have $c = e_{H_0}$.*

*Proof.* For $c \in M^1(H)$ and $\operatorname{supp}(c) \subset H_0$, we can write $c = \sum_{h_k \in H_0} a_k \varepsilon_{h_k}$ where $\sum_k a_k = 1$. Then, we have

$$c = e_{H_0} * c = \sum_{h_k \in H_0} a_k e_{H_0} * \varepsilon_{h_k} = \sum_{h_k \in H_0} a_k e_{H_0} = \left( \sum_{h_k \in H_0} a_k \right) e_{H_0} = e_{H_0}.$$

Hence we get the desired conclusion. □

Let $\omega(\ell_1)$ denote the normalized Haar measure of $H(\ell_1)$ and $\omega(\ell_2)$ denote the normalized Haar measure of $H(\ell_2)$.

**Proposition 3.3.** *For $s_0 \in S$ and $t_0 \in T$, there exist $h \in H$ and $k \in H$ such that $\varepsilon_{s_0}^- = \varepsilon_h * \varepsilon_{s_0}$ and $t_0^- = \varepsilon_k * \varepsilon_{t_0}$. Then we have $\varepsilon_{s_0}^- * \varepsilon_{s_0} = \frac{1}{2}\omega(\ell_1) + \frac{1}{2} c_1 * \varepsilon_{t_0}$, $\varepsilon_{t_0}^- * \varepsilon_{t_0} = \frac{1}{2}\omega(\ell_2) + \frac{1}{2} c_2 * \varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2} c_3 * \varepsilon_{s_0} + \frac{1}{2} c_4 * \varepsilon_{t_0}$ where $c_i \in M^1(H)$ $(i = 1,2,3,4)$ such that $c_1^- * \varepsilon_k = c_1$ and $c_2^- * \varepsilon_h = c_2$ and $\omega(\ell_1) * \omega(\ell_2) * c_i = c_i$ $(i = 1,2,3,4)$. Moreover we have $c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_k$, $c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_h$, $c_3 = c_1^-$ and $c_4 = c_2^-$.*

*Proof.* One can take $h, k \in H$ such that $\varepsilon_{s_0}^- = \varepsilon_h * \varepsilon_{s_0}$ and $t_0^- = \varepsilon_k * \varepsilon_{t_0}$ by Proposition 3.1 because $s_0^- \in S$ and $t_0^- \in T$ by the relations $\ell_1^- = \ell_1$ and $\ell_2^- = \ell_2$. It is easy to see that $\varepsilon_{s_0}^- * \varepsilon_{s_0}$ is written as

$$\varepsilon_{s_0}^- * \varepsilon_{s_0} = \frac{1}{2} c_0 + \frac{1}{2} c_1 * \varepsilon_{t_0}$$

for some $c_0, c_1 \in M^1(H)$.

First, we show the equality $c_0 = \omega(\ell_1)$. The fact $\omega(\ell_1) * \varepsilon_{s_0} = \varepsilon_{s_0}$ implies that $\omega(\ell_1) * c_0 = c_0$ and $\omega(\ell_1) * c_1 = c_1$. We suppose that $h' \notin H(\ell_1)$. Since we have $\varepsilon_{h'} * \varepsilon_{s_0} \neq \varepsilon_{s_0}$, we have $(\varepsilon_{h'} * \varepsilon_{s_0})^- \neq \varepsilon_{s_0}^-$. Then $h_0 \notin \operatorname{supp}((\varepsilon_{h'} * \varepsilon_{s_0})^- * \varepsilon_{s_0})$ by the axiom of hypergroup. Since $(\varepsilon_{h'} * \varepsilon_{s_0})^- * \varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h'}^- * c_0 + \frac{1}{2}\varepsilon_{h'}^- * c_1 * \varepsilon_{t_0}$ because $K$ is commutative, we have $h_0 \notin \operatorname{supp}(\varepsilon_{h'}^- * c_0)$. Therefore $h' \notin \operatorname{supp}(c_0)$. Hence we see that $\operatorname{supp}(c_0)$ is contained in $H(\ell_1)$. By Lemma 3.2, we get $c_0 = \omega(\ell_1)$. By the fact that $\omega(\ell_1) * \varepsilon_{s_0} = \varepsilon_{s_0}$ and $\omega(\ell_2) * \varepsilon_{t_0} = \varepsilon_{t_0}$, we see that $\omega(\ell_1) * \omega(\ell_2) * c_1 = c_1$. By the equality:

$$(\varepsilon_{s_0}^- * \varepsilon_{s_0})^- = \frac{1}{2}(\omega(\ell_1))^- + \frac{1}{2}c_1^- * \varepsilon_{t_0}^- = \frac{1}{2}\omega(\ell_1) + \frac{1}{2}c_1^- * \varepsilon_k * \varepsilon_{t_0}$$

and $(\varepsilon_{s_0}^- * \varepsilon_{s_0})^- = \varepsilon_{s_0}^- * \varepsilon_{s_0}$, we get $c_1^- * \varepsilon_k = c_1$. In a similar way to the above, we have $\varepsilon_{t_0}^- * \varepsilon_{t_0} = \frac{1}{2}\omega(\ell_2) + \frac{1}{2}c_2 * \varepsilon_{s_0}$ where $\omega(\ell_1) * \omega(\ell_2) * c_2 = c_2$ and $c_2^- * \varepsilon_h = c_2$. It is easy to see that $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}c_3 * \varepsilon_{s_0} + \frac{1}{2}c_4 * \varepsilon_{t_0}$ where $\omega(\ell_1) * \omega(\ell_2) * c_3 = c_3$ and $\omega(\ell_1) * \omega(\ell_2) * c_4 = c_4$.

Next, we show the equation $c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_k$, $c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_h$, $c_3 = c_1^-$ and $c_4 = c_2^-$. We have $\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{2}\omega(\ell_1) * \varepsilon_h^- + \frac{1}{2}c_1 * \varepsilon_h^- * \varepsilon_{t_0}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\omega(\ell_2) * \varepsilon_k^- + \frac{1}{2}c_2 * \varepsilon_k^- * \varepsilon_{s_0}$, and $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}c_3 * \varepsilon_{s_0} + \frac{1}{2}c_4 * \varepsilon_{t_0}$. It is easy to see by simple calculations that

$$(\varepsilon_{s_0} * \varepsilon_{s_0}) * \varepsilon_{t_0} = \frac{1}{4}c_1 * \varepsilon_h^- * \varepsilon_k^- + \frac{1}{4}c_1 * c_2 * \varepsilon_h^- * \varepsilon_k^- * \varepsilon_{s_0} + \frac{1}{2}\omega(\ell_1) * \omega(\ell_2) * \varepsilon_h^- * \varepsilon_{t_0},$$

$$\varepsilon_{s_0} * (\varepsilon_{s_0} * \varepsilon_{t_0}) = \frac{1}{4}c_3 * \varepsilon_h^- + \frac{1}{4}c_3 * c_4 * \varepsilon_{s_0} + \frac{1}{4}(c_1 * c_3 * \varepsilon_h^- + c_4 * c_4) * \varepsilon_{t_0}.$$

By the associativity: $(\varepsilon_{s_0} * \varepsilon_{s_0}) * \varepsilon_{t_0} = \varepsilon_{s_0} * (\varepsilon_{s_0} * \varepsilon_{t_0})$, we have $2\omega(\ell_1) * \omega(\ell_2) * \varepsilon_h^- = c_1 * c_3 * \varepsilon_h^- + c_4 * c_4$ and $c_3 = c_1 * \varepsilon_k^- = c_1^-$. In a similar way, since we have $\varepsilon_{s_0} * (\varepsilon_{t_0} * \varepsilon_{t_0}) = (\varepsilon_{s_0} * \varepsilon_{t_0}) * \varepsilon_{t_0}$, we have $c_4 = c_2 * \varepsilon_h^- = c_2^-$. By these relations, we have $2\omega(\ell_1) * \omega(\ell_2) = c_1 * c_1 * \varepsilon_k^- + c_2 * c_2 * \varepsilon_h^-$. This fact implies that $\operatorname{supp}(\omega(\ell_1) * \omega(\ell_2)) = \operatorname{supp}(c_1 * c_1 * \varepsilon_k^-) \cup \operatorname{supp}(c_2 * c_2 * \varepsilon_h^-)$. Hence we see that $\operatorname{supp}(c_1 * c_1 * \varepsilon_k^-) \subset H(\ell_1) * H(\ell_2)$ and $\operatorname{supp}(c_2 * c_2 * \varepsilon_h^-) \subset H(\ell_1) * H(\ell_2)$. Applying Lemma 3.2, we have $c_1 * c_1 * \varepsilon_k^- = \omega(\ell_1) * \omega(\ell_2)$ and $c_2 * c_2 * \varepsilon_h^- = \omega(\ell_1) * \omega(\ell_2)$. Therefore, we get $c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_k$ and $c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_h$. □

**Remark.** If $K$ is an extension of the Golden hypergroup $L = \mathbb{G}$ by a finite abelian group $H$, we can reformulate Proposition 3.3 as follows.

(0) $K$ is the disjoint union of $H = \varphi^{-1}(\ell_0)$, $S = \varphi^{-1}(\ell_1)$ and $T = \varphi^{-1}(\ell_2)$, and take $s_0 \in S$, $t_0 \in T$.

(1) $\varepsilon_{s_0}^- = \varepsilon_h * \varepsilon_{s_0}$ and $\varepsilon_{t_0}^- = \varepsilon_k * \varepsilon_{t_0}$ for $h, k \in H$.

(2) $\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{2}\omega(\ell_1) * \varepsilon_h^- + \frac{1}{2}c_1 * \varepsilon_h^- * \varepsilon_{t_0}$ for $c_1 \in M^1(H)$.

(3) $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\omega(\ell_2) * \varepsilon_k^- + \frac{1}{2}c_2 * \varepsilon_k^- * \varepsilon_{s_0}$ for $c_2 \in M^1(H)$.

(4) $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}c_1^- * \varepsilon_{s_0} + \frac{1}{2}c_2^- * \varepsilon_{t_0}$.

(5) $\omega(\ell_1) * \omega(\ell_2) * c_1 = c_1$ and $\omega(\ell_1) * \omega(\ell_2) * c_2 = c_2$.

(6) $c_1^- = c_1 * \varepsilon_k^-$ and $c_2^- = c_2 * \varepsilon_h^-$.

(7) $c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_k$ and $c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_h$.

We remark that it is easy to check that these conditions assure that $K$ is a commutative hypergroup which is an extension of $L$ by $H$. Hence we see that all extensions $K$ of $L$ by $H$ are determined in this way by

$$s_0 \in S,\ t_0 \in T,\ h, k \in H,\ c_1, c_2 \in M^1(H)$$

satisfying the above conditions (1) – (7). Therefore we denote such an extension $K$ by $K = K(s_0, t_0, h, k, c_1, c_2)$.

Let $K_1 = H \cup S_1 \cup T_1$ and $K_2 = H \cup S_2 \cup T_2$ be two extensions of $L$ by $H$ and $\varphi_1$ (resp. $\varphi_2$) be a canonical quotient mapping from $K_1$ (resp. $K_2$) onto the Golden hypergroup $L = \mathbb{G}$. Then $K_1$ is called to be equivalent to $K_2$ as extensions if there exists a hypergroup isomorphism $\psi$ from $K_1$ onto $K_2$ such that $\psi(h) = h$ for all $h \in H$ and $\varphi_2 \circ \psi = \varphi_1$.

When we take $u_0 \in S$, $v_0 \in T$, $h_1, k_1 \in H$ and $d_1, d_2 \in M^1(H)$ satisfying the above conditions (1) – (7), we have another extension $K(u_0, v_0, h_1, k_1, d_1, d_2)$ of $L$ by $H$.

**Proposition 3.4.** *Two extensions $K(s_0, t_0, h, k, c_1, c_2)$ and $K(u_0, v_0, h_1, k_1, d_1, d_2)$ of $L$ by $H$ are mutually equivalent as extensions if and only if there exist $b_1, b_2 \in H$ such that $\varepsilon_{u_0} = \varepsilon_{b_1}^- * \varepsilon_{s_0}$, $\varepsilon_{v_0} = \varepsilon_{b_2}^- * \varepsilon_{t_0}$, $d_1 = \varepsilon_{b_2} * c_1$, $d_2 = \varepsilon_{b_1} * c_2$, $\omega(\ell_1) * \varepsilon_{h_1} = \omega(\ell_1) * \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_h$ and $\omega(\ell_2) * \varepsilon_{k_1} = \omega(\ell_2) * \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_k$.*

*Proof.* Suppose that $K_1 = K(s_0, t_0, h, k, c_1, c_2)$ is equivalent to $K_2 = K(u_0, v_0, h_1, k_1, d_1, d_2)$. Then it is easy to see that both stability groups of $H$ in $K_1$ and $K_2$ at $s_0$ and $u_0$ coincide and both stability groups of $H$ at $t_0$ and $v_0$ also coincide. Hence we may assume that $\varphi_2^{-1}(\ell_1) = \varphi_1^{-1}(\ell_1) = S$ and $\varphi_2^{-1}(\ell_2) = \varphi_1^{-1}(\ell_2) = T$. For $u_0 \in S$ and $v_0 \in T$, there exist $b_1$ and $b_2 \in H$ such that $\varepsilon_{u_0} = \varepsilon_{b_1}^- * \varepsilon_{s_0}$ and $\varepsilon_{v_0} = \varepsilon_{b_2}^- * \varepsilon_{t_0}$ respectively by Proposition 3.1. By the relation that $\varepsilon_{s_0}^- = \varepsilon_h * \varepsilon_{s_0}$ and $\varepsilon_{u_0}^- = \varepsilon_{h_1} * \varepsilon_{u_0}$, we get

$$\varepsilon_{h_1} * \varepsilon_{s_0} = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_h * \varepsilon_{s_0}.$$

Hence we have $\omega(\ell_1) * \varepsilon_{h_1} = \omega(\ell_1) * \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_h$. In a similar way, we also obtain $\omega(\ell_2) * \varepsilon_{k_1} = \omega(\ell_2) * \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_k$. Since $\varepsilon_{u_0}^- * \varepsilon_{u_0} = \varepsilon_{s_0}^- * \varepsilon_{s_0}$, comparing coefficients of $t_0$ of $\varepsilon_{s_0}^- * \varepsilon_{s_0}$ and $\varepsilon_{u_0}^- * \varepsilon_{u_0}$, we get $d_1 = \varepsilon_{b_2} * c_1$. In a similar way, we see that $d_2 = \varepsilon_{b_1} * c_2$.

Conversely, if there exists $b_1, b_2 \in H$ such that $\varepsilon_{u_0} = \varepsilon_{b_1}^- * \varepsilon_{s_0}$, $\varepsilon_{v_0} = \varepsilon_{b_2}^- * \varepsilon_{t_0}$, $d_1 = \varepsilon_{b_2} * c_1$, $d_2 = \varepsilon_{b_1} * c_2$, $\omega(\ell_1) * \varepsilon_{h_1} = \omega(\ell_1) * \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_h$ and $\omega(\ell_2) * \varepsilon_{k_1} = \omega(\ell_2) * \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_k$, it is easy to check that $K(s_0, t_0, h, k, c_1, c_2)$ is equivalent to $K(u_0, v_0, h_1, k_1, d_1, d_2)$. □

Let $K$ be an extension of $L$ by a finite abelian group $H$. If there exists injective mapping $\phi$ from $L$ into $K$ such that

(1) $\varphi(\phi(\ell)) = \ell$,

(2) $\phi(e_L) = e_K$ and $\phi(\ell^-) = \phi(\ell)^-$,

(3) The set $H(\ell) = \{h \in H : h * \phi(\ell) = \phi(\ell)\}$ is a subgroup of $H$,

(4) $\phi(\delta_{\ell_i}) * \phi(\delta_{\ell_j}) = \phi(\delta_{\ell_i} \circ \delta_{\ell_j}) * \omega(\ell_i) * \omega(\ell_j)$ $(i, j = 1, 2)$,

(5) $\omega(\ell_i) * \omega(\ell_j) * \omega(\ell) = \omega(\ell_i) * \omega(\ell_j)$ if $\ell \in \mathrm{supp}(\delta_{\ell_1} \circ \delta_{\ell_2})$,

(6) $K = H * \phi(L)$, and $H \bigcap \phi(L) = \{e_K\}$,

then we call that the extension $K$ of $L$ by $H$ *splits* or $K$ is a *splitting extension* ([KST]).

**Definition** (weakly splitting)**.** We call the extension $K$ of $L$ by $H$ *weakly splitting* if the conditions (1), (2), (3), (5) are satisfied.

**Proposition 3.5.** *The extension $K = K(s_0, t_0, h, k, c_1, c_2)$ is weakly splitting if and only if there exist $b_1, b_2 \in H$ such that $c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{b_2}$, $c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{b_1}$, $\omega(\ell_1) * \varepsilon_h = \omega(\ell_1) * \varepsilon_{b_1} * \varepsilon_{b_1}$ and $\omega(\ell_2) * \varepsilon_k = \omega(\ell_2)\varepsilon_{b_2} * \varepsilon_{b_2}$. Moreover, $K$ is splitting if and only if $K$ is weakly splitting and $H(\ell_1) = H(\ell_2)$.*

*Proof.* Suppose that the extension $K$ is given by $K = K(s_0, t_0, h, k, c_1, c_2)$. We assume that $\phi(\ell_0) = h_0, \phi(\ell_1) = s_0$ and $\phi(\ell_2) = t_0$. Then we have $s_0^- = s_0$ and $t_0^- = t_0$ by weakly splitting condition (1). This implies that we can assume that $h = h_0$ and $k = k_0$ so that $c_1 = c_2 = \omega(\ell_1) * \omega(\ell_2)$. Since weakly splitting extensions are equivalent to this extension $K = K(s_0, t_0, h_0, k_0, c_1, c_2)$, we get the desired conclusion by applying Proposition 3.4.

By the structure equations (2) and (3) as described in Remark combined with splitting condition (4), we get $\omega(\ell_1) = \omega(\ell_2)$, i.e. $H(\ell_1) = H(\ell_2)$. □

**Theorem 3.6.** *Let $K$ be a commutative hypergroup extension of the Golden hypergroup $L = \{\ell_0, \ell_1, \ell_2\}$ by a finite abelian group $H$, which means that there exists a hypergroup homomorphism $\varphi$ from $K$ onto $L$ such that* Ker $\varphi = H$. *Let $H(\ell_1)$ be the stability group of $H$ at $s_0 \in S = \varphi^{-1}(\ell_1)$ and $H(\ell_2)$ be the stability group of $H$ at $t_0 \in T = \varphi^{-1}(\ell_2)$. Let $\omega(\ell_i)$ denote the normalized Haar measure of $H(\ell_i)$ $(i = 1, 2)$.*

(1) *Then we have $S = \cup_{h \in H} \mathrm{supp}(\varepsilon_h * \varepsilon_{s_0})$ and $T = \cup_{k \in H} \mathrm{supp}(\varepsilon_k * \varepsilon_{t_0})$. When $\varepsilon_{s_0}^- = \varepsilon_h * \varepsilon_{s_0}$ and $t_0^- = \varepsilon_k * \varepsilon_{t_0}$ for some $h, k \in H$, we have $\varepsilon_{s_0}^- * \varepsilon_{s_0} = \frac{1}{2}\omega(\ell_1) + \frac{1}{2}c_1 * \varepsilon_{t_0}$, $\varepsilon_{t_0}^- * \varepsilon_{t_0} = \frac{1}{2}\omega(\ell_2) + \frac{1}{2}c_2 * \varepsilon_{s_0}$ and $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}c_1^- * \varepsilon_{s_0} + \frac{1}{2}c_2^- * \varepsilon_{t_0}$ for $c_1, c_2 \in M^1(H)$ such that $\omega(\ell_1) * \omega(\ell_2) * c_i = c_i$ $(i = 1, 2)$. Moreover, $c_1^- = c_1 * \varepsilon_k^-$, $c_2^- = c_2 * \varepsilon_h^-$, $c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_k$ and $c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_h$.*

(2) *All extensions $K$ of $L$ by $H$ are characterized in this way, so that we denote such an extension $K$ by $K(s_0, t_0, h, k, c_1, c_2)$. Two extensions $K(s_0, t_0, h, k, c_1, c_2)$ and $K(u_0, v_0, h_1, k_1, d_1, d_2)$ of $L$ by $H$ are mutually equivalent as extensions if and only if there exists $b_1, b_2 \in H$ such that $\varepsilon_{u_0} = \varepsilon_{b_1}^- * \varepsilon_{s_0}$, $\varepsilon_{v_0} = \varepsilon_{b_2}^- * \varepsilon_{t_0}$, $d_1 = \varepsilon_{b_2} * c_1$, $d_2 = \varepsilon_{b_1} * c_2$, $\omega(\ell_1) * \varepsilon_{h_1} = \omega(\ell_1) * \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_h$ and $\omega(\ell_2) * \varepsilon_{k_1} = \omega(\ell_2) * \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_k$.*

(3) *Moreover, the extension $K = K(s_0, t_0, h, k, c_1, c_2)$ is weakly splitting if and only if there exist $b_1, b_2 \in H$ such that $c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{b_2}$, $c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{b_1}$, $\omega(\ell_1) * \varepsilon_h = \omega(\ell_1) * \varepsilon_{b_1} * \varepsilon_{b_1}$ and $\omega(\ell_2) * \varepsilon_k = \omega(\ell_2) * \varepsilon_{b_2} * \varepsilon_{b_2}$. The extension $K$ is splitting if and only if $K$ is weakly splitting and $H(\ell_1) = H(\ell_2)$.*

*Proof.* These statements follow immediately from Proposition 3.1, 3.2, 3.3, 3.4 and 3.5 so that we omit the details. □

3.1.2. *Applications and Examples.* Under these preparations we calculate all extensions $K$ of the Golden hypergroup $L$ by concrete abelian groups $H = \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ and $\mathbb{Z}_6$. We denote the order of $K$ by $|K|$.

**Example 3.7.** $H = \mathbb{Z}_2 = \{h_0, h_1\}$, $h_1^2 = h_0$.

(1) Case of $|K| = 6$, i.e. $H(\ell_1) = \{h_0\}$, $H(\ell_2) = \{h_0\}$ and $K^6 = H \times L$.

$K^6 = \{h_0, h_1, s_0, s_1, t_0, t_1\}$, $\varepsilon_{s_1} = \varepsilon_{h_1} * \varepsilon_{s_0}$, $\varepsilon_{t_1} = \varepsilon_{h_1} * \varepsilon_{t_0}$.

$s_0^- = s_0$, $s_1^- = s_1$, $t_0^- = t_0$, $t_1^- = t_1$,

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{t_0}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0}$,

$\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}$.

(2) Case of $|K| = 5$.

(a) When $H(\ell_1) = H$, $H(\ell_2) = \{h_0\}$, i.e.
$K_a^5 = \{h_0, h_1, s_0, t_0, t_1\}$, $\varepsilon_{t_1} = \varepsilon_{h_1} * \varepsilon_{t_0}$.

(i) $K = K_{a1}^5$ $(s_0^- = s_0, t_0^- = t_0, t_1^- = t_1)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \tfrac{1}{4}\varepsilon_{h_0} + \tfrac{1}{4}\varepsilon_{h_1} + \tfrac{1}{4}\varepsilon_{t_0} + \tfrac{1}{4}\varepsilon_{t_1},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0} + \tfrac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0} + \tfrac{1}{4}\varepsilon_{t_0} + \tfrac{1}{4}\varepsilon_{t_1}.$$

(ii) $K = K_{a2}^5$ $(s_0^- = s_0, t_0^- = t_1, t_1^- = t_0)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \tfrac{1}{4}\varepsilon_{h_0} + \tfrac{1}{4}\varepsilon_{h_1} + \tfrac{1}{4}\varepsilon_{t_0} + \tfrac{1}{4}\varepsilon_{t_1},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_1} + \tfrac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0} + \tfrac{1}{4}\varepsilon_{t_0} + \tfrac{1}{4}\varepsilon_{t_1}.$$

(b) When $H(\ell_1) = \{h_0\}$, $H(\ell_2) = H$, in a similar way, we have $K_{b1}^5$ and $K_{b2}^5$.

(3) Case of $| K |= 4$, i.e. $H(\ell_1) = H$, $H(\ell_2) = H$.
$K^4 = H \vee L = \{h_0, h_1, s_0, t_0\}$ which is the join of $H$ by $L$ and characterized by

$$s_0^- = s_0,\ t_0^- = t_0,$$
$$\varepsilon_{s_0} * \varepsilon_{s_0} = \tfrac{1}{4}\varepsilon_{h_0} + \tfrac{1}{4}\varepsilon_{h_1} + \tfrac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \tfrac{1}{4}\varepsilon_{h_0} + \tfrac{1}{4}\varepsilon_{h_1} + \tfrac{1}{2}\varepsilon_{s_0},$$
$$\varepsilon_{s_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0} + \tfrac{1}{2}\varepsilon_{t_0}.$$

Next, we consider the dual of this model. Let $\hat{K}_{a1}^5 = \{\chi_0, \chi_1, \chi_2, \chi_3, \chi_4\}$, be the dual of $K_{a1}^5$. The character table of $K_{a1}^5$ is as follows.

| | $h_0$ | $h_1$ | $s_0$ | $t_0$ | $t_1$ |
|---|---|---|---|---|---|
| $\chi_0$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_1$ | 1 | 1 | $\frac{-1+\sqrt{5}}{4}$ | $\frac{-1-\sqrt{5}}{4}$ | $\frac{-1-\sqrt{5}}{4}$ |
| $\chi_2$ | 1 | 1 | $\frac{-1-\sqrt{5}}{4}$ | $\frac{-1+\sqrt{5}}{4}$ | $\frac{-1+\sqrt{5}}{4}$ |
| $\chi_3$ | 1 | -1 | 0 | $\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ |
| $\chi_4$ | 1 | -1 | 0 | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ |

Hence the structure equations of the dual $\hat{K}_{a1}^5$ of $K_{a1}^5$ are given in the following way.

$$\varepsilon_{\chi_1} * \varepsilon_{\chi_1} = \frac{1}{2}\varepsilon_{\chi_0} + \frac{1}{2}\varepsilon_{\chi_2},\quad \varepsilon_{\chi_1} * \varepsilon_{\chi_2} = \frac{1}{2}\varepsilon_{\chi_1} + \frac{1}{2}\varepsilon_{\chi_2},\quad \varepsilon_{\chi_2} * \varepsilon_{\chi_2} = \frac{1}{2}\varepsilon_{\chi_0} + \frac{1}{2}\varepsilon_{\chi_1},$$

$$\varepsilon_{\chi_1} * \varepsilon_{\chi_3} = \frac{3-\sqrt{5}}{8}\varepsilon_{\chi_3} + \frac{5+\sqrt{5}}{8}\varepsilon_{\chi_4},\quad \varepsilon_{\chi_1} * \varepsilon_{\chi_1} = \frac{5+\sqrt{5}}{8}\varepsilon_{\chi_3} + \frac{3-\sqrt{5}}{8}\varepsilon_{\chi_4},$$

$$\varepsilon_{\chi_2} * \varepsilon_{\chi_3} = \frac{3+\sqrt{5}}{8}\varepsilon_{\chi_3} + \frac{5-\sqrt{5}}{8}\varepsilon_{\chi_4}, \quad \varepsilon_{\chi_2} * \varepsilon_{\chi_4} = \frac{5-\sqrt{5}}{8}\varepsilon_{\chi_3} + \frac{3+\sqrt{5}}{8}\varepsilon_{\chi_4},$$

$$\varepsilon_{\chi_3} * \varepsilon_{\chi_3} = \varepsilon_{\chi_4} * \varepsilon_{\chi_4} = \frac{2}{5}\varepsilon_{\chi_0} + \frac{3-\sqrt{5}}{10}\varepsilon_{\chi_1} + \frac{3+\sqrt{5}}{10}\varepsilon_{\chi_2},$$

$$\varepsilon_{\chi_3} * \varepsilon_{\chi_4} = \frac{5+\sqrt{5}}{10}\varepsilon_{\chi_1} + \frac{5-\sqrt{5}}{10}\varepsilon_{\chi_2}.$$

By this fact we see that $K^5_{a1}$ is a strong hypergroup. In a similar way, it is easy to check that $K^5_{a2}$, $K^5_{b1}$ and $K^5_{b2}$ are also strong. It is well known that $H \times L$ and $H \vee L$ are strong.

**Remark.** (1) $K$ is a splitting extension of $L$ by $H$ if and only if $K = K^6 = H \times L$ or $K^4 = H \vee L$.

(2) $K$ is a weakly splitting extension of $L$ by $H$ if and only if $K = K^6 = H \times L$, $K^4 = H \vee L$, $K^5_{a1}$, or $K^5_{b1}$

(3) Above extensions are strong.

**Example 3.8.** $H = \mathbb{Z}_3 = \{h_0, h_1, h_2\}, \quad h_1^3 = h_0,\ h_1^- = h_2,\ h_2^- = h_1.$

(1) Case of $|K| = 9$, i.e. $H(\ell_1) = \{h_0\}$, $H(\ell_2) = \{h_0\}$.
$K^9 = \{h_0, h_1, h_2, s_0, s_1, s_2, t_0, t_1, t_2\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1, 2)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2)$.

(a) $K = K^9_a = H \times L$ $(s_0^- = s_0,\ s_1^- = s_2,\ s_2^- = s_1,\ t_0^- = t_0,\ t_1^- = t_2,\ t_2^- = t_1)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0}+\tfrac{1}{2}\varepsilon_{t_0}.$$

(b) $K = K^9_b$ $(s_0^- = s_1,\ s_1^- = s_0,\ s_2^- = s_2,\ t_0^- = t_0,\ t_1^- = t_2,\ t_2^- = t_1)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_2}+\tfrac{1}{2}\varepsilon_{t_2},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{s_2},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0}+\tfrac{1}{2}\varepsilon_{t_1}.$$

(c) $K = K^9_c$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_2^- = s_0,\ t_0^- = t_0,\ t_1^- = t_2,\ t_2^- = t_1)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_1}+\tfrac{1}{2}\varepsilon_{t_1},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{s_1},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0}+\tfrac{1}{2}\varepsilon_{t_2}.$$

(d) $K = K^9_d$ $(s_0^- = s_1,\ s_1^- = s_0,\ s_2^- = s_2,\ t_0^- = t_1,\ t_1^- = t_0,\ t_2^- = t_2)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_2}+\tfrac{1}{2}\varepsilon_{t_1},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_2}+\tfrac{1}{2}\varepsilon_{s_1},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_1}+\tfrac{1}{2}\varepsilon_{t_1}.$$

(e) $K = K^9_e$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_2^- = s_0,\ t_0^- = t_1,\ t_1^- = t_0,\ t_2^- = t_2)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h_1}+\frac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_2}+\frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_1}+\frac{1}{2}\varepsilon_{t_2}.$$

(f) $K = K^9_f$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_2^- = s_0,\ t_0^- = t_2,\ t_1^- = t_1,\ t_2^- = t_0)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h_1}+\frac{1}{2}\varepsilon_{t_2},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_1}+\frac{1}{2}\varepsilon_{s_2},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_2}+\frac{1}{2}\varepsilon_{t_2}.$$

(2) Case of $|K| = 7$.

(a) When $H(\ell_1) = H,\ H(\ell_2) = \{h_0\}$, i.e.

$$K^7_a = \{h_0, h_1, h_2, s_0, t_0, t_1, t_2\} \quad \varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}\ (j = 0, 1, 2).$$

(i) $K = K^7_{a1}$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_2,\ t_2^- = t_1)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}.$$

(ii) $K = K^7_{a2}$ $(s_0^- = s_0,\ t_0^- = t_1,\ t_1^- = t_0,\ t_2^- = t_2)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}.$$

(iii) $K = K^7_{a3}$ $(s_0^- = s_0,\ t_0^- = t_2,\ t_1^- = t_1,\ t_2^- = t_0)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}.$$

(b) When $H(\ell_1) = \{h_0\},\ H(\ell_2) = H$, in a similar way, we have $K^7_{b1}$, $K^7_{b2}$ and $K^7_{b3}$.

(3) Case of $|K| = 5$, i.e. $H(\ell_1) = H,\ H(\ell_2) = H$.
$K^5 = H \vee L = \{h_0, h_1, h_2, s_0, t_0\}$ which is the join of $H$ by $L$ and characterized by

$$s_0^- = s_0,\ t_0^- = t_0,\ \varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{t_0},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}.$$

**Remark.** (1) We remark that $H \times L = K^9_a \cong K^9_b \cong K^9_c \cong K^9_d \cong K^9_e \cong K^9_f$, $K^7_{a1} \cong K^7_{a2} \cong K^7_{a3}$ and $K^7_{b1} \cong K^7_{b2} \cong K^7_{b3}$, as extensions of $L$ by $H$.

(2) $K$ is a splitting extension of $L$ by $H$ if and only if $K \cong H \times L$ or $K^5 = H \vee L$.

(3) $K$ is a weakly splitting extension of $L$ by $H$ if and only if $K \cong H \times L$, $K^5 = H \vee L$, $K^7_{a1}$, or $K^7_{b1}$.

**Example 3.9.** $H = \mathbb{Z}_4 = \{h_0, h_1, h_2, h_3\}, \quad h_1^4 = h_0, h_1^- = h_3,\ h_2^- = h_2.$

(1) Case of $|K| = 12$, i.e. $H(\ell_1) = \{h_0\}$, $H(\ell_2) = \{h_0\}$.

$K^{12} = \{h_0, h_1, h_2, h_3, s_0, s_1, s_2, s_3, t_0, t_1, t_2, t_3\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1, 2, 3)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3)$.

(a) $K = K^{12}_a = H \times L$ $(s_0^- = s_0,\ s_1^- = s_3,\ s_2^- = s_2,\ s_3^- = s_1,\ t_0^- = t_0,\ t_1^- = t_3,\ t_2^- = t_2,\ t_3^- = t_1)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0}+\tfrac{1}{2}\varepsilon_{t_0}.$$

(b) $K = K^{12}_b$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_2^- = s_0,\ s_3^- = s_3,\ t_0^- = t_0,\ t_1^- = t_3, t_2^- = t_2,\ t_3^- = t_1)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_2}+\tfrac{1}{2}\varepsilon_{t_2},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0}+\tfrac{1}{2}\varepsilon_{s_1},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0}+\tfrac{1}{2}\varepsilon_{t_3}.$$

(c) $K = K^{12}_c$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_2^- = s_0,\ s_3^- = s_3,\ t_0^- = t_2,\ t_1^- = t_1, t_2^- = t_0,\ t_3^- = t_3)$ which is characterized by

$$\varepsilon_{s_0}*\varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_2}+\tfrac{1}{2}\varepsilon_{t_3},\ \varepsilon_{t_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_2}+\tfrac{1}{2}\varepsilon_{s_3},\ \varepsilon_{s_0}*\varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_3}+\tfrac{1}{2}\varepsilon_{t_3}.$$

(2) Case of $|K| = 10$.

(a) When $H(\ell_1) = \{h_0, h_2\}$, $H(\ell_2) = \{h_0\}$, i.e.

$K^{10}_a = \{h_0, h_1, h_2, h_3, s_0, s_1, t_0, t_1, t_2, t_3\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3)$.

(i) $K = K^{10}_{a1}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_0,\ t_1^- = t_3,\ t_2^- = t_2,\ t_3^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_2},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_2}.$

(ii) $K = K^{10}_{a2}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_2,\ t_1^- = t_1,\ t_2^- = t_0,\ t_3^- = t_3)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_2},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_2}.$

(b) When $H(\ell_1) = \{h_0\}$, $H(\ell_2) = \{h_0, h_2\}$, in a similar way, we have $K^{10}_{b1}$ and $K^{10}_{b2}$.

(3) Case of $|K| = 9$.

(a) When $H(\ell_1) = H$, $H(\ell_2) = \{h_0\}$, i.e.

$K_a^9 = \{h_0, h_1, h_2, h_3, s_0, t_0, t_1, t_2, t_3\}$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3)$.

(i) $K = K_{a1}^9$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_3,\ t_2^- = t_2,\ t_3^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$.

(ii) $K = K_{a2}^9$ $(s_0^- = s_0,\ t_0^- = t_1,\ t_1^- = t_0,\ t_2^- = t_3,\ t_3^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$.

(iii) $K = K_{a3}^9$ $(s_0^- = s_0,\ t_0^- = t_2,\ t_1^- = t_1,\ t_2^- = t_0,\ t_3^- = t_3)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$.

(iv) $K = K_{a4}^9$ $(s_0^- = s_0,\ t_0^- = t_3,\ t_1^- = t_2,\ t_2^- = t_1,\ t_3^- = t_0)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{8}\varepsilon_{t_0} + \frac{1}{8}\varepsilon_{t_1} + \frac{1}{8}\varepsilon_{t_2} + \frac{1}{8}\varepsilon_{t_3}$.

(b) When $H(\ell_1) = \{h_0\}$, $H(\ell_2) = H$, in a similar way, we have $K_{b1}^9$, $K_{b2}^9$, $K_{b3}^9$ and $K_{b4}^9$.

(4) Case of $|K| = 8$, i.e. $H(\ell_1) = \{h_0, h_2\}$, $H(\ell_2) = \{h_0, h_2\}$.

$K^8 = \{h_0, h_1, h_2, h_3, s_0, s_1, t_0, t_1\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1)$.

$s_0^- = s_0$, $s_1^- = s_1$, $t_0^- = t_0$, $t_1^- = t_1$, $\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{t_0}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}$.

(5) Case of $|K| = 7$.

(a) When $H(\ell_1) = H$, $H(\ell_2) = \{h_0, h_2\}$, i.e.

$K_a^7 = \{h_0, h_1, h_2, h_3, s_0, t_0, t_1\}$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1)$.

(i) $K = K^7_{a1}$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_1)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1}.$$

(ii) $K = K^7_{a2}$ $(s_0^- = s_0,\ t_0^- = t_1,\ t_1^- = t_0)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1}.$$

(b) When $H(\ell_1) = \{h_0, h_2\},\ H(\ell_2) = H,$ in a similar way, we have $K^7_{b1}$ and $K^7_{b2}$.

(6) Case of $|K| = 6$, i.e. $H(\ell_1) = H,\ H(\ell_2) = H.$

$K^5 = H \vee L = \{h_0, h_1, h_2, h_3, s_0, t_0\}$ which is the join of $H$ by $L$ and characterized by

$$s_0^- = s_0,\ t_0^- = t_0,\ \varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{t_0},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{8}\varepsilon_{h_0} + \frac{1}{8}\varepsilon_{h_1} + \frac{1}{8}\varepsilon_{h_2} + \frac{1}{8}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}.$$

**Remark.** (1) We remark that $H \times L = K^{12}_a \cong K^{12}_b \cong K^{12}_c$, $K^9_{a1} \cong K^9_{a3}$, $K^9_{a2} \cong K^9_{a4}$, $K^9_{b1} \cong K^9_{b3}$ and $K^9_{b2} \cong K^9_{b4}$ as extensions of $L$ by $H$.

(2) $K$ is a splitting extension of $L$ by $H$ if and only if $K \cong H \times L$, $K^8$ or $K^6 = H \vee L$.

(3) $K$ is a weakly splitting extension of $L$ by $H$ if and only if $K \cong H \times L$, $K^8$, $K^6 = H \vee L$, $K^{10}_a$, $K^9_{a1}$, $K^9_{b1}$, $K^7_{a1}$ or $K^7_{b1}$.

**Example 3.10.** $H = \mathbb{Z}_5 = \{h_0, h_1, h_2, h_3, h_4\}$, $h_1^5 = h_0, h_1^- = h_4, h_2^- = h_3$.

(1) Case of $|K| = 15$, i.e. $H(\ell_1) = \{h_0\},\ H(\ell_2) = \{h_0\}.$

$K^{15}_a = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, s_2, s_3, s_4, t_0, t_1, t_2, t_3, t_4\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1, 2, 3, 4)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3, 4)$.

(a) $K = K^{15}_a = H \times L$ $(s_0^- = s_0,\ s_1^- = s_4,\ s_2^- = s_3, t_0^- = t_0,\ t_1^- = t_4,\ t_2^- = t_3)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}.$$

(b) $K = K_b^{15}$ $(s_0^- = s_1,\ s_2^- = s_4,\ s_3^- = s_3,\ t_0^- = t_0,\ t_1^- = t_4,\ t_2^- = t_3)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \tfrac{1}{2}\varepsilon_{h_4} + \tfrac{1}{2}\varepsilon_{t_4},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0} + \tfrac{1}{2}\varepsilon_{s_3},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0} + \tfrac{1}{2}\varepsilon_{t_2}.$$

In a similar way we get
$K_c^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_3^- = s_4,\ t_0^- = t_0,\ t_1^- = t_4,\ t_2^- = t_3)$,
$K_d^{15}$ $(s_0^- = s_3,\ s_1^- = s_2,\ s_4^- = s_4,\ t_0^- = t_0,\ t_1^- = t_4,\ t_2^- = t_3)$,
$K_e^{15}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ t_0^- = t_0,\ t_1^- = t_4,\ t_2^- = t_3)$,
$K_f^{15}$ $(s_0^- = s_1,\ s_2^- = s_4,\ s_3^- = s_3,\ t_0^- = t_1,\ t_2^- = t_4,\ t_3^- = t_3)$,
$K_g^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_3^- = s_4,\ t_0^- = t_1,\ t_2^- = t_4,\ t_3^- = t_3)$,
$K_h^{15}$ $(s_0^- = s_3,\ s_1^- = s_2,\ s_4^- = s_4,\ t_0^- = t_1,\ t_2^- = t_4,\ t_3^- = t_3)$,
$K_i^{15}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ t_0^- = t_1,\ t_2^- = t_4,\ t_3^- = t_3)$,
$K_j^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_3^- = s_4,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_4)$,
$K_k^{15}$ $(s_0^- = s_3,\ s_1^- = s_2,\ s_4^- = s_4,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_4)$,
$K_l^{15}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_4)$,
$K_m^{15}$ $(s_0^- = s_3,\ s_1^- = s_2,\ s_4^- = s_4,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_4)$,
$K_n^{15}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_4)$
and
$K_o^{15}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2)$.

(2) Case of $|K| = 11$.

(a) When $H(\ell_1) = H,\ H(\ell_2) = \{h_0\}$, i.e.

$K_a^{11} = \{h_0, h_1, h_2, h_3, h_4, s_0, t_0, t_1, t_2, t_3, t_4\}$,
$\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3, 4)$.

(i) $K = K_{a1}^{11}$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_4,\ t_2^- = t_3)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \tfrac{1}{10}\varepsilon_{h_0} + \tfrac{1}{10}\varepsilon_{h_1} + \tfrac{1}{10}\varepsilon_{h_2} + \tfrac{1}{10}\varepsilon_{h_3} + \tfrac{1}{10}\varepsilon_{h_4} + \tfrac{1}{10}\varepsilon_{t_0} + \tfrac{1}{10}\varepsilon_{t_1} + \tfrac{1}{10}\varepsilon_{t_2} + \tfrac{1}{10}\varepsilon_{t_3} + \tfrac{1}{10}\varepsilon_{t_4},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{h_0} + \tfrac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \tfrac{1}{2}\varepsilon_{s_0} + \tfrac{1}{10}\varepsilon_{t_0} + \tfrac{1}{10}\varepsilon_{t_1} + \tfrac{1}{10}\varepsilon_{t_2} + \tfrac{1}{10}\varepsilon_{t_3} + \tfrac{1}{10}\varepsilon_{t_4}.$$

In a similar way we get
$K_{a2}^{11}$ $(s_0^- = s_0,\ t_0^- = t_1,\ t_2^- = t_4,\ t_3^- = t_3)$,
$K_{a3}^{11}$ $(s_0^- = s_0,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_4)$,
$K_{a4}^{11}$ $(s_0^- = s_0,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_4)$ and
$K_{a5}^{11}$ $(s_0^- = s_0,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2)$.

Moreover, we get $K_{b1}^{11}$, $K_{b2}^{11}$, $K_{b3}^{11}$, $K_{b4}^{11}$ and $K_{b5}^{11}$.

(3) Case of $|K| = 6$ i.e. $H(\ell_1) = H,\ H(\ell_2) = H$.

$K = H \vee L = \{h_0, h_1, h_2, h_3, h_4, s_0, t_0\}$ which is the join of $H$ by $L$ and characterized by

$s_0^- = s_0,\ t_0^- = t_0,\ \varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{10}\varepsilon_{h_0} + \frac{1}{10}\varepsilon_{h_1} + \frac{1}{10}\varepsilon_{h_2} + \frac{1}{10}\varepsilon_{h_3} + \frac{1}{10}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{10}\varepsilon_{h_0} + \frac{1}{10}\varepsilon_{h_1} + \frac{1}{10}\varepsilon_{h_2} + \frac{1}{10}\varepsilon_{h_3} + \frac{1}{10}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}$.

**Remark.** (1) We remark that $H \times L = K_a^{15} \cong K_b^{15} \cong K_c^{15} \cong K_d^{15} \cong K_e^{15} \cong K_f^{15} \cong K_g^{15} \cong K_h^{15} \cong K_i^{15} \cong K_j^{15} \cong K_k^{15} \cong K_l^{15} \cong K_m^{15} \cong K_n^{15} \cong K_o^{15}$, $K_{a1}^{11} \cong K_{a2}^{11} \cong K_{a3}^{11} \cong K_{a4}^{11} \cong K_{a5}^{11}$ and $K_{b1}^{11} \cong K_{b2}^{11} \cong K_{b3}^{11} \cong K_{b4}^{11} \cong K_{b5}^{11}$ as extensions of $L$ by $H$.

(2) $K$ is a splitting extension of $L$ by $H$ if and only if $K \cong H \times L$ or $H \vee L$.

(3) $K$ is a weakly splitting extension of $L$ by $H$ if and only if $K \cong H \times L$, $H \vee L$, $K_{a1}^{11}$ or $K_{b1}^{11}$.

**Example 3.11.** $H = \mathbb{Z}_6 = \{h_0, h_1, h_2, h_3, h_4, h_5\}$, $h_1^6 = h_0, h_1^- = h_5, h_2^- = h_4, h_3^- = h_3$.

(1) Case of $|K| = 18$ i.e. $H(\ell_1) = \{h_0\}$, $H(\ell_2) = \{h_0\}$.

$K^{18} = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, s_2, s_3, s_4, s_5, t_0, t_1, t_2, t_3, t_4, t_5\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1, 2, 3, 4, 5)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3, 4, 5)$.

(a) $K = K_a^{18} = H \times L$ $(s_0^- = s_0,\ s_1^- = s_5,\ s_2^- = s_4,\ s_3^- = s_3,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$ $= H \times L$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}.$$

(b) $K = K_b^{18}$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_3^- = s_5,\ s_4^- = s_4,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{2}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{t_4},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_1},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_5}.$$

In a similar way we get

$K_c^{18}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ s_5^- = s_5,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$,

$K_d^{18}$ $(s_0^- = s_2,\ s_1^- = s_1,\ s_3^- = s_5,\ s_4^- = s_4,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5,\ t_4^- = t_4)$,

$K_e^{18}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ s_5^- = s_5,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5,\ t_4^- = t_4)$ and

$K_f^{18}$ $(s_0^- = s_4,\ s_1^- = s_3,\ s_2^- = s_2,\ s_5^- = s_5,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2,\ t_5^- = t_5)$.

(2) Case of $|K| = 15$.

(a) When $H(\ell_1) = \{h_0, h_3\}$, $H(\ell_2) = \{h_0\}$ i.e.

$K_a^{15} = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, s_2, t_0, t_1, t_2, t_3, t_4, t_5\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1, 2)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3, 4, 5)$.

(i) $K = K_{a1}^{15}$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_3},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_3}$.

(ii) $K = K_{a2}^{15}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{4}\varepsilon_{t_2} + \frac{1}{4}\varepsilon_{t_5},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_2},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_1} + \frac{1}{4}\varepsilon_{t_4}$.

In a similar way we get
$K_{a3}^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$,
$K_{a4}^{15}$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_1,\ t_2^- = t_5, t_3^- = t_4)$,
$K_{a5}^{15}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_1,\ t_2^- = t_5,\ t_3^- = t_4)$,
$K_{a6}^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_1,\ t_2^- = t_5,\ t_3^- = t_4)$,
$K_{a7}^{15}$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5, t_4^- = t_4)$,
$K_{a8}^{15}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5, t_4^- = t_4)$,
$K_{a9}^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5, t_4^- = t_4)$,
$K_{a10}^{15}$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_5)$,
$K_{a11}^{15}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_5)$,
$K_{a12}^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_5)$,
$K_{a13}^{15}$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2,\ t_5^- = t_5)$,
$K_{a14}^{15}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2,\ t_5^- = t_5)$,
$K_{a15}^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2,\ t_5^- = t_5)$,
$K_{a16}^{15}$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_5,\ t_1^- = t_4,\ t_2^- = t_3)$,
$K_{a17}^{15}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_5,\ t_1^- = t_4,\ t_2^- = t_3)$ and
$K_{a18}^{15}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_5,\ t_1^- = t_4,\ t_2^- = t_3)$.

(b) When $H(\ell_1) = \{h_0\}$, $H(\ell_2) = \{h_0, h_3\}$, in a similar way, we have $K_{b1}^{15}$, $K_{b2}^{15}$, …, $K_{b18}^{15}$.

(3) Case of $|K| = 14$.

(a) When $H(\ell_1) = \{h_0, h_2, h_4\}$, $H(\ell_2) = \{h_0\}$ i.e.

$K_a^{14} = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, t_0, t_1, t_2, t_3, t_4, t_5\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3, 4, 5)$.

(i) $K = K_{a1}^{14}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_2} + \frac{1}{6}\varepsilon_{t_4},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_2} + \frac{1}{6}\varepsilon_{t_4}.$$

(ii) $K = K_{a2}^{14}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5,\ t_4^- = t_4)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_2} + \frac{1}{6}\varepsilon_{t_4},$$
$$\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_2} + \frac{1}{6}\varepsilon_{t_4}.$$

(iii) $K = K_{a3}^{14}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2,\ t_5^- = t_5)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_2} + \frac{1}{6}\varepsilon_{t_4},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_2} + \frac{1}{6}\varepsilon_{t_4}.$$

(b) When $H(\ell_1) = \{h_0\}$ , $H(\ell_2) = \{h_0, h_2, h_4\}$, in a similar way, we have $K_{b1}^{14}$, $K_{b2}^{14}$ and $K_{b3}^{14}$.

(4) Case of $|K| = 13$.

(a) When $H(\ell_1) = H,\ H(\ell_2) = \{h_0\}$ i.e.

$K_a^{13} = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, t_0, t_1, t_2, t_3, t_4, t_5\}$,
$\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2, 3, 4, 5)$.

(i) $K = K_{a1}^{13}$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_5,\ t_2^- = t_4,\ t_3^- = t_3)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_0} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}.$$

(ii) $K = K_{a2}^{13}$ $(s_0^- = s_0,\ t_0^- = t_1,\ t_2^- = t_5,\ t_3^- = t_4)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}.$$

(iii) $K = K_{a3}^{13}$ $(s_0^- = s_0,\ t_0^- = t_2,\ t_1^- = t_1,\ t_3^- = t_5)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5},\ \varepsilon_{t_0} * \varepsilon_{t_0} =$$

$\frac{1}{2}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$.

(iv) $K = K^{13}_{a4}$ $(s_0^- = s_0,\ t_0^- = t_3,\ t_1^- = t_2,\ t_4^- = t_5)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$.

(v) $K = K^{13}_{a5}$ $(s_0^- = s_0,\ t_0^- = t_4,\ t_1^- = t_3,\ t_2^- = t_2,\ t_5^- = t_5)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_2} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$.

(vi) $K = K^{13}_{a6}$ $(s_0^- = s_0,\ t_0^- = t_5,\ t_1^- = t_4,\ t_2^- = t_3)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{12}\varepsilon_{t_0} + \frac{1}{12}\varepsilon_{t_1} + \frac{1}{12}\varepsilon_{t_2} + \frac{1}{12}\varepsilon_{t_3} + \frac{1}{12}\varepsilon_{t_4} + \frac{1}{12}\varepsilon_{t_5}$.

(b) When $H(\ell_1) = \{h_0\}$, $H(\ell_2) = H$, in a similar way, we have $K^{13}_{b1}$, $K^{13}_{b2}$, $K^{13}_{b3}$, $K^{13}_{b4}$, $K^{13}_{b5}$ and $K^{13}_{b6}$.

(5) Case of $|K| = 12$ i.e. $H(\ell_1) = \{h_0, h_3\}$, $H(\ell_2) = \{h_0, h_3\}$.

$K^{12} = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, s_2, t_0, t_1, t_2\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1, 2)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1, 2)$.

(a) $K = K^{12}_a$ $(s_0^- = s_0,\ s_1^- = s_2,\ t_0^- = t_0,\ t_1^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{t_0}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}$.

(b) $K = K^{12}_b$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_0,\ t_1^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_1}$.

(c) $K = K_c^{12}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_0,\ t_1^- = t_2)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_2}.$$

(d) $K = K_d^{12}$ $(s_0^- = s_1,\ s_2^- = s_2,\ t_0^- = t_1,\ t_2^- = t_2)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{s_1},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_1} + \frac{1}{2}\varepsilon_{t_1}.$$

(e) $K = K_e^{12}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_1,\ t_2^- = t_2)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_1} + \frac{1}{2}\varepsilon_{t_2}.$$

(f) $K = K_f^{12}$ $(s_0^- = s_2,\ s_1^- = s_1,\ t_0^- = t_2,\ t_1^- = t_1)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{t_2},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{s_2},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_2} + \frac{1}{2}\varepsilon_{t_2}.$$

(6) Case of $|K| = 11$.

(a) When $H(\ell_1) = \{h_0, h_2, h_4\}$, $H(\ell_2) = \{h_0, h_3\}$ i.e.

$$K_a^{11} = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, t_0, t_1, t_2\},$$
$$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}\ (k = 0, 1)\ ,\ \varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}\ (j = 0, 1, 2).$$

(i) $K = K_{a1}^{11}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_0,\ t_1^- = t_2)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}.$$

(ii) $K = K_{a2}^{11}$ $(s_0^- = s_1,\ t_0^- = t_0,\ t_1^- = t_2)$ which is characterized by

$$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_3} + \frac{1}{6}\varepsilon_{h_5} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}.$$

(iii) $K = K_{a3}^{11}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_1,\ t_2^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$.

(iv) $K = K^{11}_{a4}$ $(s_0^- = s_1,\ t_0^- = t_1,\ t_2^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_3} + \frac{1}{6}\varepsilon_{h_5} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$.

(v) $K = K^{11}_{a5}$ $(s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_2,\ t_1^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$.

(vi) $K = K^{11}_{a6}$ $(s_0^- = s_1,\ t_0^- = t_2,\ t_1^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_3} + \frac{1}{6}\varepsilon_{h_5} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$.

(b) When $H(\ell_1) = \{h_0, h_3\}$, $H(\ell_2) = \{h_0, h_2, h_4\}$, in a similar way, we have $K^{11}_{b1}$, $K^{11}_{b2}$, $K^{11}_{b3}$, $K^{11}_{b4}$, $K^{11}_{b5}$ and $K^{11}_{b6}$.

(7) Case of $|K| = 10$.

(a) When $H(\ell_1) = H$, $H(\ell_2) = \{h_0, h_3\}$ i.e.

$K^{10}_a = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, t_0, t_1, t_2\}$,
$\varepsilon_{t_k} = \varepsilon_{h_k} * \varepsilon_{t_0}$ $(k = 0, 1, 2)$.

(i) $K = K^{10}_{a1}$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_0} + \frac{1}{4}\varepsilon_{h_3} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$.

(ii) $K = K^{10}_{a2}$ $(s_0^- = s_0,\ t_0^- = t_1,\ t_2^- = t_2)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$, $\varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_2} + \frac{1}{4}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{s_0}$, $\varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}$.

(iii) $K = K^{10}_{a3}$ $(s_0^- = s_0,\ t_0^- = t_2,\ t_1^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{h_1} + \frac{1}{4}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{6}\varepsilon_{t_0} + \frac{1}{6}\varepsilon_{t_1} + \frac{1}{6}\varepsilon_{t_2}.$

(b) When $H(\ell_1) = \{h_0, h_3\}$ , $H(\ell_2) = H,$ in a similar way, we have $K^{10}_{b1}$, $K^{10}_{b2}$ and $K^{10}_{b3}$.

(c) When $H(\ell_1) = \{h_0, h_2, h_4\}$, $H(\ell_2) = \{h_0, h_2, h_4\}$ i.e.

$K^{10}_c = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, s_1, t_0, t_1\}$,
$\varepsilon_{s_k} = \varepsilon_{h_k} * \varepsilon_{s_0}$ $(k = 0, 1)$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1)$.

$s_0^- = s_0,\ s_1^- = s_1,\ t_0^- = t_0,\ t_1^- = t_1,\ \varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{4}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{s_1} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1}.$

(8) Case of $|K| = 9$.

(a) When $H(\ell_1) = H$, $H(\ell_2) = \{h_0, h_2, h_4\}$ i.e.

$K^9_a = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, t_0, t_1\}$, $\varepsilon_{t_j} = \varepsilon_{h_j} * \varepsilon_{t_0}$ $(j = 0, 1)$.

(i) $K = K^9_{a1}$ $(s_0^- = s_0,\ t_0^- = t_0,\ t_1^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{6}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_2} + \frac{1}{6}\varepsilon_{h_4} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1}.$

(ii) $K = K^9_{a2}$ $(s_0^- = s_0,\ t_0^- = t_1)$ which is characterized by

$\varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{6}\varepsilon_{h_1} + \frac{1}{6}\varepsilon_{h_3} + \frac{1}{6}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{4}\varepsilon_{t_0} + \frac{1}{4}\varepsilon_{t_1}.$

(b) When $H(\ell_1) = \{h_0, h_2, h_4\}$, $H(\ell_2) = H,$ in a similar way, we have $K^9_{b1}$ and $K^9_{b2}$.

(9) Case of $|K| = 8$ i.e. $H(\ell_1) = H$, $H(\ell_2) = H$.

$K = H \vee L = \{h_0, h_1, h_2, h_3, h_4, h_5, s_0, t_0\}$ which is the join of $H$ by $L$ and characterized by

$s_0^- = s_0,\ t_0^- = t_0,\ \varepsilon_{s_0} * \varepsilon_{s_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{t_0},\ \varepsilon_{t_0} * \varepsilon_{t_0} = \frac{1}{12}\varepsilon_{h_0} + \frac{1}{12}\varepsilon_{h_1} + \frac{1}{12}\varepsilon_{h_2} + \frac{1}{12}\varepsilon_{h_3} + \frac{1}{12}\varepsilon_{h_4} + \frac{1}{12}\varepsilon_{h_5} + \frac{1}{2}\varepsilon_{s_0},\ \varepsilon_{s_0} * \varepsilon_{t_0} = \frac{1}{2}\varepsilon_{s_0} + \frac{1}{2}\varepsilon_{t_0}.$

**Remark.** (1) We remark that $H \times L = K_a^{18} \cong K_b^{18} \cong K_c^{18} \cong K_d^{18} \cong K_e^{18} \cong K_f^{18}$, $K_{a1}^{15} \cong K_{a2}^{15} \cong K_{a3}^{15} \cong K_{a4}^{15} \cong K_{a5}^{15} \cong K_{a6}^{15} \cong K_{a7}^{15} \cong K_{a8}^{15} \cong K_{a9}^{15} \cong K_{a10}^{15} \cong K_{a11}^{15} \cong K_{a12}^{15} \cong K_{a13}^{15} \cong K_{a14}^{15} \cong K_{a15}^{15} \cong K_{a16}^{15} \cong K_{a17}^{15} \cong K_{a18}^{15}$, $K_{b1}^{15} \cong K_{b2}^{15} \cong K_{b3}^{15} \cong K_{b4}^{15} \cong K_{b5}^{15} \cong K_{b6}^{15} \cong K_{b7}^{15} \cong K_{b8}^{15} \cong K_{b9}^{15} \cong K_{b10}^{15} \cong K_{b11}^{15} \cong K_{b12}^{15} \cong K_{b13}^{15} \cong K_{b14}^{15} \cong K_{b15}^{15} \cong K_{b16}^{15} \cong K_{b17}^{15} \cong K_{b18}^{15}$, $K_{a1}^{14} \cong K_{a2}^{14} \cong K_{a3}^{14}$, $K_{b1}^{14} \cong K_{b2}^{14} \cong K_{b3}^{14}$, $K_{a1}^{13} \cong K_{a3}^{13} \cong K_{a5}^{13}$, $K_{a2}^{13} \cong K_{a4}^{13} \cong K_{a6}^{13}$, $K_{b1}^{13} \cong K_{b3}^{13} \cong K_{b5}^{13}$, $K_{b2}^{13} \cong K_{b4}^{13} \cong K_{b6}^{13}$, $K_a^{12} \cong K_b^{12} \cong K_c^{12} \cong K_d^{12} \cong K_e^{12} \cong K_f^{12}$, $K_{a1}^{11} \cong K_{a2}^{11} \cong K_{a3}^{11}$ and $K_{b1}^{11} \cong K_{b2}^{11} \cong K_{b3}^{11}$ as extensions of $L$ by $H$.

(2) $K$ is a splitting extension of $L$ by $H$ if and only if $K \cong H \times L$, $K_a^{12}$, $K_c^{10}$or $H \vee L$.

(3) $K$ is a weakly splitting extension of $L$ by $H$ if and only if $K \cong H \times L$, $K_a^{12}$, $K_c^{10}$, $H \vee L$, $K_{a1}^{15}$, $K_{b1}^{15}$, $K_{a1}^{14}$, $K_{b1}^{14}$, $K_{a1}^{13}$, $K_{b1}^{13}$, $K_{a1}^{11}$, $K_{b1}^{11}$, $K_{a1}^{10}$, $K_{b1}^{10}$, $K_{a1}^{9}$, $K_{b1}^{9}$ or $H \vee L$.

## 3.2. **Extensions of hypergroups of order two by locally compact abelian groups.**

3.2.1. *The structure of extension hypergroups.* Let $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ be a hypergroup of order two with the convolution $\circ$ on $M^b(L)$ where $\ell_0$ is unit of $L$. Since the hypergroup structure of $L$ is determined by

$$\delta_{\ell_1} \circ \delta_{\ell_1} = q\delta_{\ell_0} + (1-q)\delta_{\ell_1}, \quad 0 < q \le 1$$

where $\delta_{\ell_i}$ is the Dirac measure at $\ell_i \in L$. Let $H$ be a locally compact abelian group with unit $h_0$.

We will investigate the structure of extensions $K$ of $L = \mathbb{Z}_q(2)$ by $H$. Let $\varphi$ be a continuous homomorphism from a commutative hypergroup $K$ onto $L$ such that Ker $\varphi = H$, where $H$ is assumed to be a closed subgroup of $K$. Then $K$ is written as the disjoint union of the sets $H = \varphi^{-1}(\ell_0)$ and $S_1 := \varphi^{-1}(\ell_1)$. Fix $s_0 \in S_1$.

**Lemma 3.12.** *For each $s \in S_1$, there exists $h \in H$ such that $\varepsilon_s = \varepsilon_h * \varepsilon_{s_0}$.*

*Proof.* For $s_i' \in S_i$, there exists $h \in H$ such that $h \in \text{supp}(\varepsilon_{s_i}^- * \varepsilon_{s_i'})$ since $\varphi(\varepsilon_{s_i}^- * \varepsilon_{s_i'}) = \delta_{\ell_i}^- \circ \delta_{\ell_i}$. Hence we see that $h_0 = h^- h \in \text{supp}((\varepsilon_h * \varepsilon_{s_i})^- * \varepsilon_{s_i'})$. This implies that $s_i' \in \text{supp}(\varepsilon_h * \varepsilon_{s_i})$. Then

$$\text{supp}(\varepsilon_h^- * \varepsilon_{s_i'}) \subset \text{supp}(\varepsilon_h^- * \varepsilon_h * \varepsilon_{s_i}) = \text{supp}(\varepsilon_{h_0} * \varepsilon_{s_i}) = \text{supp}(\varepsilon_{s_i}) = \{s_i\}.$$

Hence we see that $\varepsilon_h^- * \varepsilon_{s_i'} = \varepsilon_{s_i}$, namely $\varepsilon_{s_i'} = \varepsilon_h * \varepsilon_{s_i}$. $\square$

Let $H(\ell_1)$ denote the stability group of $H$ at $s_0 \in S_1$, i.e.

$$H(\ell_1) = \{h \in H : \varepsilon_h * \varepsilon_{s_0} = \varepsilon_{s_0}\}.$$

**Lemma 3.13.** $H \cap \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0}) = H(\ell_1)$.

*Proof.* Take $h \in H \cap \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0})$. Then $h_0 = h^- h \in \mathrm{supp}(\varepsilon_h^- * \varepsilon_{s_0}^- * \varepsilon_{s_0})$ $= \mathrm{supp}((\varepsilon_h * \varepsilon_{s_0})^- * \varepsilon_{s_0})$. Hence we get $s_0 \in \mathrm{supp}(\varepsilon_h * \varepsilon_{s_0})$. Therefore

$$\mathrm{supp}(\varepsilon_h^- * \varepsilon_{s_0}) \subset \mathrm{supp}(\varepsilon_h^- * \varepsilon_h * \varepsilon_{s_0}) = \mathrm{supp}(\varepsilon_{h_0} * \varepsilon_{s_0}) = \mathrm{supp}(\varepsilon_{s_0}) = \{s_0\},$$

since $H$ is a group. Then, we see that $\varepsilon_h^- * \varepsilon_{s_0} = \varepsilon_{s_0}$, namely $\varepsilon_h * \varepsilon_{s_0} = \varepsilon_{s_0}$ which implies that $h \in H(\ell_1)$.

Conversely, we show that $H(\ell_1) \subset H \cap \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0})$. Take $k \in H(\ell_1)$, then $\varepsilon_k * \varepsilon_{s_0} = \varepsilon_{s_0}$. Since $h_0 \in \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0})$, we see

$$k \in \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_k * \varepsilon_{s_0}) = \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0}).$$

□

**Lemma 3.14.** *$H(\ell_1)$ is a compact subgroup of $H$.*

*Proof.* Since $\mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0})$ is compact by the axiom (3) of locally compact hypergroups and $H$ is a closed subgroup of $K$, $H \cap \mathrm{supp}(\varepsilon_{s_0}^- * \varepsilon_{s_0})$ must be compact. Hence we have that $H(\ell_1)$ is a compact subgroup of $H$ by Lemma 3.13. □

Let $\omega(\ell_1)$ denote the normalized Haar measure of $H(\ell_1)$. We note that $\omega(\ell_1)$ has the following properties.

(1) $\omega(\ell_1) * \varepsilon_h = \omega(\ell_1)$ for $h \in H(\ell_1)$.

(2) $\omega(\ell_1) * \omega(\ell_1) = \omega(\ell_1)$.

(3) $\omega(\ell_1)^- = \omega(\ell_1)$.

We denote $H/H(\ell_1)$ by $Q(\ell_1)$.

**Proposition 3.15.** *If $K$ is a commutative hypergroup extension of a hypergroup $\mathbb{Z}_q(2)$ of order two by a locally compact abelian group $H$ and $\varphi$ a continuous homomorphism from $K$ onto $\mathbb{Z}_q(2)$ such that Ker $\varphi = H$, we have the conditions* (0) – (3) *as follows.*

(0) *$K$ is the disjoint union of the sets $H = \varphi^{-1}(\ell_0)$ and $S_1 = \varphi^{-1}(\ell_1)$.*

(1) $\varepsilon_{s_0}^- = \varepsilon_h^- * \varepsilon_{s_0}$ *for some $h \in Q(\ell_1)$.*

(2) $\varepsilon_{s_0} * \varepsilon_{s_0} = q\varepsilon_h * \omega(\ell_1) + (1-q)\varepsilon_h * c * \varepsilon_{s_0}$ *for some $c \in M^1(H)$.*

(3) $c * \omega(\ell_1) = c$ *and* $c^- = \varepsilon_h * c$.

*Proof.* (1) Since ${s_0}^- \in S_1$ by the relation $\ell_1^- = \ell_1$, one can take $h \in Q(\ell_1)$ such that

$$\varepsilon_{s_0}^- = \varepsilon_h^- * \varepsilon_{s_0}$$

by Lemma 3.12.

(2) It is easy to see that $\varepsilon_{s_0}^- * \varepsilon_{s_0}$ is written as

$$\varepsilon_{s_0}^- * \varepsilon_{s_0} = qc_0 + (1-q)c * \varepsilon_{s_0}$$

for some $c_0, c \in M^1(H)$. By the fact that $\omega(\ell_1)*c_0 = c_0$ and $\mathrm{supp}(c_0) = H(\ell_1)$ by Lemma 3.13, we have $c_0 = \omega(\ell_1)$. Hence we obtain

$$\varepsilon_{s_0}^- * \varepsilon_{s_0} = q\omega(\ell_1) + (1-q)c * \varepsilon_{s_0},$$

namely $\varepsilon_{s_0} * \varepsilon_{s_0} = q\varepsilon_h * \omega(\ell_1) + (1-q)\varepsilon_h * c * \varepsilon_{s_0}$ by (1).

(3) We can take $c \in M^1(H)$ as $c * \omega(\ell_1)$. Then we obtain $c * \omega(\ell_1) = c$ and $c^- = \varepsilon_h * c$. □

We see that all extensions $K$ of $\mathbb{Z}_q(2)$ by $H$ are characterized by

$$H(\ell_1),\ s_0 \in S_1,\ h \in H,\ c \in M^1(H)$$

satisfying the conditions described in Proposition 3.15. Therefore we denote such an extension $K$ by $K(H(\ell_1), s_0, h, c)$.

When we take $H_1(\ell_1)$, $r_0 \in S_1$, $k \in H$ and $d \in M^1(H)$ satisfying the conditions (0) – (3) in Proposition 3.15, we have another extension $K(H_1(\ell_1), r_0, k, d)$ of $\mathbb{Z}_q(2)$ by $H$.

**Proposition 3.16.** *Two extensions $K(H(\ell_1), s_0, h, c)$ and $K(H_1(\ell_1), r_0, k, d)$ of $\mathbb{Z}_q(2)$ by $H$ are mutually equivalent as extensions if and only if $H(\ell_1) = H_1(\ell_1)$ and there exists $b \in H$ such that $\varepsilon_k * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \varepsilon_h * \omega(\ell_1)$ and $d = \varepsilon_b * c$.*

*Proof.* Suppose that $K_1 = K(H(\ell_1), s_0, h, c)$ is equivalent to $K_2 = K(H_1(\ell_1), r_0, k, d)$ as extensions. Let $\varphi_i$ be a continuous homomorphism from $K_i$ onto $\mathbb{Z}_q(2)$ $(i = 1, 2)$. Let $K_1 = H \cup S_1$ and $K_2 = H \cup R_1$ where $S_1 = \varphi_1^{-1}(\ell_1)$ and $R_1 = \varphi_2^{-1}(\ell_1)$. Let $\psi$ be an isomorphism from $K_1$ to $K_2$ such that $\psi(h) = h$ for any $h \in H$ and $\varphi_2 \circ \psi = \varphi_1$. Put $\psi(s_0) = u_0 \in R_1$. Since $\varepsilon_h * \varepsilon_{u_0} = \varepsilon_{u_0}$ for $h \in H(\ell_1)$, we see that $H(\ell_1) = H_1(\ell_1)$. For $u_0 \in R_1$, there exists $b \in Q(\ell_1)$ such that $\varepsilon_{u_0} = \varepsilon_b * \varepsilon_{r_0}$ by Lemma 3.12. Then, by $\psi(\varepsilon_{s_0}^- * \varepsilon_{s_0}) = \varepsilon_{r_0}^- * \varepsilon_{r_0}$, it is easy to see that $b$ satisfies $\varepsilon_k * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \varepsilon_h * \omega(\ell_1)$ and $d = \varepsilon_b * c$.

Conversely, we assume that $H(\ell_1) = H_1(\ell_1)$ and there exists $b \in H$ such that

$$\varepsilon_k * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \varepsilon_h * \omega(\ell_1) \text{ and } d = \varepsilon_b * c.$$

Take $u_0 \in R_1$ given by $\varepsilon_{u_0} = \varepsilon_b * \varepsilon_{r_0}$. We put a map $\psi$ from $K_1$ to $K_2$ such that

$$\psi(\varepsilon_h) = \varepsilon_h \text{ and } \psi(\varepsilon_{s_0}) = \varepsilon_{u_0}$$

for any $h \in H$. Let $\varphi_i$ be a continuous homomorphism from $K_i$ onto $\mathbb{Z}_q(2)$ $(i = 1, 2)$. Then it is clear that $\psi$ is isomorphism from $K_1$ to $K_2$ such that $\varphi_2 \circ \psi = \varphi_1$. □

3.2.2. *Construction of the model.* Let $H$ be a locally compact abelian group with unit $h_0$ and $L$ a hypergroup of order two $\mathbb{Z}_q(2)$ with unit $\ell_0$. Take a compact subgroup $H(\ell_1)$ of $H$ and denote the quotient space $H/H(\ell_1)$ by $Q(\ell_1)$. The normalized Haar measure of $H(\ell_1)$ is denoted by $\omega(\ell_1)$. Let $K$ be the disjoint union of the sets $H$ and $Q(\ell_1)$, namely

$$\begin{aligned} K &= H \cup Q(\ell_1) \\ &= \{(\ell_0, h_1), (\ell_1, h_2 * H(\ell_1)) : h_1, h_2 \in H\}. \end{aligned}$$

The Dirac measures at $(\ell_0, h_1)$ and $(\ell_1, h_2 * H(\ell_1)) \in K$ are realized respectively in $M^b(L) \otimes M^b(H)$ by

$$\delta_{\ell_0} \otimes \varepsilon_{h_1} \text{ and } \delta_{\ell_1} \otimes (\varepsilon_{h_2} * \omega(\ell_1)).$$

Take and fix $f \in H$. We define the involution $^-$ of $K$ by

$$(\ell_0, h)^- = (\ell_0, h^{-1}) \text{ and } (\ell_1, h * H(\ell_1))^- = (\ell_1, h^{-1} * f^{-1} * H(\ell_1)).$$

Moreover we define the convolution $*_c$ of $K$ in $M^b(L) \otimes M^b(H)$ associated with $c \in M^1(H)$ such that $c * \omega(\ell_1) = c$ and $(\delta_{\ell_0} \otimes c)^- = (\delta_{\ell_0} \otimes \varepsilon_f * c)$.

(1) $(\delta_{\ell_0} \otimes \varepsilon_{h_1}) *_c (\delta_{\ell_0} \otimes \varepsilon_{h_2}) = \delta_{\ell_0} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2})$.

(2) $(\delta_{\ell_0} \otimes \varepsilon_{h_1}) *_c (\delta_{\ell_1} \otimes (\varepsilon_{h_2} * \omega(\ell_1))) = (\delta_{\ell_1} \otimes (\varepsilon_{h_2} * \omega(\ell_1))) *_c (\delta_{\ell_0} \otimes \varepsilon_{h_1})$
$= \delta_{\ell_1} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2} * \omega(\ell_1))$.

(3) $(\delta_{\ell_1} \otimes (\varepsilon_{h_1} * \omega(\ell_1))) *_c (\delta_{\ell_1} \otimes (\varepsilon_{h_2} * \omega(\ell_1)))$
$= q\delta_{\ell_0} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2} * \varepsilon_f * \omega(\ell_1)) + (1 - q)\delta_{\ell_1} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2} * \varepsilon_f * c)$.

Since the model $K$ is determined by the compact subgroups $H(\ell_1)$ of $H$, $f \in H$ and $c \in M^1(H)$, we denote $K$ by $K(H(\ell_1), f, c)$.

Now we arrive at the main theorem of Section 3.2.

**Theorem 3.17.** *Under the preceding arguments we have the following.*

(1) *The model* $K(H(\ell_1), f, c)$ *is a commutative hypergroup and an extension of* $\mathbb{Z}_q(2)$ *by* $H$.

(2) *All extensions* $K$ *of* $L$ *by* $H$ *are equivalent to* $K(H(\ell_1), f, c)$ *as extensions.*

(3) *The extensions* $K(H(\ell_1), f, c)$ *and* $K(H_1(\ell_1), g, d)$ *are equivalent as extensions if and only if there exists* $b \in H$ *such that* $\varepsilon_g * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \varepsilon_f * \omega(\ell_1)$ *and* $d = \varepsilon_b^- * c$.

(4) *The extension* $K(H(\ell_1), f, c)$ *is splitting if and only if there exists* $b \in H$ *such that* $\varepsilon_f * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \omega(\ell_1)$ *and* $c = \varepsilon_b^- * \omega(\ell_1)$.

*Proof.* (1) Since $H$ is a locally compact group and $H(\ell_1)$ is a compact subgroup of $H$, the quotient space $Q(\ell_1) = H/H(\ell_1)$ is a locally compact space. Then the disjoint union $K(H(\ell_1), f, c) = H \cup Q(\ell_1)$ is also a locally compact space. It is clear that the definition of the convolution $*$ and the involution $^-$ is well defined. By the definition of $K(H(\ell_1), f, c)$, we know that the convolution $*$ and the involution $^-$ are continuous from the fact that group operation and inverse operation of $H$ as well as an action of $H$ on $Q(\ell_1)$ are all continuous.

The compactness of the support of $(\delta_{\ell_1} \otimes \omega(\ell_1)) * (\delta_{\ell_1} \otimes \omega(\ell_1))$ is assured by the fact that $H(\ell_1)$ is compact. Since it is easy to check other axioms of hypergroup, we know that $K(H(\ell_1), f, c)$ holds axioms of a hypergroup.

Let $\varphi$ be a mapping from $K(H(\ell_1), f, c)$ onto $\mathbb{Z}_q(2)$ such that $\varphi(\ell_0, h) = \ell_0$ and $\varphi(\ell_1, h * H(\ell_1)) = \ell_1$ for $h \in H$. Then it is easy to see that $\varphi$ is a continuous hypergroup homomorphism from $K(H(\ell_1), f, c)$ onto $\mathbb{Z}_q(2)$ such that $\operatorname{Ker} \varphi = H$. This implies that $K(H(\ell_1), f, c)$ is an extension of $L$ by $H$.

(2) Take an extension $K$ of $\mathbb{Z}_q(2)$ by $H$. Then $K$ is characterized as $K = K(H(\ell_1), s_0, h, c)$ by Proposition 3.15. Put $\psi$ be a mapping from $K$ onto the model $K(H(\ell_1), f, c)$ given by $\psi(\varepsilon_h) = \delta_{\ell_0} \otimes \varepsilon_h$ and $\psi(\varepsilon_h * \varepsilon_{s_0}) = \delta_{\ell_1} \otimes (\varepsilon_h * \omega(\ell_2))$ for $h \in H$. It is easy to see that the mapping $\psi$ is an involutive isomorphism such that $\varphi_2 \circ \psi = \varphi$ where $\varphi_2$ is the continuous homomorphism $K(H(\ell_1), f, c)$ onto $\mathbb{Z}_q(2)$.

(3) We note that $K_1 = K(H(\ell_1), f, c)$ is equal to $K(H(\ell_1), s_0, h, c)$ such that $h = (\ell_0, h), s_0 = (\ell_1, H(\ell_1))$ and $[h] = [f^-]$ in $Q(\ell_1)$, and $K_2 = K(H_1(\ell_1), g, d)$ is also similar. We assume that $K_1$ is equivalent to $K_2$ as extensions. By Proposition 3.16, there exists $b \in H$ such that

$$\delta_{\ell_0} \otimes \varepsilon_f^- * \omega(\ell_1) = \delta_{\ell_0} \otimes (\varepsilon_b * \varepsilon_b * \varepsilon_g^- * \omega(\ell_1)),$$

$$\delta_{\ell_0} \otimes c = \delta_{\ell_0} \otimes (\varepsilon_b * d).$$

Hence we get $\varepsilon_g * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \varepsilon_f * \omega(\ell_1)$ and $d = \varepsilon_b^- * c$.

The converse assertion is clear by Proposition 3.16.

(4) We assume that $K$ is a splitting extension. Then, there exists an injective mapping $\phi$ from $\mathbb{Z}_q(2)$ into $K(H(\ell_1), f, c)$ such that $\phi(\ell_0) = (\ell_0, h_0)$ and $\phi(\ell_1) = (\ell_1, b^- * H(\ell_1))$ for some $b \in Q(\ell_1)$. Since

$$\phi(\delta_{\ell_1}) * \phi(\delta_{\ell_1}) = q\delta_{\ell_0} \otimes (\varepsilon_b^- * \varepsilon_b^- * \varepsilon_f * \omega(\ell_1)) + (1-q)\delta_{\ell_1} \otimes (\varepsilon_b^- * \varepsilon_b^- * \varepsilon_f * c)$$

and

$$\phi(\delta_{\ell_1} \circ \delta_{\ell_1}) * \omega(\ell_1) = q\delta_{\ell_0} \otimes \omega(\ell_1) + (1-q)\delta_{\ell_1} \otimes (\varepsilon_b^- * \omega(\ell_1)),$$

we get

$$\varepsilon_b^- * \varepsilon_b^- * \varepsilon_f * \omega(\ell_1) = \omega(\ell_1),$$
$$\varepsilon_b^- * \varepsilon_b^- * \varepsilon_f * c = \varepsilon_b^- * \omega(\ell_1)$$

by the splitting condition (1). Then we see that $\varepsilon_f * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \omega(\ell_1)$ for the first term. Hence we have $\varepsilon_b^- * \varepsilon_b^- * \varepsilon_f * c = c$. Therefore we know that

$$\varepsilon_f * \omega(\ell_1) = \varepsilon_b * \varepsilon_b * \omega(\ell_1) \text{ and } c = \varepsilon_b^- * \omega(\ell_1).$$

It is easy to check the converse. □

3.2.3. *Applications and examples.* Under these discussions we calculate all extensions $K$ of hypergroups $\mathbb{Z}_q(2)$ of order two by concrete locally compact abelian groups $H$.

**Example 3.18.** $H = \mathbb{R}^n$.

Since the trivial subgroup $\{0\}$ of $\mathbb{R}^n$ is the only compact subgroup of $\mathbb{R}^n$, we get extensions $K$ as follows.

$K(c) = \mathbb{R}^n \cup \mathbb{R}^n := \{(0,h), (1,s) : h, s \in \mathbb{R}^n\}$, where $c \in M^1(\mathbb{R}^n)$ with $c^- = c$.

$$\varepsilon_{(0,h)}^- = \varepsilon_{(0,-h)}, \quad \varepsilon_{(1,s)}^- = \varepsilon_{(1,-s)}, \quad \varepsilon_{(0,h)} * \varepsilon_{(0,k)} = \varepsilon_{(0,h+k)},$$
$$\varepsilon_{(0,h)} * \varepsilon_{(1,s)} = \varepsilon_{(1,h+s)}, \quad \varepsilon_{(1,s)} * \varepsilon_{(1,t)} = q\varepsilon_{(0,0)} + (1-q)c * \varepsilon_{(1,0)}.$$

**Remark.** When $c \in M^1(\mathbb{R}^n)$ is taking by $\varepsilon_{(0,0)}$, then $K(c) = H \times L$ which is a splitting extension. M. Voit determined commutative hypergroup structures on two disjoint real lines $\mathbb{R} \cup \mathbb{R}$ ([V]). We note that the hypergroup structure obtained here coincide with Voit's result since the hypergroup structure of the real line is known to be unique by Hm. Zeuner ([Z]).

**Example 3.19.** $H = \mathbb{Z}^n$.

Since the trivial subgroup $\{0\}$ of $\mathbb{Z}^n$ is the only compact subgroup of $\mathbb{Z}^n$, we get extensions $K = \mathbb{Z}^n \cup \mathbb{Z}^n$ as follows.

Take $f = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \mathbb{Z}^n$ where $\varepsilon_j = 0$ or $1$ for $j = 1, 2, \cdots, n$ and $c \in M^1(\mathbb{Z}^n)$ such that $c^- = \varepsilon_{(0,f)} * c$.

$K(f, c) = \mathbb{Z}^n \cup \mathbb{Z}^n := \{(0, h), (1, s) : h, s \in \mathbb{Z}^n\}$.

$$\varepsilon^-_{(0,h)} = \varepsilon_{(0,-h)}, \quad \varepsilon^-_{(1,s)} = \varepsilon_{(1,-f-s)}, \quad \varepsilon_{(0,h)} * \varepsilon_{(0,k)} = \varepsilon_{(0,h+k)},$$
$$\varepsilon_{(0,h)} * \varepsilon_{(1,s)} = \varepsilon_{(1,s)} * \varepsilon_{(0,h)} = \varepsilon_{(1,h+s)},$$
$$\varepsilon_{(1,s)} * \varepsilon_{(1,t)} = q\varepsilon_{(0,s+t+f)} + (1-q)c * \varepsilon_{(1,s+t+f)}.$$

**Remark.** If $c = \varepsilon_{(0,0)}$, then $K(c) = H \times L$ which is a splitting extension.

**Example 3.20.** $H = \mathbb{T}$.

For a natural number $m$, a real number $h$ is written in the form $h = \frac{2\pi}{m}k + r$, where $k$ is an integer and $0 \leq r < \frac{2\pi}{m}$. Then we denote the residue $r$ by $[h]_m$ i.e. $r = [h]_m$.

We identify $\mathbb{T}$ with $[0, 2\pi)$ by

$$\mathbb{T} \ni e^{i\theta} \longleftrightarrow [\theta]_1 \in [0, 2\pi).$$

Then the product $e^{i\theta_1}e^{i\theta_2}$ in $\mathbb{T}$ corresponds to $[\theta_1 + \theta_2]_1$ in $[0, 2\pi)$. For $h_1, h_2 \in [0, 2\pi)$, we can write $\varepsilon_{h_1} * \varepsilon_{h_2} = \varepsilon_{[h_1+h_2]_1}$.

(1) Case of $H(\ell_1) = \{0\}$.

Then we get extensions $K_1(c) = \mathbb{T} \cup \mathbb{T}$ with $c \in M^1(\mathbb{T})$ such that $c^- = c$, which are similar to the case $H = \mathbb{R}^n$ in Example 3.18.

(2) Case of $H(\ell_1) = H$.

The extension $K_2$ of $L$ by $H$ is the hypergroup join $H \vee L$.

Since $Q(\ell_1) = \{0\}$, $K_2 = \mathbb{T} \cup \{0\} := \{(0, h), (1, 0) : h \in [0, 2\pi)\}$.

$$\varepsilon^-_{(0,h)} = \varepsilon_{(0,[-h]_1)}, \quad \varepsilon^-_{(1,0)} = \varepsilon_{(1,0)}, \quad \varepsilon_{(0,h)} * \varepsilon_{(0,k)} = \varepsilon_{(0,[h+k]_1)},$$
$$\varepsilon_{(0,h)} * \varepsilon_{(1,0)} = \varepsilon_{(1,0)}, \quad \varepsilon_{(1,0)} * \varepsilon_{(1,0)} = qe_H + (1-q)\varepsilon_{(1,0)}$$

where $e_H$ is the normalized Haar measure of $H$.

(3) Case of $H(\ell_1) \cong \mathbb{Z}_n$.

Since $Q(\ell_1) = [0, \frac{1}{n}2\pi)$, $K_3(c) = \mathbb{T} \cup S_1 := \{(0, h), (1, s) : h \in [0, 2\pi), s \in [0, \frac{1}{n}2\pi)\}$ where $c \in M^1(\mathbb{T})$ such that $c^- = c$.

$$\varepsilon^-_{(0,h)} = \varepsilon_{(0,[-h]_1)}, \quad \varepsilon^-_{(1,s)} = \varepsilon_{(1,[-s]_n)}, \quad \varepsilon_{(0,h)} * \varepsilon_{(0,k)} = \varepsilon_{(0,[h+k]_1)},$$
$$\varepsilon_{(0,h)} * \varepsilon_{(1,s)} = \varepsilon_{(1,[h+s]_n)},$$
$$\varepsilon_{(1,s)} * \varepsilon_{(1,t)} = \frac{q}{n}\sum_{l=0}^{n-1} \varepsilon_{(0,[\frac{l}{n}2\pi+s+t]_1)} + (1-q)c * \varepsilon_{(1,[s+t]_n)}.$$

**Remark.** $K_3(c)$ is homeomorphic with $\mathbb{T} \cup \mathbb{T}$. M. Voit [V] determined all commutative hypergroup structures on two disjoint tori $\mathbb{T} \cup \mathbb{T}$. We remark that these extensions obtained here also agree with his result since the hypergroup structure of the one-dimensional torus is known to be unique by Hm. Zeuner [Z]. When we identify $\varepsilon_{(0,h)}$ with Voit's notation $\delta_{(0,e^{ih})}$ and $\varepsilon_{(1,s)}$ with $\delta_{(1,e^{ins})}$, it is easy to check that the both are same.

**Example 3.21.** $H = \mathbb{T}^2$.

We identify $\mathbb{T}^2 = \{(e^{i\theta_1}, e^{i\theta_2}) : \theta_1, \theta_2 \in [0, 2\pi)\}$ with $[0, 2\pi) \times [0, 2\pi)$.

(1) Case of $H(\ell_1) = \{(0, 0)\}$.

Then we get extensions $K_1(c) = \mathbb{T}^2 \cup \mathbb{T}^2$ for $c \in M^1(\mathbb{T}^2)$ with $c^- = c$, which are similar to Example 3.18 and (1) in Example 3.20.

(2) Case of $H(\ell_1) = H$.

The extension $K_2$ of $L$ by $H$ is the hypergroup join $H \vee L$.

(3) Case of $H(\ell_1) \cong \mathbb{Z}_n \times \{0\}$.

Since $S_1 = [0, \frac{1}{n}2\pi) \times [0, 2\pi)$, $K_3(c) = \mathbb{T}^2 \cup S_1 := \{(0, h_1, h_2), (1, s_1, s_2) : h_1, h_2, s_2 \in [0, 2\pi), s_1 \in [0, \frac{1}{n}2\pi)\}$ where $c \in M^1(\mathbb{T}^2)$ such that $c^- = c$.

$\varepsilon^-_{(0,h_1,h_2)} = \varepsilon_{(0,[-h_1]_1,[-h_2]_1)}$, $\varepsilon^-_{(1,s_1,s_2)} = \varepsilon_{(1,[-s_1]_n,[-s_2]_1)}$, $\varepsilon_{(0,h_1,h_2)} * \varepsilon_{(0,k_1,k_2)} = \varepsilon_{(0,[h_1+k_1]_1,[h_2+k_2]_1)}$, $\varepsilon_{(0,h_1,h_2)} * \varepsilon_{(1,s_1,s_2)} = \varepsilon_{(1,[h_1+s_1]_n,[h_2+s_2]_1)}$, $\varepsilon_{(1,s_1,s_2)} * \varepsilon_{(1,t_1,t_2)} = q \cdot \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_{(0,[\frac{j}{n}2\pi+s_1+t_1]_1,[s_2+t_2]_1)} + (1-q)c * \varepsilon_{(1,[s_1+t_1]_n,[s_2+t_2]_1)}$.

(4) Case of $H(\ell_1) \cong \mathbb{Z}_n \times \mathbb{Z}_m$.

Since $S_1 = [0, \frac{1}{n}2\pi) \times [0, \frac{1}{m}2\pi)$, $K_4(c) = \mathbb{T}^2 \cup S_1 := \{(0, h_1, h_2), (1, s_1, s_2) : h_1, h_2 \in [0, 2\pi), s_1 \in [0, \frac{1}{n}2\pi), s_2 \in [0, \frac{1}{m}2\pi)\}$ where $c \in M^1(\mathbb{T}^2)$ such that $c^- = c$.

$\varepsilon^-_{(0,h_1,h_2)} = \varepsilon_{(0,[-h_1]_1,[-h_2]_1)}$, $\varepsilon^-_{(1,s_1,s_2)} = \varepsilon_{(1,[-s_1]_n,[-s_2]_m)}$,

$\varepsilon_{(0,h_1,h_2)} * \varepsilon_{(0,k_1,k_2)} = \varepsilon_{(0,[h_1+k_1]_1,[h_2+k_2]_1)}$,

$\varepsilon_{(0,h_1,h_2)} * \varepsilon_{(1,s_1,s_2)} = \varepsilon_{(1,[h_1+s_1]_n,[h_2+s_2]_m)}$,

$$\varepsilon_{(1,s_1,s_2)} * \varepsilon_{(1,t_1,t_2)} = q \cdot \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \varepsilon_{(0,[\frac{i}{n}2\pi+s_1+t_1]_1,[\frac{j}{m}2\pi+s_2+t_2]_1)} + (1-q)c * \varepsilon_{(1,[s_1+t_1]_n,[s_2+t_2]_m)}.$$

(5) Case of $H(\ell_1) \cong \mathbb{Z}_n \times \mathbb{T}$.

Since $S_1 = [0, \frac{1}{n}2\pi) \times \{0\}$, $K_5(c) = \mathbb{T}^2 \cup S_1 := \{(0, h_1, h_2), (1, s_1, 0) : h_1, h_2 \in [0, 2\pi), s_1 \in [0, \frac{1}{n}2\pi)\}$ where $c \in M^1(\mathbb{T}^2)$ such that $c^- = c$.

$\varepsilon^-_{(0,h_1,h_2)} = \varepsilon_{(0,[-h_1]_1,[-h_2]_1)}$, $\varepsilon^-_{(1,s_1,0)} = \varepsilon_{(1,[-s_1]_n,0)}$,

$\varepsilon_{(0,h_1,h_2)} * \varepsilon_{(0,k_1,k_2)} = \varepsilon_{(0,[h_1+k_1]_1,[h_2+k_2]_1)}$,

$\varepsilon_{(0,h_1,h_2)} * \varepsilon_{(1,s_1,0)} = \varepsilon_{(1,[h_1+s_1]_n,0)}$,

$$\varepsilon_{(1,s_1,0)} * \varepsilon_{(1,t_1,0)} = \frac{q}{n} \sum_{l=0}^{n-1} \varepsilon_{(0,[\frac{l}{n}2\pi+s_1+t_1]_1,0)} * \omega_{(0,0,\mathbb{T})} + (1-q)c * \varepsilon_{(1,[s_1+t_1]_n,0)}.$$

**Remark.** For $K_1(c)$, if $c = \varepsilon_{(0,0)}$, then $K_1(c) = H \times L$ which is a splitting extension. $K_2$ is also a splitting extension.

## 3.3. **Extensions of the Golden hypergroups by locally compact abelian groups.**

3.3.1. *The structures of extension hypergroups.* Let $L = \{\ell_0, \ell_1, \ell_2\}$ be the Golden hypergroup $\mathbb{G}$ with the convolution $\circ$ on $M^b(L)$ where $\ell_0$ is unit of $L$. The hypergroup structure of $L$ is determined by

$$\delta_{\ell_1} \circ \delta_{\ell_1} = \frac{1}{2}\delta_{\ell_0} + \frac{1}{2}\delta_{\ell_2}, \quad \ell_1^- = \ell_1,$$

$$\delta_{\ell_2} \circ \delta_{\ell_2} = \frac{1}{2}\delta_{\ell_0} + \frac{1}{2}\delta_{\ell_1}, \quad \ell_2^- = \ell_2,$$

$$\delta_{\ell_1} \circ \delta_{\ell_2} = \delta_{\ell_2} \circ \delta_{\ell_1} = \frac{1}{2}\delta_{\ell_1} + \frac{1}{2}\delta_{\ell_2},$$

where $\delta_{\ell_i}$ is the Dirac measure at $\ell_i \in L$. Let $H$ be a locally compact abelian group with unit $h_0$.

We will investigate the structure of extensions $K$ of $L$ by $H$. Let $\varphi$ be a continuous homomorphism from a commutative hypergroup $K$ onto $L$ such that Ker $\varphi = H$, where $H$ is assumed to be a closed subgroup of $K$. Then $K$ is written as the disjoint union of the sets $H = \varphi^{-1}(\ell_0)$, $S_1 := \varphi^{-1}(\ell_1)$ and $S_2 := \varphi^{-1}(\ell_2)$. Fix $s_1 \in S_1$ and $s_2 \in S_2$.

Let $H(\ell_i)$ denote the stability group of $H$ at $s_i \in S_i$, i.e.

$$H(\ell_i) := \{h \in H : \varepsilon_h * \varepsilon_{s_i} = \varepsilon_{s_i}\}.$$

We note that $H(\ell_i)$ does not depend on the choice of $s_i \in S_i$ and that $H(\ell_i)$ is a compact subgroup of $H$ by Lemma 3.14 for $i = 1, 2$.

Let $\omega_{H_0}$ denote the normalized Haar measure of a compact subgroup $H_0$ of $H$. The next lemma is useful for our arguments hereafter.

**Lemma 3.22.** *For a compact subgroup $H_0$ of $H$, if a probability measure $\mu$ on $H$ satisfies that supp($\mu$) $\subset H_0$ and $\omega_{H_0} * \mu = \mu$, then we have $\mu = \omega_{H_0}$.*

*Proof.* For $\mu \in M^1(H)$ with $\mathrm{supp}(\mu) \subset H_0$, we can write $\mu = \displaystyle\int_{H_0} \varepsilon_h d\mu(h)$. We assume that $\mu = \omega_{H_0} * \mu$. Then, we have

$$\mu = \omega_{H_0} * \mu = \omega_{H_0} * \int_{H_0} \varepsilon_h d\mu(h) = \int_{H_0} \omega_{H_0} * \varepsilon_h d\mu(h) = \int_{H_0} \omega_{H_0} d\mu(h)$$

$$= \omega_{H_0} \int_{H_0} 1 d\mu(h) = \omega_{H_0} * \mu(H_0) = \omega_{H_0} * 1 = \omega_{H_0}.$$

Hence we get the desired conclusion. ∎

Let $\omega(\ell_i)$ $(i=1,2)$ denote the normalized Haar measure of $H(\ell_i)$. We note that $\omega(\ell_i)$ has the following properties.

(1) $\omega(\ell_i)*\varepsilon_h=\omega(\ell_i)$ for $h\in H(\ell_i)$.

(2) $\omega(\ell_i)*\omega(\ell_i)=\omega(\ell_i)$.

(3) $\omega(\ell_i)^-=\omega(\ell_i)$.

**Proposition 3.23.** *If $K$ is a commutative hypergroup extension of the Golden hypergroup $\mathbb{G}=\{\ell_0,\ell_1,\ell_2\}$ by a locally compact abelian group $H$ and $\varphi$ is a continuous homomorphism from $K$ onto $\mathbb{G}$, we have the conditions* (0) – (7) *as follows.*

(0) *$K$ is the disjoint union of the sets $H=\varphi^{-1}(\ell_0)$, $S_1=\varphi^{-1}(\ell_1)$ and $S_2=\varphi^{-1}(\ell_2)$.*

*Let $H(\ell_i)$ denote the stability group of $H$ at $s_i\in S_i$ and $\omega(\ell_i)$ the normalized Haar measure of $H(\ell_i)$ for $i=1,2$. Fix $s_1\in S_1$ and $s_2\in S_2$.*

(1) $\varepsilon_{s_1}^-=\varepsilon_{h_1}^-*\varepsilon_{s_1}$ *and* $\varepsilon_{s_2}^-=\varepsilon_{h_2}^-*\varepsilon_{s_2}$ *for some $h_1,h_2\in H$.*

(2) $\varepsilon_{s_1}*\varepsilon_{s_1}=\frac{1}{2}\varepsilon_{h_1}*\omega(\ell_1)+\frac{1}{2}\varepsilon_{h_1}*c_1*\varepsilon_{s_2}$ *for some $c_1\in M^1(H)$.*

(3) $\varepsilon_{s_2}*\varepsilon_{s_2}=\frac{1}{2}\varepsilon_{h_2}*\omega(\ell_2)+\frac{1}{2}\varepsilon_{h_2}*c_2*\varepsilon_{s_1}$ *for some $c_2\in M^1(H)$.*

(4) $\varepsilon_{s_1}*\varepsilon_{s_2}=\frac{1}{2}c_1^-*\varepsilon_{s_1}+\frac{1}{2}c_2^-*\varepsilon_{s_2}$.

(5) $\omega(\ell_1)*\omega(\ell_2)*c_1=c_1$ *and* $\omega(\ell_1)*\omega(\ell_2)*c_2=c_2$.

(6) $c_1^-=c_1*\varepsilon_{h_2}$ *and* $c_2^-=c_2*\varepsilon_{h_1}$.

(7) $c_1*c_1=\omega(\ell_1)*\omega(\ell_2)*\varepsilon_{h_2}^-$ *and* $c_2*c_2=\omega(\ell_1)*\omega(\ell_2)*\varepsilon_{h_1}^-$.

*Proof.* (1) Since $\varepsilon_{s_i}^-\in S_i$ by the relations $\ell_i{}^-=\ell_i$ $(i=1,2)$, one can take $h_i\in H$ such that $\varepsilon_{s_i}^-=\varepsilon_{h_i}^-*\varepsilon_{s_i}$ by Lemma 3.12.

(2) and (3) It is easy to see that $\varepsilon_{s_1}^-*\varepsilon_{s_1}$ is written as

$$\varepsilon_{s_1}^-*\varepsilon_{s_1}=\frac{1}{2}c_0+\frac{1}{2}c_1*\varepsilon_{s_2}$$

for some $c_0,c_1\in M^1(H)$. By the fact that $\omega(\ell_1)*\varepsilon_{s_1}=\varepsilon_{s_1}$, we have that $\omega(\ell_1)*c_0=c_0$ and $\omega(\ell_1)*c_1=c_1$. Since $\mathrm{supp}(c_0)=H\cap\,\mathrm{supp}(\varepsilon_{s_1}^-*\varepsilon_{s_1})=H(\ell_1)$ by Lemma 3.13 and $\omega(\ell_1)*c_0=c_0$, we get $c_0=\omega(\ell_1)$ by Lemma 3.22. Hence we obtain

$$\varepsilon_{s_1}^-*\varepsilon_{s_1}=\frac{1}{2}\omega(\ell_1)+\frac{1}{2}c_1*\varepsilon_{s_2},$$

namely $\varepsilon_{s_1}*\varepsilon_{s_1}=\frac{1}{2}\varepsilon_{h_1}*\omega(\ell_1)+\frac{1}{2}\varepsilon_{h_1}*c_1*\varepsilon_{s_2}$ by (1) $\varepsilon_{s_1}=\varepsilon_{h_1}*\varepsilon_{s_1}^-$. In a similar way, we obtain $\varepsilon_{s_2}*\varepsilon_{s_2}=\frac{1}{2}\varepsilon_{h_2}*\omega(\ell_2)+\frac{1}{2}\varepsilon_{h_2}*c_2*\varepsilon_{s_1}$.

(5) and (6) We may suppose that $\omega(\ell_1)*\omega(\ell_2)*c_1=c_1$ by the fact that $\omega(\ell_i)*\varepsilon_{s_i}=\varepsilon_{s_i}$. From the equality: $\varepsilon_{s_1}^-*\varepsilon_{s_1}=\frac{1}{2}\omega(\ell_1)+\frac{1}{2}c_1*\varepsilon_{s_2}$, we have

$$(\varepsilon_{s_1}^-*\varepsilon_{s_1})^-=\frac{1}{2}(\omega(\ell_1))^-+\frac{1}{2}c_1^-*\varepsilon_{s_2}^-=\frac{1}{2}\omega(\ell_1)+\frac{1}{2}c_1^-*\varepsilon_{h_2}^-*\varepsilon_{s_2}.$$

Since $(\varepsilon_{s_1}^-*\varepsilon_{s_1})^-=\varepsilon_{s_1}^-*\varepsilon_{s_1}$, we get $c_1^-*\varepsilon_{h_2}^-=c_1$, namely $c_1^-=c_1*\varepsilon_{h_2}$. In a similar way to the above, we have $\omega(\ell_1)*\omega(\ell_2)*c_2=c_2$ and $c_2^-=c_2*\varepsilon_{h_1}$.

(4) and (7) It is easy to see that $\varepsilon_{s_1} * \varepsilon_{s_2} = \frac{1}{2}c_3 * \varepsilon_{s_1} + \frac{1}{2}c_4 * \varepsilon_{s_2}$ for some $c_3, c_4 \in M^1(H)$ such that $\omega(\ell_1) * \omega(\ell_2) * c_3 = c_3$ and $\omega(\ell_1) * \omega(\ell_2) * c_4 = c_4$.

Then

$$(\varepsilon_{s_1} * \varepsilon_{s_1}) * \varepsilon_{s_2} = \frac{1}{4}\varepsilon_{h_1} * \varepsilon_{h_2} * c_1 + \frac{1}{4} * \varepsilon_{h_1} * \varepsilon_{h_2} * c_1 * c_2 * \varepsilon_{s_1} + \frac{1}{2}\omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_1} * \varepsilon_{s_2},$$

$$\varepsilon_{s_1} * (\varepsilon_{s_1} * \varepsilon_{s_2}) = \frac{1}{4}\varepsilon_{h_1} * c_3 + \frac{1}{4}c_3 * c_4 * \varepsilon_{s_1} + \frac{1}{4}(\varepsilon_{h_1} * c_1 * c_3 + c_4 * c_4) * \varepsilon_{s_2}.$$

By the associativity: $(\varepsilon_{s_1} * \varepsilon_{s_1}) * \varepsilon_{s_2} = \varepsilon_{s_1} * (\varepsilon_{s_1} * \varepsilon_{s_2})$, we have $c_1 * \varepsilon_{h_2} = c_3$ from the first term and $2\omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_1} = c_1 * c_3 * \varepsilon_{h_1} + c_4 * c_4$ from the last term. Since $c_1^- = c_1 * \varepsilon_{h_2}$, we see that $c_3 = c_1^-$. In a similar way, by the associativity: $\varepsilon_{s_1} * (\varepsilon_{s_2} * \varepsilon_{s_2}) = (\varepsilon_{s_1} * \varepsilon_{s_2}) * \varepsilon_{s_2}$, we have $c_4 = c_2 * \varepsilon_{h_1} = c_2^-$. Then we see that $\varepsilon_{s_1} * \varepsilon_{s_2} = \frac{1}{2}c_1^- * \varepsilon_{s_1} + \frac{1}{2}c_2^- * \varepsilon_{s_2}$. From these equalities we obtain

$$\begin{aligned} 2\omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_1} &= c_1 * c_3 * \varepsilon_{h_1} + c_4 * c_4 \\ &= c_1 * (c_1 * \varepsilon_{h_2}) * \varepsilon_{h_1} + (c_2 * \varepsilon_{h_1}) * (c_2 * \varepsilon_{h_1}), \end{aligned}$$

namely

$$2\omega(\ell_1) * \omega(\ell_2) = c_1 * c_1 * \varepsilon_{h_2} + c_2 * c_2 * \varepsilon_{h_1}.$$

This fact implies that $\mathrm{supp}(\omega(\ell_1) * \omega(\ell_2)) = \mathrm{supp}(c_1 * c_1 * \varepsilon_{h_2}) \cup \mathrm{supp}(c_2 * c_2 * \varepsilon_{h_1})$. Hence we see that $\mathrm{supp}(c_1 * c_1 * \varepsilon_{h_2}) \subset H(\ell_1)H(\ell_2)$ and $\mathrm{supp}(c_2 * c_2 * \varepsilon_{h_1}) \subset H(\ell_1)H(\ell_2)$. Since $\omega(\ell_1) * \omega(\ell_2) * c_i = c_i$, we have $c_1 * c_1 * \varepsilon_{h_2} = \omega(\ell_1) * \omega(\ell_2)$ and $c_2 * c_2 * \varepsilon_{h_1} = \omega(\ell_1) * \omega(\ell_2)$ by Lemma 3.22. Therefore, we get $c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_2}^-$ and $c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_1}^-$. □

We see that any extension $K$ of $L$ by $H$ is characterized by

$$H(\ell_1),\ H(\ell_2),\ s_1 \in S_1,\ s_2 \in S_2,\ h_1, h_2 \in H,\ c_1, c_2 \in M^1(H)$$

satisfying the conditions described in Proposition 3.23. Therefore we denote such an extension $K$ by $K(H(\ell_1), H(\ell_2), s_1, s_2, h_1, h_2, c_1, c_2)$.

When we take $H_1(\ell_1), H_1(\ell_2), t_1 \in S_1,\ t_2 \in S_2,\ \ k_1, k_2 \in H$ and $d_1, d_2 \in M^1(H)$ satisfying the conditions (0)–(7) in Proposition 3.23, we have another extension $K(H_1(\ell_1), H_1(\ell_2), t_1, t_2, k_1, k_2, d_1, d_2)$ of $L$ by $H$.

**Proposition 3.24.** *Two extensions $K(H(\ell_1), H(\ell_2), s_1, s_2, h_1, h_2, c_1, c_2)$ and $K(H_1(\ell_1), H_1(\ell_2), t_1, t_2, k_1, k_2, d_1, d_2)$ of $L = \mathbb{G}$ by $H$ are mutually equivalent as extensions if and only if $H(\ell_1) = H_1(\ell_1), H(\ell_2) = H_1(\ell_2)$ and there exist $b_1, b_2 \in H$ such that $\varepsilon_{k_1} * \omega(\ell_1) = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_{h_1} * \omega(\ell_1)$, $\varepsilon_{k_2} * \omega(\ell_2) = \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_{h_2} * \omega(\ell_2)$, $d_1 = \varepsilon_{b_2}^- * c_1$ and $d_2 = \varepsilon_{b_1}^- * c_2$.*

*Proof.* Suppose that $K_1 = K(H(\ell_1), H(\ell_2), s_1, s_2, h_1, h_2, c_1, c_2)$ is equivalent to $K_2 = K(H_1(\ell_1), H_1(\ell_2), t_1, t_2, k_1, k_2, d_1, d_2)$ as extensions. Let $\varphi_i$ be a continuous homomorphism from $K_i$ onto $L$ $(i = 1, 2)$. Let $K_1 = H \cup S_1 \cup S_2$ and $K_2 = H \cup T_1 \cup T_2$ where $S_i = \varphi_1^{-1}(\ell_i)$ and $T_i = \varphi_2^{-1}(\ell_i)$. Let $\psi$ be an isomorphism from $K_1$ to $K_2$ such that $\varphi_2 \circ \psi = \varphi_1$. Put $\psi(s_1) = u_1 \in T_1$ and $\psi(s_2) = u_2 \in T_2$. Since $\varepsilon_h * \varepsilon_{u_i} = \varepsilon_{u_i}$ for any $h \in H(\ell_i)$, we see that $H(\ell_i) = H_1(\ell_i) (i = 1, 2)$. For $u_1 \in T_1$ and $u_2 \in T_2$, there exist $b_1$ and $b_2 \in H$ such that $\varepsilon_{u_1} = \varepsilon_{b_1}^- * \varepsilon_{t_1}$ and $\varepsilon_{u_2} = \varepsilon_{b_2}^- * \varepsilon_{t_2}$ by Lemma 3.12. Then,

$$\varepsilon_{u_1}^- = (\varepsilon_{b_1}^- * \varepsilon_{t_1})^- = \varepsilon_{b_1} * \varepsilon_{t_1}^- = \varepsilon_{b_1} * \varepsilon_{k_1}^- * \varepsilon_{t_1} = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_{k_1}^- * \varepsilon_{u_1}.$$

By the relation that $\varepsilon_{s_1}^- = \varepsilon_{h_1}^- * \varepsilon_{s_1}$, we have $\varepsilon_{u_1}^- = \varepsilon_{h_1}^- * \varepsilon_{u_1}$ since $\psi(\varepsilon_{h_1}^- * \varepsilon_{s_1}) = \psi(\varepsilon_{h_1})^- * \psi(\varepsilon_{s_1}) = \varepsilon_{h_1}^- * \varepsilon_{u_1}$. Hence we have

$$\varepsilon_{h_1}^- * \varepsilon_{u_1} = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_{k_1}^- * \varepsilon_{u_1},$$

namely

$$\varepsilon_{k_1} * \varepsilon_{u_1} = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_{h_1} * \varepsilon_{u_1}.$$

Since $\varepsilon_{u_1} * \omega(\ell_1) = \varepsilon_{u_1}$, we obtain

$$\varepsilon_{k_1} * \omega(\ell_1) = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_{h_1} * \omega(\ell_1).$$

In a similar way, we also get

$$\varepsilon_{k_2} * \omega(\ell_2) = \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_{h_2} * \omega(\ell_2).$$

Since $\varepsilon_{u_1}^- * \varepsilon_{u_1} = \varepsilon_{t_1}^- * \varepsilon_{t_1}$ and $\varepsilon_{t_2} = \varepsilon_{b_2} * \varepsilon_{u_2}$, we have

$$\varepsilon_{u_1}^- * \varepsilon_{u_1} = \frac{1}{2}\omega(\ell_1) + \frac{1}{2}d_1 * \varepsilon_{t_2} = \frac{1}{2}\omega(\ell_1) + \frac{1}{2}\varepsilon_{b_2} * d_1 * \varepsilon_{u_2}.$$

Since

$$\psi(\varepsilon_{s_1}^- * \varepsilon_{s_1}) = \psi(\varepsilon_{s_1})^- * \psi(\varepsilon_{s_1}) = \varepsilon_{u_1}^- * \varepsilon_{u_1}$$

and

$$\begin{aligned} \psi(\varepsilon_{s_1}^- * \varepsilon_{s_1}) &= \psi\left(\frac{1}{2}\omega(\ell_1) + \frac{1}{2}c_1 * \varepsilon_{s_2}\right) = \frac{1}{2}\omega(\ell_1) + \frac{1}{2}c_1 * \psi(\varepsilon_{s_2}) \\ &= \frac{1}{2}\omega(\ell_1) + \frac{1}{2}c_1 * \varepsilon_{u_2}, \end{aligned}$$

we have

$$\varepsilon_{u_1}^- * \varepsilon_{u_1} = \frac{1}{2}\omega(\ell_1) + \frac{1}{2}c_1 * \varepsilon_{u_2}.$$

Hence we get $d_1 * \varepsilon_{b_2} = c_1$ from the last term, namely $d_1 = \varepsilon_{b_2}^- * c_1$. In a similar way, we see that $d_2 = \varepsilon_{b_1}^- * c_2$.

Conversely, we assume that $H(\ell_1) = H_1(\ell_1), H(\ell_2) = H_1(\ell_2)$ and there exist $b_1, b_2 \in H$ such that

$$\varepsilon_{k_1} * \omega(\ell_1) = \varepsilon_{b_1} * \varepsilon_{b_1} * \varepsilon_{h_1} * \omega(\ell_1),\ \varepsilon_{k_2} * \omega(\ell_2) = \varepsilon_{b_2} * \varepsilon_{b_2} * \varepsilon_{h_2} * \omega(\ell_2),$$

$$d_1 = \varepsilon_{b_2}^- * c_1 \text{ and } d_2 = \varepsilon_{b_1}^- * c_2.$$

Take $u_1 \in T_1$ and $u_2 \in T_2$ by $\varepsilon_{u_1} = \varepsilon_{b_1}^- * \varepsilon_{t_1}$ and $\varepsilon_{u_2} = \varepsilon_{b_2}^- * \varepsilon_{t_2}$. Then we have

$\varepsilon_{u_1}^- = (\varepsilon_{b_1}^- * \varepsilon_{t_1})^- = \varepsilon_{b_1} * \varepsilon_{t_1}^- = \varepsilon_{b_1} * \varepsilon_{k_1}^- * \varepsilon_{t_1} = \varepsilon_{b_1} * \varepsilon_{b_1}^- * \varepsilon_{b_1}^- * \varepsilon_{h_1}^- * \varepsilon_{b_1} * \varepsilon_{u_1} = \varepsilon_{h_1}^- * \varepsilon_{u_1}$

by the relation $\varepsilon_{k_1}^- * \omega(\ell_1) = \varepsilon_{b_1}^- * \varepsilon_{b_1}^- * \varepsilon_{h_1}^- * \omega(\ell_1)$ and

$$\begin{aligned}
\varepsilon_{u_1} * \varepsilon_{u_1} &= \varepsilon_{b_1}^- * \varepsilon_{b_1}^- * \varepsilon_{t_1} * \varepsilon_{t_1} \\
&= \frac{1}{2}\varepsilon_{b_1}^- * \varepsilon_{b_1}^- * \varepsilon_{k_1} * \omega(\ell_1) + \frac{1}{2}\varepsilon_{b_1}^- * \varepsilon_{b_1}^- * \varepsilon_{k_1} * d_1 * \varepsilon_{t_2} \\
&= \frac{1}{2}\varepsilon_{h_1} * \omega(\ell_1) + \frac{1}{2}\varepsilon_{h_1} * c_1 * \varepsilon_{u_2}
\end{aligned}$$

by the relation $d_1 = \varepsilon_{b_2}^- * c_1$ and $\varepsilon_{t_2} = \varepsilon_{b_2} * \varepsilon_{u_2}$. In a similar way, we have $\varepsilon_{u_2}^- = \varepsilon_{h_2}^- * \varepsilon_{u_2}, \varepsilon_{u_2} * \varepsilon_{u_2} = \frac{1}{2}\varepsilon_{h_2} * \omega(\ell_2) + \frac{1}{2}\varepsilon_{h_2} * c_2 * \varepsilon_{u_1}$ and $\varepsilon_{u_1} * \varepsilon_{u_2} = \frac{1}{2}c_1^- * \varepsilon_{u_1} + \frac{1}{2}c_2^- * \varepsilon_{u_2}$.

We put a map $\psi$ from $K_1$ to $K_2$ such that

$$\psi(\varepsilon_h) = \varepsilon_h,\ \psi(\varepsilon_h * \varepsilon_{s_1}) = \varepsilon_h * \varepsilon_{u_1} \text{ and } \psi(\varepsilon_h * \varepsilon_{s_2}) = \varepsilon_h * \varepsilon_{u_2}$$

for $h \in H$. It is clear that $\psi(S_i) = T_i$ for $i = 1, 2$. Since

$$\begin{aligned}
\psi(\varepsilon_{s_1} * \varepsilon_{s_1}) &= \psi\left(\frac{1}{2}\varepsilon_{h_1} * \omega(\ell_1) + \frac{1}{2}\varepsilon_{h_1} * c_1 * \varepsilon_{s_2}\right) \\
&= \frac{1}{2}\varepsilon_{h_1} * \omega(\ell_1) + \frac{1}{2}\varepsilon_{h_1} * c_1 * \varepsilon_{u_2} = \varepsilon_{u_1} * \varepsilon_{u_1},
\end{aligned}$$

we have $\psi(\varepsilon_{s_1} * \varepsilon_{s_1}) = \psi(\varepsilon_{s_1}) * \psi(\varepsilon_{s_1})$. In a similar way, we know that $\psi$ is a homomorphism. Since

$$\psi(\varepsilon_{s_1}^-) = \psi(\varepsilon_{h_1}^- * \varepsilon_{s_1}) = \varepsilon_{h_1}^- * \varepsilon_{u_1} = \varepsilon_{u_1}^-$$

and $\psi(\varepsilon_{s_1})^- = \varepsilon_{u_1}^-$, we get $\psi(\varepsilon_{s_1}^-) = \psi(\varepsilon_{s_1})^-$. In a similar way, we obtain that $\psi(\varepsilon_{s_2}^-) = \psi(\varepsilon_{s_2})^-$. Moreover, for a continuous homomorphism $\varphi_i$ from $K_i$ onto $L$ $(i = 1, 2)$, it is easy to check that $\varphi_2 \circ \psi = \varphi_1$. □

3.3.2. *Construction of the model.* Let $H$ be a locally compact abelian group with unit $h_0$ and $L = \{\ell_0, \ell_1, \ell_2\}$ be the Golden hypergroup $\mathbb{G}$ with unit $\ell_0$. Take any compact subgroup $H(\ell_i)$ of $H$ where $H(\ell_0) = \{h_0\}$ and denote the quotient space $H/H(\ell_i)$ by $Q(\ell_i)$ for $i = 1, 2$. The normalized Haar measure of $H(\ell_i)$ is denote by $\omega(\ell_i)$ $(i = 0, 1, 2)$. Let $K$ be the disjoint union of the sets $H, Q(\ell_1)$ and $Q(\ell_2)$, namely

$$\begin{aligned}
K &= H \cup Q(\ell_1) \cup Q(\ell_2) \\
&= \{(\ell_i, h * H(\ell_i)) : \ell_i \in L, h \in H\}.
\end{aligned}$$

The Dirac measure at $(\ell_i, h * H(\ell_i)) \in K$ is realized in $M^b(L) \otimes M^b(H)$ by

$$\delta_{\ell_i} \otimes (\varepsilon_h * \omega(\ell_i)).$$

Take and fix $f_1, f_2 \in H(\ell_1)H(\ell_2)$. We define the involution $^-$ of $K$ by

$$(\ell_i, h * H(\ell_i))^- = (\ell_i, h^- * f_i{}^- * H(\ell_i)),$$

where $f_0 = h_0$. Moreover we define the convolution $*_c$ of $K$ in $M^b(L) \otimes M^b(H)$ by the following.

(1) $(\delta_{\ell_0} \otimes \varepsilon_{h_1}) *_c (\delta_{\ell_i} \otimes (\varepsilon_{h_2} * \omega(\ell_i))) = (\delta_{\ell_i} \otimes (\varepsilon_{h_2} * \omega(\ell_i))) *_c (\delta_{\ell_0} \otimes \varepsilon_{h_1})$
$= \delta_{\ell_i} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2} * \omega(\ell_i))$ for $i = 0, 1, 2$.

(2) $(\delta_{\ell_1} \otimes (\varepsilon_{h_1} * \omega(\ell_1))) *_c (\delta_{\ell_1} \otimes (\varepsilon_{h_2} * \omega(\ell_1)))$
$= \frac{1}{2}\delta_{\ell_0} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2} * \varepsilon_{f_1} * \omega(\ell_1)) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{h_1} * \varepsilon_{h_2} * \omega(\ell_1) * \omega(\ell_2))$.

(3) $(\delta_{\ell_2} \otimes (\varepsilon_{k_1} * \omega(\ell_2))) *_c (\delta_{\ell_2} \otimes (\varepsilon_{k_2} * \omega(\ell_2)))$
$= \frac{1}{2}\delta_{\ell_0} \otimes (\varepsilon_{k_1} * \varepsilon_{k_2} * \varepsilon_{f_2} * \omega(\ell_2)) + \frac{1}{2}\delta_{\ell_1} \otimes (\varepsilon_{k_1} * \varepsilon_{k_2} * \omega(\ell_1) * \omega(\ell_2))$.

(4) $(\delta_{\ell_1} \otimes (\varepsilon_h * \omega(\ell_1))) *_c (\delta_{\ell_2} \otimes (\varepsilon_k * \omega(\ell_2)))$
$= (\delta_{\ell_2} \otimes (\varepsilon_k * \omega(\ell_2))) *_c (\delta_{\ell_1} \otimes (\varepsilon_h * \omega(\ell_1)))$
$= \frac{1}{2}\delta_{\ell_1} \otimes (\varepsilon_h * \varepsilon_k * \omega(\ell_1) * \omega(\ell_2)) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_h * \varepsilon_k * \omega(\ell_1) * \omega(\ell_2))$.

Since the model $K$ is determined by the compact subgroups $H(\ell_1), H(\ell_2)$ of $H$ and $f_1, f_2 \in H(\ell_1)H(\ell_2)$, we denote $K$ by $K(H(\ell_1), H(\ell_2), f_1, f_2)$. Put $P(\ell_i) = (H(\ell_1)H(\ell_2))/H(\ell_i)$, $P^2(\ell_i) = \{p^2 : p \in P(\ell_i)\}$ and $P_2(\ell_i) = P(\ell_i)/P^2(\ell_i)$ for $i = 1, 2$. Now we arrive at the main theorem of the section 3.2.

**Theorem 3.25.** *Under the preceding arguments we have the following.*

(1) *The model $K(H(\ell_1), H(\ell_2), f_1, f_2)$ is a commutative hypergroup and an extension of $L$ by $H$.*
(2) *All extensions $K$ of $L$ by $H$ are equivalent to $K(H(\ell_1), H(\ell_2), f_1, f_2)$ as extensions.*
(3) *The extensions $K(H(\ell_1), H(\ell_2), f_1, f_2)$ and $K(H_1(\ell_1), H_1(\ell_2), g_1, g_2)$ are equivalent as extensions if and only if $[f_i] = [g_i]$ in $P_2(\ell_i)$ for $i = 1, 2$.*

*Proof.* (1) Since $H$ is a locally compact group and $H(\ell_i)$ is a compact subgroup of $H$, the quotient space $Q(\ell_i) = H/H(\ell_i)$ is a locally compact space. Then the disjoint union $K(H(\ell_1), H(\ell_2), f_1, f_2) = H \cup Q(\ell_1) \cup Q(\ell_2)$ is also a locally compact space. It is clear that the definition of the convolution $*_c$ and the involution $^-$ is well defined. By the definition of $K(H(\ell_1), H(\ell_2), f_1, f_2)$, we know that the convolution $*_c$ and the involution $^-$ are continuous from the fact that group operation and inverse operation of $H$ as well as an action of $H$ on $Q(\ell_i)$ are all continuous for $i = 1, 2$.

We check the associativity of the convolution. It is easy to see that

$$\begin{aligned}&\{(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c (\delta_{\ell_1}\otimes\omega(\ell_1))\} *_c (\delta_{\ell_2}\otimes\omega(\ell_2))\\&=\frac{1}{4}\delta_{\ell_0}\otimes(\varepsilon_{f_2}*\omega(\ell_1)*\omega(\ell_2))+\frac{1}{4}\delta_{\ell_1}\otimes(\omega(\ell_1)*\omega(\ell_2))\\&\quad+\frac{1}{2}\delta_{\ell_2}\otimes(\varepsilon_{f_1}*\omega(\ell_1)*\omega(\ell_2))\end{aligned}$$

and

$$\begin{aligned}&(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c \{(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c (\delta_{\ell_2}\otimes\omega(\ell_2))\}\\&=\frac{1}{4}\delta_{\ell_0}\otimes(\varepsilon_{f_1}*\omega(\ell_1)*\omega(\ell_2))+\frac{1}{4}\delta_{\ell_1}\otimes(\omega(\ell_1)*\omega(\ell_2))\\&\quad+\frac{1}{2}\delta_{\ell_2}\otimes(\omega(\ell_1)*\omega(\ell_2)).\end{aligned}$$

Since $f_1, f_2 \in H(\ell_1)H(\ell_2)$, we obtain $\{(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c (\delta_{\ell_1}\otimes\omega(\ell_1))\} *_c (\delta_{\ell_2}\otimes\omega(\ell_2)) = (\delta_{\ell_1}\otimes\omega(\ell_1)) *_c \{(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c (\delta_{\ell_2}\otimes\omega(\ell_2))\}$. In a similar way, we know that the associativity of other convolutions holds. For the involution, it is easy to see that

$$\{(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c (\delta_{\ell_1}\otimes\omega(\ell_1))\}^- = \frac{1}{2}\delta_{\ell_0}\otimes(\varepsilon_{f_1}^- *\omega(\ell_1)) + \frac{1}{2}\delta_{\ell_2}\otimes(\omega(\ell_1)*\omega(\ell_2))$$

and

$$\begin{aligned}&(\delta_{\ell_1}\otimes\omega(\ell_1))^- *_c (\delta_{\ell_1}\otimes\omega(\ell_1))^-\\&=\frac{1}{2}\delta_{\ell_0}\otimes(\varepsilon_{f_1}*\varepsilon_{f_1}^- *\varepsilon_{f_1}^- *\omega(\ell_1))+\frac{1}{2}\delta_{\ell_2}\otimes(\varepsilon_{f_1}^- *\varepsilon_{f_1}^- *\omega(\ell_1)*\omega(\ell_2)).\end{aligned}$$

Since $f_1, f_2 \in H(\ell_1)H(\ell_2)$, we have $\{(\delta_{\ell_1}\otimes\omega(\ell_1)) *_c (\delta_{\ell_1}\otimes\omega(\ell_1))\}^- = (\delta_{\ell_1}\otimes\omega(\ell_1))^- *_c (\delta_{\ell_1}\otimes\omega(\ell_1))^-$. In a similar way, we know that the other properties of the involution hold.

The compactness of the support of $(\delta_{\ell_i}\otimes\omega(\ell_i)) *_c (\delta_{\ell_j}\otimes\omega(\ell_j))$ is assured by the fact that $H(\ell_i)$ and $H(\ell_j)$ are compact and $L$ is finite. It is easy to check other axioms of a hypergroup. We omit the detail. Hence we see that $K(H(\ell_1), H(\ell_2), f_1, f_2)$ is a commutative hypergroup.

Let $\varphi$ be a mapping from $K(H(\ell_1), H(\ell_2), f_1, f_2)$ onto $L$ such that $\varphi(\ell_i, h * H(\ell_i)) = \ell_i$ for $h \in H$ and $\ell_i \in L$. Then it is easy to see that $\varphi$ is a continuous hypergroup homomorphism from $K(H(\ell_1), H(\ell_2), f_1, f_2)$ onto $L$ such that Ker $\varphi = H$. This implies that $K(H(\ell_1), H(\ell_2), f_1, f_2)$ is an extension of $L$ by $H$.

(2) Take an extension $K$ of $L$ by $H$. Then $K$ is characterized as $K = K(H(\ell_1), H(\ell_2), s_1, s_2, h_1, h_2, c_1, c_2)$ by Proposition 3.23. By the conditions (1) and (7) in Proposition 3.23:

$$\varepsilon_{s_1}^- = \varepsilon_{h_1}^- * \varepsilon_{s_1},\ \varepsilon_{s_2}^- = \varepsilon_{h_2}^- * \varepsilon_{s_2},$$

$$c_1 * c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_2}^-,\ c_2 * c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{h_1}^-,$$

we know that there exist $a_1, a_2 \in H$ and $f_1, f_2 \in H(\ell_1)H(\ell_2)$ such that

$$c_1 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{a_2}^-,\ c_2 = \omega(\ell_1) * \omega(\ell_2) * \varepsilon_{a_1}^-,$$

$$\varepsilon_{h_1} = \varepsilon_{a_1} * \varepsilon_{a_1} * \varepsilon_{f_1},\ \varepsilon_{h_2} = \varepsilon_{a_2} * \varepsilon_{a_2} * \varepsilon_{f_2}.$$

Put $\psi$ be a mapping from $K$ to the model extension $K(H(\ell_1), H(\ell_2), f_1, f_2)$ such that $\psi(\varepsilon_h) = \delta_{\ell_0} \otimes \varepsilon_h$, $\psi(\varepsilon_h * \varepsilon_{s_1}) = \delta_{\ell_1} \otimes (\varepsilon_{a_1} * \varepsilon_h * \omega(\ell_1))$ and $\psi(\varepsilon_h * \varepsilon_{s_2}) = \delta_{\ell_2} \otimes (\varepsilon_{a_2} * \varepsilon_h * \omega(\ell_2))$ for $h \in H$. It is easy to see that the mapping $\psi$ is well-defined and bijective.

We have

$$\psi(\varepsilon_{s_1} * \varepsilon_{s_1}) = \frac{1}{2}\delta_{\ell_0} \otimes (\varepsilon_{h_1} * \omega(\ell_1)) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{h_1} * \varepsilon_{a_2} * c_1)$$

and

$$\psi(\varepsilon_{s_1}) *_c \psi(\varepsilon_{s_1}) = \frac{1}{2}\delta_{\ell_0} \otimes (\varepsilon_{a_1} * \varepsilon_{a_1} * \varepsilon_{f_1} * \omega(\ell_1)) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{a_1} * \varepsilon_{a_1} * \omega(\ell_1) * \omega(\ell_2)).$$

Since $\varepsilon_{h_1} = \varepsilon_{a_1} * \varepsilon_{a_1} * \varepsilon_{f_1}$ and $\varepsilon_{a_2} * c_1 = \omega(\ell_1) * \omega(\ell_2)$, we have $\psi(\varepsilon_{s_1} * \varepsilon_{s_1}) = \psi(\varepsilon_{s_1}) *_c \psi(\varepsilon_{s_1})$. In a similar way, we see that $\psi$ is a homomorphism. Moreover,

$$\psi(\varepsilon_{s_1}^-) = \psi(\varepsilon_{h_1}^- * \varepsilon_{s_1}) = \delta_{\ell_1} \otimes (\varepsilon_{a_1} * \varepsilon_{h_1}^- * \omega(\ell_1)).$$

By the definition of the model $K(H(\ell_1), H(\ell_2), f_1, f_2)$,

$$\psi(\varepsilon_{s_1})^- = \delta_{\ell_1} \otimes (\varepsilon_{a_1}^- * \varepsilon_{f_1}^- * \omega(\ell_1)).$$

Since $\varepsilon_{h_1}^- = \varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \varepsilon_{f_1}^-$, we have $\psi(\varepsilon_{s_1}^-) = \psi(\varepsilon_{s_1})^-$. In a similar way, we know that $\psi$ preserves the involution. Hence $\psi$ is an involutive isomorphism.

If we take a continuous homomorphism $\varphi_2$ from $K(H(\ell_1), H(\ell_2), f_1, f_2)$ onto $L$ such that $\varphi_2((\ell_i, h * H(\ell_i))) = \ell_i$ for $\ell_i \in L$, then it is clear that $\varphi_2 \circ \psi = \varphi$.

(3) We note that $K_1 = K(H(\ell_1), H(\ell_2), f_1, f_2)$ is equal to $K(H(\ell_1), H(\ell_2), s_1, s_2, h_1, h_2, c_1, c_2)$ such that $h = (\ell_0, h)$, $s_1 = (\ell_1, H(\ell_1))$, $s_2 = (\ell_2, H(\ell_2))$, $h_1 = f_1$, $h_2 = f_2$, $c_1 = \omega(\ell_1) * \omega(\ell_2)$ and $c_2 = \omega(\ell_1) * \omega(\ell_2)$ and $K_2 = K(H_1(\ell_1), H_1(\ell_2), g_1, g_2)$ is also similar. We assume that $K_1$ is equivalent to $K_2$ as extensions. Applying Proposition 3.24, there exists $a_1 \in H$ such that

$$\delta_{\ell_0} \otimes (\varepsilon_{f_1} * \omega(\ell_1)) = \delta_{\ell_0} \otimes (\varepsilon_{a_1} * \varepsilon_{a_1} * \varepsilon_{g_1} * \omega(\ell_1)),$$

$$\delta_{\ell_0} \otimes (\omega(\ell_1) * \omega(\ell_2)) = \delta_{\ell_0} \otimes (\varepsilon_{a_1} * \omega(\ell_1) * \omega(\ell_2)).$$

Hence we get $a_1 \in H(\ell_1)H(\ell_2)$ and $f_1 * H(\ell_1) = a_1^2 g_1 * H(\ell_1)$ i.e. $[f_1] = [g_1]$ in $P_2(\ell_1)$. In a similar way, we obtain $[f_2] = [g_2]$ in $P_2(\ell_2)$.

The converse assertion is clear by Proposition 3.24. □

**Definition.** Let $L = \{\ell_0, \ell_1, \cdots, \ell_n\}$ be a finite commutative hypergroup and $H$ a locally compact abelian group with unit $h_0$. Let $K$ be an extension of $L$ by a locally compact abelian group $H$ and let $\varphi$ be a continuous homomorphism from $K$ onto $L$. Let $H(\ell_i)$ be a compact subgroup of $H$ such that $H(\ell_0) = \{h_0\}, H(\ell_i^-) = H(\ell_i)$ and let $\omega(\ell_i)$ denote the normalized Haar measure of $H(\ell_i)$ for $\ell_i \in L$. If there exists an injective mapping $\phi$ from $L$ into $K$ such that

(1) The mapping $\phi$ is a cross section of $\varphi$ i.e. $\varphi(\phi(\ell)) = \ell$ for $\ell \in L$ and $\phi(\ell_0) = h_0$,

(2) $\phi(\delta_{\ell_i}) * \phi(\delta_{\ell_j}) = \phi(\delta_{\ell_i} \circ \delta_{\ell_j}) * \omega(\ell_i) * \omega(\ell_j)$ for $\ell_i, \ell_j \in L$,

then we call that the extension $K$ of $L$ by $H$ *splits* or $K$ is a *splitting* extension. Moreover, If a splitting extension $K$ has a property:

(1) $\omega(\ell_i) * \omega(\ell_j) * \omega(\ell) = \omega(\ell_i) * \omega(\ell_j)$ for $\ell \in \text{supp}(\delta_{\ell_i} \circ \delta_{\ell_j})$,

then we call that the extension $K$ of $L$ by $H$ is *strong splitting*.

**Theorem 3.26.** *The extension $K(H(\ell_1), H(\ell_2), f_1, f_2)$ is splitting if and only if $[f_i] = [h_0]$ in $P_2(\ell_i)$ $(i = 1, 2)$ where $h_0$ is unit element of $H$. Moreover, $K$ is strong splitting if and only if $K$ is splitting and $H(\ell_1) = H(\ell_2)$.*

*Proof.* We assume that $K$ is splitting. Then, there exists an injective mapping $\phi$ from $L$ into $K(H(\ell_1), H(\ell_2), f_1, f_2)$ such that $\phi(\ell_0) = (\ell_0, h_0), \phi(\ell_1) = (\ell_1, a_1^- * H(\ell_1))$, and $\phi(\ell_2) = (\ell_2, a_2^- * H(\ell_2))$ for some $a_1, a_2 \in H$. Since

$$\phi(\delta_{\ell_1}) * \phi(\delta_{\ell_1}) = \frac{1}{2}\delta_{\ell_0} \otimes (\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \varepsilon_{f_1} * \omega(\ell_1)) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \omega(\ell_1) * \omega(\ell_2))$$

and

$$\phi(\delta_{\ell_1} \circ \delta_{\ell_1}) * \omega(\ell_1) * \omega(\ell_1) = \frac{1}{2}\delta_{\ell_0} \otimes \omega(\ell_1) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{a_2}^- * \omega(\ell_1) * \omega(\ell_2)),$$

we get

$$\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \varepsilon_{f_1} * \omega(\ell_1) = \omega(\ell_1),$$
$$\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \omega(\ell_1) * \omega(\ell_2) = \varepsilon_{a_2}^- * \omega(\ell_1) * \omega(\ell_2)$$

by the splitting condition (1). Hence we know that

$$f_1 * H(\ell_1) = a_1^2 * H(\ell_1) \text{ and } a_2 \in H(\ell_1)H(\ell_2).$$

In a similar way, we get

$$f_2 * H(\ell_2) = a_2^2 * H(\ell_2) \text{ and } a_1 \in H(\ell_1)H(\ell_2)$$

by the equation $\phi(\delta_{\ell_2}) *_c \phi(\delta_{\ell_2}) = \phi(\delta_{\ell_2} \circ \delta_{\ell_2}) * \omega(\ell_2)$. Therefore, we obtain $[f_i] = [h_0]$ in $P_2(\ell_i)$ for $i = 1, 2$.

Conversely, we assume that $[f_i] = [h_0]$ in $P_2(\ell_i)$ for $i = 1, 2$. Then there exist $a_1, a_2 \in H(\ell_1)H(\ell_2)$ such that

$$f_1 * H(\ell_1) = a_1^2 * H(\ell_1) \text{ and } f_2 * H(\ell_2) = a_2^2 * H(\ell_2).$$

Put $\phi$ a mapping from $L$ into $K(H(\ell_1), H(\ell_2), f_1, f_2)$ such that

$$\phi(\ell_0) = (\ell_0, h_0), \phi(\ell_1) = (\ell_1, a_1^- * H(\ell_1)), \phi(\ell_2) = (\ell_2, a_2^- * H(\ell_2)).$$

It is clear that $\phi$ is a cross section of $\varphi$. It is easy to see that

$$\phi(\delta_{\ell_1}) *_c \phi(\delta_{\ell_1}) = \frac{1}{2}\delta_{\ell_0} \otimes (\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \varepsilon_{f_1} * \omega(\ell_1)) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \omega(\ell_1) * \omega(\ell_2))$$

and

$$\phi(\delta_{\ell_1} \circ \delta_{\ell_1}) * \omega(\ell_1) = \frac{1}{2}\delta_{\ell_0} \otimes \omega(\ell_1) + \frac{1}{2}\delta_{\ell_2} \otimes (\varepsilon_{a_2}^- * \omega(\ell_1) * \omega(\ell_2)).$$

Since $\varepsilon_{a_1}^- * \varepsilon_{a_1}^- * \varepsilon_{f_1} * \omega(\ell_1) = \omega(\ell_1)$ and $a_2 \in H(\ell_1)H(\ell_2)$, we get

$$\phi(\delta_{\ell_1}) *_c \phi(\delta_{\ell_1}) = \phi(\delta_{\ell_1} \circ \delta_{\ell_1}) * \omega(\ell_1).$$

In a similar way, we obtain $\phi(\delta_{\ell_2}) *_c \phi(\delta_{\ell_2}) = \phi(\delta_{\ell_2} \circ \delta_{\ell_2}) * \omega(\ell_2)$ and $\phi(\delta_{\ell_1}) *_c \phi(\delta_{\ell_2}) = \phi(\delta_{\ell_1} \circ \delta_{\ell_2}) * \omega(\ell_1) * \omega(\ell_2)$. Therefore $K(H(\ell_1), H(\ell_2), f_1, f_2)$ is splitting.

Suppose that $K(H(\ell_1), H(\ell_2), f_1, f_2)$ is strong splitting. Then we have

$$\omega(\ell_1) * \omega(\ell_1) * \omega(\ell_2) = \omega(\ell_1) * \omega(\ell_1).$$

Since

$$\text{supp}(\delta_{\ell_1} \circ \delta_{\ell_1}) = \{\ell_0, \ell_2\},$$

we get $\omega(\ell_1) * \omega(\ell_2) = \omega(\ell_1)$. In a similar way, we get $\omega(\ell_1) * \omega(\ell_2) = \omega(\ell_2)$. Therefore we obtain

$$\omega(\ell_1) = \omega(\ell_2).$$

The converse is clear. □

3.3.3. *Applications and examples.* Under these discussions we calculate all extensions $K$ of the Golden hypergroup $L$ by concrete locally compact abelian groups $H$.

**Example 3.27.** $H = \mathbb{R}^n$.

Since the trivial subgroup $\{0\}$ of $\mathbb{R}^n$ is the only compact subgroup of $\mathbb{R}^n$, we get only one extension $K$ which is $H \times L$.

**Example 3.28.** $H = \mathbb{Z}^n$.

Since the trivial subgroup $\{0\}$ of $\mathbb{Z}^n$ is the only compact subgroup of $\mathbb{Z}^n$, we get only one extension $K$ which is $H \times L$.

**Example 3.29.** $H = \mathbb{T}$.

For a natural number $m$, a real number $h$ is written in the form $h = \frac{2\pi}{m}k + r$, where $k$ is an integer and $0 \le r < \frac{2\pi}{m}$. Then we denote the residue $r$ by $[h]_m$ i.e. $r = [h]_m$. For $h_1, h_2 \in [0, 2\pi), \varepsilon_{h_1} * \varepsilon_{h_2} = \varepsilon_{[h_1+h_2]_1}$.

We identify $\mathbb{T}$ with $[0, 2\pi)$ by

$$\mathbb{T} \ni e^{i\theta} \longleftrightarrow \theta \in [0, 2\pi).$$

Then the product $e^{i\theta_1}e^{i\theta_2}$ in $\mathbb{T}$ corresponds to $[\theta_1 + \theta_2]_1$ in $[0, 2\pi)$.

(1) Case of $H(\ell_1) = \{0\}$ and $H(\ell_2) = \{0\}$.
The extension $K$ of $L$ by $H$ must be $H \times L$.

(2) Case of $H(\ell_1) = H$ and $H(\ell_2) = H$. Then $K = \mathbb{T} \cup S_1 \cup S_2 = \mathbb{T} \cup \{0\} \cup \{0\}$.
The extension $K$ of $L$ by $H$ is the hypergroup join $H \vee L$.
We identify $\mathbb{T}, S_1$ and $S_2$ with $\{(0, h) : h \in [0, 2\pi)\}$, $\{(1, 0)\}$, and $\{(2, 0)\}$ respectively. We denote by $\varepsilon_{(j,h)}$ a Dirac measure of $(j, h) \in K$ and by $e_H$ the normalized Haar measure of $H$.

$\varepsilon_{(1,0)}^- = \varepsilon_{(1,0)}, \quad \varepsilon_{(2,0)}^- = \varepsilon_{(2,0)}, \quad \varepsilon_{(0,h_1)} * \varepsilon_{(0,h_2)} = \varepsilon_{(0,[h_1+h_2]_1)},$
$\varepsilon_{(0,h)} * \varepsilon_{(1,0)} = \varepsilon_{(1,0)}, \quad \varepsilon_{(0,h)} * \varepsilon_{(2,0)} = \varepsilon_{(2,0)},$
$\varepsilon_{(1,0)} * \varepsilon_{(1,0)} = \frac{1}{2}e_H + \frac{1}{2}\varepsilon_{(2,0)}, \quad \varepsilon_{(2,0)} * \varepsilon_{(2,0)} = \frac{1}{2}e_H + \frac{1}{2}\varepsilon_{(1,0)},$
$\varepsilon_{(1,0)} * \varepsilon_{(2,0)} = \frac{1}{2}\varepsilon_{(1,0)} + \frac{1}{2}\varepsilon_{(2,0)}.$

(3) Case of $H(\ell_1) \cong \mathbb{Z}_{m_1}$ and $H(\ell_2) \cong \mathbb{Z}_{m_2}$. Then $K = \mathbb{T} \cup S_1 \cup S_2 \cong \mathbb{T} \cup \mathbb{T} \cup \mathbb{T}$.
We identify $S_1$ and $S_2$ with $\{(1, s_1) : s_1 \in [0, \frac{1}{m_1}2\pi)\}$ and $\{(2, s_2) : s_2 \in [0, \frac{1}{m_2}2\pi)\}$ respectively. Let $d$ be the greatest common divisor of $m_1$ and $m_2$ and put $p_1 = \frac{m_1}{d}$ and $p_2 = \frac{m_2}{d}$.

(a) Case that both $p_1$ and $p_2$ are odd numbers.
We get one extension which is given by

$\varepsilon_{(1,s_1)}^- = \varepsilon_{(1,[-s_1]_{m_1})}, \quad \varepsilon_{(2,s_2)}^- = \varepsilon_{(2,[-s_2]_{m_2})},$
$\varepsilon_{(0,h_1)} * \varepsilon_{(0,h_2)} = \varepsilon_{(0,[h_1+h_2]_1)}, \quad \varepsilon_{(0,h)} * \varepsilon_{(1,s_1)} = \varepsilon_{(1,[h+s_1]_{m_1})},$
$\varepsilon_{(0,h)} * \varepsilon_{(2,s_2)} = \varepsilon_{(2,[h+s_2]_{m_2})},$
$\varepsilon_{(1,s_1)} * \varepsilon_{(1,t_1)} = \frac{1}{2m_1}\sum_{k=0}^{m_1-1} \varepsilon_{(0,[\frac{k}{m_1}2\pi+s_1+t_1]_1)} + \frac{1}{2p_1}\sum_{k=0}^{p_1-1} \varepsilon_{(2,[\frac{k}{m_2}2\pi+s_1+t_1]_{m_2})},$
$\varepsilon_{(2,s_2)} * \varepsilon_{(2,t_2)} = \frac{1}{2m_2}\sum_{k=0}^{m_2-1} \varepsilon_{(0,[\frac{k}{m_2}2\pi+s_2+t_2]_1)} + \frac{1}{2p_2}\sum_{k=0}^{p_2-1} \varepsilon_{(1,[\frac{k}{m_1}2\pi+s_2+t_2]_{m_1})},$
$\varepsilon_{(1,s_1)} * \varepsilon_{(2,s_2)} = \frac{1}{2p_2}\sum_{k=0}^{p_2-1} \varepsilon_{(1,[\frac{k}{m_1}2\pi+s_1+s_2]_{m_1})} + \frac{1}{2p_1}\sum_{k=0}^{p_1-1} \varepsilon_{(2,[\frac{k}{m_2}2\pi+s_1+s_2]_{m_2})}.$

This extension is splitting.

(b) Case that either $p_1$ or $p_2$ is an even number.
We get two extensions up to equivalence as extensions. One is the same in the above (1). Another one is given as follows.

We assume that $p_2$ is even number. Then we take $f_1 \in H(\ell_1)H(\ell_2)$ such that $[f_1] \neq [h_0]$ in $P_2(\ell_1)$, for example, $f_1 = \frac{1}{p_1 p_2 d}2\pi$.

$\varepsilon^{-}_{(1,s_1)} = \varepsilon_{(1,[-f_1-s_1]_{m_1})}$, $\varepsilon^{-}_{(2,s_2)} = \varepsilon_{(2,[-s_2]_{m_2})}$, $\varepsilon_{(0,h_1)} * \varepsilon_{(0,h_2)} = \varepsilon_{(0,[h_1+h_2]_1)}$,
$\varepsilon_{(0,h)} * \varepsilon_{(1,s_1)} = \varepsilon_{(1,[h+s_1]_{m_1})}$, $\varepsilon_{(0,h)} * \varepsilon_{(2,s_2)} = \varepsilon_{(2,[h+s_2]_{m_2})}$,
$\varepsilon_{(1,s_1)}*\varepsilon_{(1,t_1)} = \frac{1}{2m_1}\sum_{k=0}^{m_1-1}\varepsilon_{(0,[\frac{k}{m_1}2\pi+f_1+s_1+t_1]_1)}+\frac{1}{2p_1}\sum_{k=0}^{p_1-1}\varepsilon_{(2,[\frac{k}{m_2}2\pi+s_1+t_1]_{m_2})}$,
$\varepsilon_{(2,s_2)}*\varepsilon_{(2,t_2)} = \frac{1}{2m_2}\sum_{k=0}^{m_2-1}\varepsilon_{(0,[\frac{k}{m_2}2\pi+s_2+t_2]_1)}+\frac{1}{2p_2}\sum_{k=0}^{p_2-1}\varepsilon_{(1,[\frac{k}{m_1}2\pi+s_2+t_2]_{m_1})}$,
$\varepsilon_{(1,s_1)}*\varepsilon_{(2,s_2)} = \frac{1}{2p_2}\sum_{k=0}^{p_2-1}\varepsilon_{(1,[\frac{k}{m_1}2\pi+s_1+s_2]_{m_1})}+\frac{1}{2p_1}\sum_{k=0}^{p_1-1}\varepsilon_{(2,[\frac{k}{m_2}2\pi+s_1+s_2]_{m_2})}$.

(4) Case of $H(\ell_1) \cong \mathbb{Z}_m$ and $H(\ell_2) = H$. Then $K = \mathbb{T} \cup S_1 \cup \{0\} \cong \mathbb{T} \cup \mathbb{T} \cup \{0\}$.

We identify $S_1$ with $\{(1, s_1) : s_1 \in [0, \frac{1}{m}2\pi)\}$.

$\varepsilon^{-}_{(1,s_1)} = \varepsilon_{(1,[-s_1]_{m_1})}$, $\varepsilon^{-}_{(2,0)} = \varepsilon_{(2,0)}$, $\varepsilon_{(0,h_1)} * \varepsilon_{(0,h_2)} = \varepsilon_{(0,[h_1+h_2]_1)}$,
$\varepsilon_{(0,h)} * \varepsilon_{(1,s_1)} = \varepsilon_{(1,[h+s_1]_{m_1})}$, $\varepsilon_{(0,h)} * \varepsilon_{(2,0)} = \varepsilon_{(2,0)}$,
$\varepsilon_{(1,s_1)} * \varepsilon_{(1,t_1)} = \frac{1}{2m}\sum_{k=0}^{m-1}\varepsilon_{(0,[\frac{k}{m}2\pi+s_1+t_1]_1)} + \frac{1}{2}\varepsilon_{(2,0)}$,
$\varepsilon_{(2,0)} * \varepsilon_{(2,0)} = \frac{1}{2}e_H + \frac{1}{2m}e_H * \varepsilon_{(1,0)}$,
$\varepsilon_{(1,s_1)} * \varepsilon_{(2,0)} = \frac{1}{2m}e_H * \varepsilon_{(1,0)} + \frac{1}{2}\varepsilon_{(2,0)}$.

In the case that $H(\ell_1) = H$ and $H(\ell_2) \cong \mathbb{Z}_m$, we obtain similar conclusion for $K \cong \mathbb{T} \cup \{0\} \cup \mathbb{T}$.

**Example 3.30.** $H = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

If $n$ is a prime number, then there are two extension i.e. $K = H \times L$ or $K = H \vee L$. If $n$ is an odd number, then the extensions are similar to 3-(a) in Example 3.29. If $n$ is an even number, then the extensions are similar to 3 in Example 3.29.

**Example 3.31.** $H = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n}$.

If $H(\ell_1) = H$ and $H(\ell_2) = \{h_0\}$, then we know that $P_2(\ell_1) = \{p_0\}$ and $P_2(\ell_2) = H = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n}$. In this case we obtain extensions associated with each element of $H$, which are not mutually equivalent as extensions.

## 4. Signed Actions of Finite Hypergroups and the Extension Problem

4.1. **Signed actions of signed hypergroups.** For a finite set $X = \{x_1, x_2, \cdots, x_m\}$, $B(M^b(X))$ denotes the algebra of all (bounded) linear operators on the linear space $M^b(X)$ over $\mathbb{C}$.

**Definition.** We call $\alpha$ a *signed action* of a finite signed hypergroup $K$ on a set $X$ if $\alpha$ satisfies the following conditions.

(1) $\alpha$ is a homomorphism from $M^b(K)$ to $B(M^b(X))$ as algebras such that $\alpha(\varepsilon_{c_0})$ is the identity mapping on $M^b(X)$.
(2) For $c_i \in K$ and $\mu \in M^1(X)$, $\alpha(\varepsilon_{c_i})\mu \in M^1_{\mathbb{R}}(X)$.
(3) For the normalized Haar measure $e_K$ of $K$ and $\mu \in M^1(X)$, $\alpha(e_K)\mu \in M^1(X)$.

Moreover, if the condition

(2') For $c_i \in K$ and $\mu \in M^1(X)$, $\alpha(\varepsilon_{c_i})\mu \in M^1(X)$

holds, then we call $\alpha$ an *action* of $K$ on $X$.

We denote $\alpha(\varepsilon_{c_i})$ by $\alpha(c_i)$. A subset $S$ of $X$ is called *invariant* under the signed action $\alpha$ of $K$ if $\mathrm{supp}(\alpha(e_K)\delta_x) \subset S$ for any $x \in S$.

**Definition.** A signed action $\alpha$ of a finite signed hypergroup $K$ on $X$ is called *irreducible* if a non-empty subset $S$ of $X$ which is invariant under the signed action $\alpha$ must be $X$.

For a signed hypergroup $K$, When we take $X = K$ and $\rho^K(c_i)\varepsilon_{c_j} = \varepsilon_{c_i} * \varepsilon_{c_j}$ for $c_i, c_j \in K$, we get a signed action $\rho^K$ of $K$ on $K$. We call this signed action $\rho^K$ the *(left) regular action* of $K$. It is easy to check that the regular action $\rho^K$ is irreducible.

**Lemma 4.1.** *If a non-negative measure $\mu$ on $X$ is invariant under an irreducible signed action $\alpha$ of $K$ on $X$, then supp$(\mu) = \emptyset$ or supp$(\mu) = X$.*

*Proof.* Let $\mu$ be a non-negative measure on $X$ such that $\mu$ is invariant under an irreducible signed action $\alpha$ of $K$ on $X$ and $\mu \neq 0$. Put $S = \mathrm{supp}(\mu)$. Then $S \neq \emptyset$ because $\mu \neq 0$. The measure $\mu$ is written by

$$\mu = t_1\delta_{x_1} + \cdots + t_m\delta_{x_m}$$

where $t_j \geq 0$ $(j = 1, 2, \cdots, m)$. Then we have

$$\begin{aligned}\alpha(e_K)\mu &= \alpha(e_K)(t_1\delta_{x_1} + \cdots + t_m\delta_{x_m}) \\ &= t_1\alpha(e_K)\delta_{x_1} + \cdots + t_m\alpha(e_K)\delta_{x_m}.\end{aligned}$$

Since $\alpha(e_K)\delta_{x_1}, \cdots, \alpha(e_K)\delta_{x_m}$ are non-negative probability measures by the condition (3) of the definition of a signed action, $\alpha(e_K)\mu$ must be a non-negative measure. Then for any $x \in S$, we have

$$\mathrm{supp}(\alpha(e_K)\delta_x) \subset \mathrm{supp}(\alpha(e_K)\mu) = \mathrm{supp}(\mu) = S.$$

Hence $\mathrm{supp}(\mu) = X$ by irreducibility of the signed action $\alpha$. □

**Proposition 4.2.** *An irreducible signed action $\alpha$ of a finite signed hypergroup $K$ has the unique invariant probability measure on $X$.*

*Proof.* For the normalized Haar measure $e_K$ on $K$ and $x \in X$, put $\mu = \alpha(e_K)\delta_x$. It is easy to check that $\mu$ is an $\alpha$-invariant probability measure on X.

Assume that $\mu_1$ and $\mu_2$ are $\alpha$-invariant probability measures on $X$ written by

$$\mu_1 = t_1\delta_{x_1} + t_2\delta_{x_2} + \cdots + t_m\delta_{x_m},$$
$$\mu_2 = s_1\delta_{x_1} + s_2\delta_{x_2} + \cdots + s_m\delta_{x_m}.$$

We note that $t_1, t_2, \cdots, t_m$ and $s_1, s_2, \cdots, s_m$ are all positive real numbers by Lemma 4.1. Take the minimum value $\frac{t_i}{s_i}$ among $\frac{t_1}{s_1}, \frac{t_2}{s_2}, \cdots, \frac{t_m}{s_m}$ and put $\mu = \mu_1 - \frac{t_i}{s_i}\mu_2$. Then, $\mu$ is a non-negative measure on $X$ and $x_i \notin \mathrm{supp}(\mu)$ by the fact that

$$\begin{aligned} \mu &= s_1\left(\frac{t_1}{s_1} - \frac{t_i}{s_i}\right)\delta_{x_1} + \cdots + s_i\left(\frac{t_i}{s_i} - \frac{t_i}{s_i}\right)\delta_{x_i} + \cdots + s_m\left(\frac{t_m}{s_m} - \frac{t_i}{s_i}\right)\delta_{x_m} \\ &= s_1\left(\frac{t_1}{s_1} - \frac{t_i}{s_i}\right)\delta_{x_1} + \cdots + 0 \cdot \delta_{x_i} + \cdots + s_m\left(\frac{t_m}{s_m} - \frac{t_i}{s_i}\right)\delta_{x_m}. \end{aligned}$$

Hence $\mathrm{supp}(\mu) \neq X$. It is easy to see that $\mu$ is $\alpha$-invariant. Then we have $\mathrm{supp}(\mu) = \phi$ by Lemma 4.1. This implies that $\mu = 0$. Therefore we obtain $\mu_1 = \frac{t_i}{s_i}\mu_2$. Since $\mu_1(X) = 1$ and $\mu_2(X) = 1$, we obtain $\frac{t_i}{s_i} = 1$. Hence $\mu_1 = \mu_2$. □

**Remark.** When the $\alpha$-invariant probability measure $\mu$ on $X$ is written by

$$\mu = t_1\delta_{x_1} + t_2\delta_{x_2} + \cdots + t_m\delta_{x_m},$$

where $t_j > 0$ $(j = 1, 2, \cdots, m)$ and $\sum_{j=1}^{m} t_j = 1$, the representing matrix of $\alpha(e_K)$ associated with the basis $\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_m}$ is

$$\alpha(e_K) = \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_2 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_m & t_m & \cdots & t_m \end{pmatrix}$$

by the fact that $\alpha(e_K)\delta_{x_1} = \mu$, $\alpha(e_K)\delta_{x_2} = \mu$, $\cdots$, $\alpha(e_K)\delta_{x_m} = \mu$. We note that $\alpha(e_K)$ is a rank one projection.

**Definition.** A signed action $\alpha$ of a finite signed hypergroup $K$ on $X$ is called to be *equivalent* to a signed action $\beta$ of $K$ on $Y$ if there exists a bijection $\psi$ from $X$ to $Y$ such that

$$\beta(c_j) = \psi_* \circ \alpha(c_j) \circ \psi_*^{-1}$$

for all $c_j \in K$ where $\psi_*$ is a linear isomorphism from $M^b(X)$ to $M^b(Y)$ given by $\psi_*(\delta_x) = \delta_{\psi(x)}$ for $x \in X$.

In this Chapter we report to succeed to determine all irreducible signed actions of signed hypergroups $\mathbb{Z}_q(2)$ $(q > 0)$ of order two and all two dimensional irreducible signed actions of hypergroups of order three.

4.2. **Irreducible signed actions of signed hypergroup of order two.** Let $K = \{c_0, c_1\}$ be a signed hypergroup of order two with unit $c_0$ where the structure is characterized by a parameter $q$ $(q > 0)$ given by

$$\varepsilon_{c_1} * \varepsilon_{c_1} = q\varepsilon_{c_0} + (1-q)\varepsilon_{c_1}.$$

We denote this hypergroup $K$ by $\mathbb{Z}_q(2)$. The total weight $w(K)$ of $K$ is $w(K) = \frac{1+q}{q}$ and the normalized Haar measure $e_K$ of $K$ is given by

$$e_K = \frac{q}{1+q}\varepsilon_{c_0} + \frac{1}{1+q}\varepsilon_{c_1}.$$

Let $\alpha$ be an irreducible signed action of $K$ on $X = \{x_1, x_2, \cdots, x_m\}$ and $\mu$ the unique $\alpha$-invariant probability measure on $X$ which is written by

$$\mu = t_1\delta_{x_1} + t_2\delta_{x_2} + \cdots + t_m\delta_{x_m}$$

where $0 < t_j < 1$ $(j = 1, 2, \cdots, m)$ and $t_1 + t_2 + \cdots + t_m = 1$. For $t = (t_1, t_2, \cdots, t_m)$, $\alpha$ is characterized by a parameter $t$. We denote $\alpha$ by $\alpha^t$. Then we see that

$$\alpha^t(c_1) = (1+q)\alpha^t(e_K) - q\alpha^t(c_0)$$

and the representation matrices of $\alpha^t(e_K)$ and $\alpha^t(c_0)$ associated with the basis $\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_m}$ in $M^b(X)$ are

$$\alpha^t(e_K) = \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_2 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_m & t_m & \cdots & t_m \end{pmatrix} \text{ and } \alpha^t(c_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then we obtain

$$\alpha^t(c_1) = \begin{pmatrix} (1+q)t_1 - q & (1+q)t_1 & \cdots & (1+q)t_1 \\ (1+q)t_2 & (1+q)t_2 - q & \cdots & (1+q)t_2 \\ \vdots & \vdots & \ddots & \vdots \\ (1+q)t_m & (1+q)t_m & \cdots & (1+q)t_m - q \end{pmatrix} \cdots (*)$$

with a parameter $t = (t_1, t_2, \cdots, t_m)$, where $0 < t_j < 1$ $(j = 1, 2, \cdots, m)$ and $t_1 + t_2 + \cdots + t_m = 1$. Let $S_m$ be the symmetric group of order $m$. For $\sigma \in S_m$ and $t = (t_1, t_2, \cdots, t_m)$, we denote $(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)})$ by $\sigma(t)$. Then we have the following proposition on irreducible signed actions of $\mathbb{Z}_q(2)$.

**Proposition 4.3.** (1) *When $m \geq 2$, $0 < t_j < 1$ $(j = 1, 2, \cdots, m)$ and $t_1 + t_2 + \cdots + t_m = 1$, the action $\alpha^t$ given by $(*)$ with the parameter $t = (t_1, t_2, \cdots, t_m)$ is an irreducible action of $\mathbb{Z}_q(2)$.*

(2) *All irreducible signed actions of $\mathbb{Z}_q(2)$ are obtained in this way.*

(3) *For two irreducible signed actions $\alpha^t$ and $\alpha^{t'}$ of $\mathbb{Z}_q(2)$, $\alpha^t$ is equivalent to $\alpha^{t'}$ if and only if there exists $\sigma \in S_m$ such that $t = \sigma(t')$.*

**Remark.** The signed action $\alpha^t$ of a hypergroup $\mathbb{Z}_q(2)$ $(0 < q \leq 1)$ is an action if and only if $m \leq \frac{1+q}{q}$ and $\frac{q}{1+q} \leq t_j \leq \frac{1}{1+q}$ when $m \geq 2$.

4.3. **Two-dimensional irreducible signed action of a signed hypergroup of order three.** Let $K = \{c_0, c_1, c_2\}$ be a signed hypergroup of order three with unit $c_0$ and $\hat{K} = \{\chi_0, \chi_1, \chi_2\}$ where $\chi_0(c) = 1$ for $c \in K$. Let $\alpha$ be an irreducible two dimensional signed action of $K$ on $X = \{x_1, x_2\}$ and $\mu = t\delta_{x_1} + (1-t)\delta_{x_2}$ $(0 < t < 1)$ be the unique $\alpha$-invariant probability measure on $X$.

**Lemma 4.4.** *Under the above situation there exists a measure $\nu$ on $X$ such that $\alpha(c)\nu = \chi(c)\nu$ for some $\chi \in \hat{K}$ where $\chi \neq \chi_0$. Moreover, $\alpha(e_K)\nu = 0$ for the normalized Haar measure $e_K$ of $K$.*

*Proof.* We may assume that there exists an eigen vector $\nu \in M^b(X)$ with an eigen value $\lambda(c_1) \neq 1$ such that $\alpha(c_1)\nu = \lambda(c_1)\nu$ for $c_1 \in K$ by irreducibility of the action $\alpha$ of $K$ on $X$. Then we see that

$$\alpha(e_K)\nu = \alpha(e_K)\alpha(c_1)\nu = \lambda(c_1)\alpha(e_K)\nu.$$

The fact $\lambda(c_1) \neq 1$ implies that $\alpha(e_K)\nu = 0$. Since $\alpha(e_K)$ is a linear combination of $\alpha(c_0), \alpha(c_1)$ and $\alpha(c_2)$, we obtain $\alpha(c_2)\nu = \lambda(c_2)\nu$ for some $\lambda(c_2) \in \mathbb{C}$.

By the fact that $\alpha(c_ic_j) = \alpha(c_i)\alpha(c_j)$, we see that $\lambda(c_ic_j) = \lambda(c_i)\lambda(c_j)$. Hence $\lambda(c) = \chi(c)$ for some $\chi \in \hat{K}$ such that $\chi \neq \chi_0$. □

The representing matrices of $\alpha(e_K), \alpha(c_0), \alpha(c_1)$ and $\alpha(c_2)$ associated with eigen vectors $\mu$ and $\nu$ on $M^b(X)$ are

$$\alpha(e_K) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \alpha(c_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\alpha(c_1) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \ \alpha(c_2) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1 = \chi(c_1)$ and $\lambda_2 = \chi(c_2)$.

The representing matrix $E(t)$ of $\alpha(e_K)$ associated with $\delta_{x_1}$ and $\delta_{x_2}$ is

$$E(t) = \begin{pmatrix} t & t \\ 1-t & 1-t \end{pmatrix}.$$

Take a matrix $T(t)$ which satisfies that

$$E(t) = T(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T(t)^{-1}.$$

For example, we take $T(t) = \begin{pmatrix} t & -1 \\ 1-t & 1 \end{pmatrix}$ and put

$$A(t, \lambda) := T(t) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} T(t)^{-1}.$$

Then we have

$$A(t, \lambda) = \begin{pmatrix} \lambda + (1-\lambda)t & (1-\lambda)t \\ (1-\lambda) - (1-\lambda)t & 1-(1-\lambda)t \end{pmatrix}.$$

We note that $A(t, \lambda)$ does not depend on the choice of $T(t)$. Hence we obtain irreducible signed actions $\alpha_1^t$ and $\alpha_2^r$ of $K$ on $X = \{x_1, x_2\}$ whose representing matrices associated with $\delta_{x_1}$ and $\delta_{x_2}$ are respectively

(1) $\alpha_1^t(c_1) = A(t, \chi_1(c_1))$ and $\alpha_1^t(c_2) = A(t, \chi_1(c_2))$,
(2) $\alpha_2^r(c_1) = A(r, \chi_2(c_1))$ and $\alpha_2^r(c_2) = A(r, \chi_2(c_2))$.

**Proposition 4.5.** (1) *The action $\alpha$ given by $\alpha^t(c_i) = A(t, \chi(c_i))$ with the parameter $0 < t < 1$ is a two-dimensional irreducible signed actions of $K$ on $X$.*

(2) *All two dimensional irreducible signed actions of $K$ are obtained in this way.*

(3) *For the character $\chi_1, \chi_2 \in \hat{K}$, the actions $\alpha^t$ and $\beta^r$ given by $\alpha^t(c_i) = A(t, \chi_1(c_i))$ and $\beta^t(c_i) = A(t, \chi_2(c_i))$ respectively are never mutually equivalent.*

(4) *The action $\alpha^t$ (resp. $\beta^r$) is equivalent to $\alpha^{t'}$ (resp. $\beta^{r'}$) if and only if $t' = t$ or $t' = 1-t$ (resp. $r' = r$ or $r' = 1-r$).*

**Example 4.6.** The case that $K$ is the Golden hypergroup $\mathbb{G} = \{c_0, c_1, c_2\}$ with unit $c_0$. The structure equations of $\mathbb{G}$ are given by

$$\varepsilon_{c_1} * \varepsilon_{c_1} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_2},\ \varepsilon_{c_2} * \varepsilon_{c_2} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_1},\ \varepsilon_{c_1} * \varepsilon_{c_2} = \frac{1}{2}\varepsilon_{c_1} + \frac{1}{2}\varepsilon_{c_2}.$$

Let $\hat{\mathbb{G}} = \{\chi_0, \chi_1, \chi_2\}$ be the dual of $\mathbb{G}$ such that $\chi_1(c_1) = a = \frac{-1+\sqrt{5}}{4}$, $\chi_1(c_2) = b = \frac{-1-\sqrt{5}}{4}$, $\chi_2(c_1) = b$ and $\chi_2(c_2) = a$. Then we have

$$\alpha^t(c_1) = \begin{pmatrix} a + (1-a)t & (1-a)t \\ (1-a) - (1-a)t & 1 - (1-a)t \end{pmatrix},$$

$$\alpha^t(c_2) = \begin{pmatrix} b + (1-b)t & (1-b)t \\ (1-b) - (1-b)t & 1 - (1-b)t \end{pmatrix}$$

and

$$\beta^r(c_1) = \begin{pmatrix} b + (1-b)r & (1-b)r \\ (1-b) - (1-b)r & 1 - (1-b)r \end{pmatrix},$$

$$\beta^r(c_2) = \begin{pmatrix} a + (1-a)r & (1-a)r \\ (1-a) - (1-a)r & 1 - (1-a)r \end{pmatrix}$$

where $0 < t < 1$ and $0 < r < 1$.

**Remark.** The signed action $\alpha^t$ (resp. $\beta^r$) of $\mathbb{G}$ is an action if and only if $\frac{-b}{1-b} \leq t \leq \frac{1}{1-b}$ (resp. $\frac{-b}{1-b} \leq r \leq \frac{1}{1-b}$).

**Example 4.7.** The case that $K$ is the conjugacy class hypergroup $K(S_3) = \{c_0, c_1, c_2\}$ of $S_3$ with unit $c_0$. The structure equations of $K(S_3)$ are

$$\varepsilon_{c_1} * \varepsilon_{c_1} = \frac{1}{2}\varepsilon_{c_0} + \frac{1}{2}\varepsilon_{c_1},\ \varepsilon_{c_2} * \varepsilon_{c_2} = \frac{1}{3}\varepsilon_{c_0} + \frac{2}{3}\varepsilon_{c_1},\ \varepsilon_{c_1} * \varepsilon_{c_2} = \varepsilon_{c_2}.$$

Let $\widehat{K(S_3)} = \{\chi_0, \chi_1, \chi_2\}$ be the dual of $K(S_3)$ such that $\chi_1(c_1) = 1$, $\chi_1(c_2) = -1$, $\chi_2(c_1) = -\frac{1}{2}$ and $\chi_2(c_2) = 0$. Then we have

$$\alpha^t(c_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \alpha^t(c_2) = \begin{pmatrix} -1 + 2t & 2t \\ 2 - 2t & 1 - 2t \end{pmatrix}$$

and

$$\beta^r(c_1) = \begin{pmatrix} -\frac{1}{2} + \frac{3}{2}r & \frac{3}{2}r \\ \frac{3}{2} - \frac{3}{2}r & 1 - \frac{3}{2}r \end{pmatrix},\ \beta^r(c_2) = \begin{pmatrix} r & r \\ 1 - r & 1 - r \end{pmatrix}$$

where $0 < t < 1$ and $0 < r < 1$.

**Remark.** The signed action $\alpha^t$ of $K(S_3)$ is an action if and only if $t = \frac{1}{2}$. The signed action $\beta^r$ of $K(S_3)$ is an action if and only if $\frac{1}{3} \leq r \leq \frac{2}{3}$.

**Example 4.8.** The case that $K$ is the character hypergroup $K(\hat{S}_3) = \{c_0, c_1, c_2\}$ of $S_3$ with unit $c_0$. The structure equations of $K(\hat{S}_3)$ are

$$\varepsilon_{c_1} * \varepsilon_{c_1} = \varepsilon_{c_0},\ \varepsilon_{c_2} * \varepsilon_{c_2} = \frac{1}{4}\varepsilon_{c_0} + \frac{1}{4}\varepsilon_{c_1} + \frac{1}{2}\varepsilon_{c_2},\ \varepsilon_{c_1} * \varepsilon_{c_2} = \varepsilon_{c_2}.$$

Let $\widehat{K(\hat{S}_3)} = \{\chi_0, \chi_1, \chi_2\}$ be the dual of $K(\hat{S}_3)$ such that $\chi_1(c_1) = 1$, $\chi_1(c_2) = -\frac{1}{2}$, $\chi_2(c_1) = -1$ and $\chi_2(c_2) = 0$. Then we have

$$\alpha^t(c_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha^t(c_2) = \begin{pmatrix} -\frac{1}{2} + \frac{3}{2}t & \frac{3}{2}t \\ \frac{3}{2} - \frac{3}{2}t & 1 - \frac{3}{2}t \end{pmatrix}$$

and

$$\beta^r(c_1) = \begin{pmatrix} -1 + 2r & 2r \\ 2 - 2r & 1 - 2r \end{pmatrix}, \ \beta^r(c_2) = \begin{pmatrix} r & r \\ 1 - r & 1 - r \end{pmatrix}$$

where $0 < t < 1$ and $0 < r < 1$.

**Remark.** The signed action $\alpha^t$ of $K(\hat{S}_3)$ is an action if and only if $\frac{1}{3} \leq t \leq \frac{2}{3}$. The signed action $\beta^r$ of $K(\hat{S}_3)$ is an action if and only if $r = \frac{1}{2}$.

4.4. **Applications to the extension problem.** Our strategy to solve the extension problem is to apply irreducible actions which are already determined. Let $H$ and $L$ be finite commutative hypergroups and $K$ be an extension of $L$ by $H$, i.e. the sequence

$$1 \longrightarrow H \longrightarrow K \overset{\varphi}{\longrightarrow} L \longrightarrow 1$$

is exact. We note that $K$ is a finite commutative hypergroup. We denote $L = \{\ell_0, \ell_1, \cdots, \ell_p\}$ and the unit by $\ell_0$. Put $S(\ell_j) = \varphi^{-1}(\ell_j)$ for $\ell_j \in L$. Then $K$ is decomposed as $K = \bigcup_{j=0}^{p} S(\ell_j)$ where $S(\ell_0) = H$.

Next proposition plays an essential role to our strategy.

**Proposition 4.9.** *Let $\rho^K$ be the regular action of $K$ and $\rho_H^K$ be the action of $H$ which is the restriction of $\rho^K$ to $H$. Then $\rho_H^K$ is decomposed as actions of $H$ by*

$$(\rho_H^K, K) = \sum_{j=0}^{p} \oplus \, (\rho_j, S(\ell_j))$$

*where $\rho_j$ is an irreducible action of $H$ on $S(\ell_j)$ and $\rho_0$ is the regular action $\rho^H$ of $H$.*

**Remark.** Let $\nu_j$ be the invariant probability measure on $S(\ell_j) = \{s_1, s_2, \cdots, s_m\}$ under the action $\rho_j$ of $H$, which is written by

$$\nu_j = t_1\varepsilon_{s_1} + t_2\varepsilon_{s_2} + \cdots + t_m\varepsilon_{s_m}$$

where $t_i > 0$ $(i = 1, 2, \cdots, m)$ and $\sum_{i=1}^{m} t_i = 1$. Then we note that the weight $w(s_i)$ is given by $w(s_i) = t_i w(S(\ell_j)) = t_i w(\ell_j) w(H)$, refer to [IK2].

**Our strategy.**

(1) The irreducible action $\rho_j$ gives the convolution $\varepsilon_h * \varepsilon_s$ of $h \in H$ and $s \in S(\ell_j)$ by $\varepsilon_h * \varepsilon_s := \rho_j(h)\varepsilon_s$.
(2) The invariant probability measure $\nu_j$ under $\rho_j$ gives each weight $w(s)$ for $s \in S(\ell_j)$, so that the normalized Haar measure $e_K$ of $K$ is determined.
(3) The other structure comes from the conditions of commutativity of the regular action $\rho^K$ of $K$.

**Example 4.10.** The case that $H = \mathbb{Z}_q(2) = \{h_0, h_1\}$ $(0 < q \leq 1)$, $L = \mathbb{Z}_2 = \{\ell_0, \ell_1\}$, $K = H \cup S(\ell_1) = \{h_0, h_1, s_1, s_2\}$.

(1) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$.

By Proposition 4.3 and Remark, all two dimensional irreducible actions $\rho^t$ of $\mathbb{Z}_q(2)$ on $S(\ell_1)$ are given by

$$\rho^t(h_1) = \begin{pmatrix} (1+q)t - q & (1+q)t \\ (1+q)(1-t) & (1+q)(1-t) - q \end{pmatrix},$$

where $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$ and the invariant probability measure $\nu$ on $S(\ell_1)$ under the action $\rho^t$ is

$$\nu = t\varepsilon_{s_1} + (1-t)\varepsilon_{s_2}.$$

Since $w(S(\ell_1)) = \frac{1+q}{q}$, we have $w(s_1) = \frac{(1+q)t}{q}$ and $w(s_2) = \frac{(1+q)(1-t)}{q}$. We obtain the structure equations :

$$\begin{aligned}
\varepsilon_{h_1} * \varepsilon_{s_1} &= \rho^t(h_1)\varepsilon_{s_1} = ((1+q)t - q)\varepsilon_{s_1} + (1+q)(1-t)\varepsilon_{s_2}, \\
\varepsilon_{h_1} * \varepsilon_{s_2} &= \rho^t(h_1)\varepsilon_{s_2} = (1+q)t\varepsilon_{s_1} + ((1+q)(1-t) - q)\varepsilon_{s_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_2} &= \varepsilon_{h_1}, \\
\varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{q}{(1+q)t}\varepsilon_{h_0} + \left(1 - \frac{q}{(1+q)t}\right)\varepsilon_{h_1}, \\
\varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{q}{(1+q)(1-t)}\varepsilon_{h_0} + \left(1 - \frac{q}{(1+q)(1-t)}\right)\varepsilon_{h_1}
\end{aligned}$$

where $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$.

(2) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that $w(s_1) = w(s_2)$. Hence we get $t = \frac{1}{2}$, so the structure equations are

$$\begin{aligned}
\varepsilon_{h_1} * \varepsilon_{s_1} &= \rho^{\frac{1}{2}}(h_1)\varepsilon_{s_1} = \frac{1}{2}(1-q)\varepsilon_{s_1} + \frac{1}{2}(1+q)\varepsilon_{s_2}, \\
\varepsilon_{h_1} * \varepsilon_{s_2} &= \rho^{\frac{1}{2}}(h_1)\varepsilon_{s_2} = \frac{1}{2}(1+q)\varepsilon_{s_1} + \frac{1}{2}(1-q)\varepsilon_{s_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{2q}{1+q}\varepsilon_{h_0} + \frac{1-q}{1+q}\varepsilon_{h_1}, \ \varepsilon_{s_1} * \varepsilon_{s_1} = \varepsilon_{s_2} * \varepsilon_{s_2} = \varepsilon_{h_1}.
\end{aligned}$$

**Example 4.11.** The case that $H = \mathbb{Z}_q(2) = \{h_0, h_1\}$ $(0 < q \leq 1)$, $L = \mathbb{Z}_p(2) = \{\ell_0, \ell_1\}$ $(0 < p \leq 1)$, $K = H \cup S(\ell_1) = \{h_0, h_1, s_1, s_2\}$. By similar arguments to Example 4.10, we have the following.

(1) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$.

We have the irreducible action $\alpha^t$ of $\mathbb{Z}_q(2)$ on $S(\ell_1)$ with $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$, $w(s_0) = \frac{(1+q)t}{pq}$ and $w(s_1) = \frac{(1+q)(1-t)}{pq}$. We put

$$
\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{pq}{(1+q)t}\varepsilon_{h_0} + (p - \frac{pq}{(1+q)t})\varepsilon_{h_1} \\
&\quad +(1-p)d_0\varepsilon_{s_1} + (1-p)d_1\varepsilon_{s_2}, \\
\varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{pq}{(1+q)(1-t)}\varepsilon_{h_0} + (p - \frac{pq}{(1+q)(1-t)})\varepsilon_{h_1} \\
&\quad +(1-p)f_0\varepsilon_{s_1} + (1-p)f_1\varepsilon_{s_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_2} &= p\varepsilon_{h_1} + (1-p)g_0\varepsilon_{s_1} + (1-p)g_1\varepsilon_{s_2}
\end{aligned}
$$

where the parameters satisfy that $d_i, f_i, g_i \geq 0$ $(i = 0, 1)$, $d_0 + d_1 = 1, f_0 + f_1 = 1$ and $g_0 + g_1 = 1$. Then the regular action $\rho^K(s_1)$ and $\rho^K(s_2)$ are given by

$$
\rho^K(s_1) = \begin{pmatrix} 0 & 0 & \frac{pq}{(1+q)t} & 0 \\ 0 & 0 & p - \frac{pq}{(1+q)t} & p \\ 1 & (1+q)t - q & (1-p)d_0 & (1-p)g_0 \\ 0 & (1+q)(1-t) & (1-p)d_1 & (1-p)g_1 \end{pmatrix},
$$

$$
\rho^K(s_2) = \begin{pmatrix} 0 & 0 & 0 & \frac{pq}{(1+q)(1-t)} \\ 0 & 0 & p & p - \frac{pq}{(1+q)(1-t)} \\ 0 & (1+q)t & (1-p)g_0 & (1-p)f_0 \\ 1 & (1+q)(1-t) - q & (1-p)g_1 & (1-p)f_1 \end{pmatrix}.
$$

One can determine the structure by applying the commutativity condition $\rho^K(s_1)\rho^K(s_2) = \rho^K(s_2)\rho^K(s_1)$ as follows.

$$
\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{pq}{(1+q)t}\varepsilon_{h_0} + \left(p - \frac{pq}{(1+q)t}\right)\varepsilon_{h_1} \\
&\quad +(1-p)\left(1 - \frac{(1-t)r}{t}\right)\varepsilon_{s_1} + \frac{(1-p)(1-t)r}{t}\varepsilon_{s_2}, \\
\varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{pq}{(1+q)(1-t)}\varepsilon_{h_0} + \left(p - \frac{pq}{(1+q)(1-t)}\right)\varepsilon_{h_1} \\
&\quad +(1-p)\frac{t(1-r)}{1-t}\varepsilon_{s_1} + (1-p)\left(1 - \frac{t(1-r)}{1-t}\right)\varepsilon_{s_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_2} &= p\varepsilon_{h_1} + (1-p)(r\varepsilon_{s_1} + (1-r)\varepsilon_{s_2})
\end{aligned}
$$

where $\frac{q}{1+q} \leq t \leq \frac{1}{2}$ and $0 \leq r \leq \frac{t}{1-t}$, or $\frac{1}{2} \leq t \leq \frac{1}{1+q}$ and $\frac{2t-1}{t} \leq r \leq 1$. We denote $K$ by $K = K(t, u)$.

(2) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that $w(s_1) = w(s_2)$ and $t = \frac{1}{2}$. Therefore we obtain the structure equations :

$$\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{2pq}{1+q}\varepsilon_{h_0} + \left(p - \frac{2pq}{1+q}\right)\varepsilon_{h_1} + \frac{1-p}{2}\varepsilon_{s_1} + \frac{1-p}{2}\varepsilon_{s_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_1} &= \varepsilon_{s_2} * \varepsilon_{s_2} = p\varepsilon_{h_1} + \frac{1-p}{2}\varepsilon_{s_1} + \frac{1-p}{2}\varepsilon_{s_2}.
\end{aligned}$$

**Example 4.12.** The case that $H = \mathbb{G} = \{h_0, h_1, h_2\}$, $L = \mathbb{Z}_2 = \{\ell_0, \ell_1\}$, $K = H \cup S(\ell_1) = \{h_0, h_1, h_2, s_1, s_2\}$.

(1) By Example 4.7, a two-dimensional irreducible signed actions $\alpha^t$ of $H$ are given by

$$\alpha^t(h_1) = \begin{pmatrix} a + (1-a)t & (1-a)t \\ (1-a)(1-t) & 1-(1-a)t \end{pmatrix},$$

$$\alpha^t(h_2) = \begin{pmatrix} b + (1-b)t & (1-b)t \\ (1-b)(1-t) & 1-(1-b)t \end{pmatrix}$$

where $a = \frac{-1+\sqrt{5}}{4}$, $b = \frac{-1-\sqrt{5}}{4}$ and $\frac{-b}{1-b} \leq t \leq \frac{1}{1-b}$.

(a) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$. We obtain the structure equations :

$$\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{2}{5}(1-a)\varepsilon_{h_1} + \frac{2}{5}(1-b)\varepsilon_{h_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{1}{5t}\varepsilon_{h_0} + \frac{2}{5}\left((1-a) + \frac{1}{t}a\right)\varepsilon_{h_1} \\
&\quad + \frac{2}{5}\left((1-b) + \frac{1}{t}b\right)\varepsilon_{h_2}, \\
\varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{1}{5(1-t)}\varepsilon_{h_0} + \frac{2}{5}\left(1 + \frac{at}{1-t}\right)\varepsilon_{h_1} \\
&\quad + \frac{2}{5}\left(1 + \frac{bt}{1-t}\right)\varepsilon_{h_2}
\end{aligned}$$

where $\frac{-b}{1-b} \leq t \leq \frac{1}{1-b}$.

(b) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that the structure equations are

$$\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{2}{5}\varepsilon_{h_0} + \frac{2}{5}(1+a)\varepsilon_{h_1} + \frac{2}{5}(1+b)\varepsilon_{h_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_1} &= \varepsilon_{s_2} * \varepsilon_{s_2} = \frac{2}{5}(1-a)\varepsilon_{h_1} + \frac{2}{5}(1-b)\varepsilon_{h_2}.
\end{aligned}$$

(2) By Example 4.7, the other two-dimensional irreducible signed actions $\beta^r$ of $H$ are given by

$$\beta^r(h_1) = \begin{pmatrix} b+(1-b)r & (1-b)r \\ (1-b)(1-r) & 1-(1-b)r \end{pmatrix},$$

$$\beta^r(h_2) = \begin{pmatrix} a+(1-a)r & (1-a)r \\ (1-a)(1-r)r & 1-(1-a)r \end{pmatrix}$$

where $a = \frac{-1+\sqrt{5}}{4}$, $b = \frac{-1-\sqrt{5}}{4}$ and $\frac{-b}{1-b} \leq r \leq \frac{1}{1-b}$.

(a) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$. We obtain the structure equations :

$$\begin{aligned} \varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{2}{5}(1-b)\varepsilon_{h_1} + \frac{2}{5}(1-a)\varepsilon_{h_2}, \\ \varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{1}{5r}\varepsilon_{h_0} + \frac{2}{5}\left((1-b) + \frac{1}{r}b\right)\varepsilon_{h_1} \\ &\quad + \frac{2}{5}\left((1-a) + \frac{1}{r}a\right)\varepsilon_{h_2}, \\ \varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{1}{5(1-r)}\varepsilon_{h_0} + \frac{2}{5}\left(1 + \frac{br}{1-r}\right)\varepsilon_{h_1} \\ &\quad + \frac{2}{5}\left(1 + \frac{ar}{1-r}\right)\varepsilon_{h_2} \end{aligned}$$

where $\frac{-b}{1-b} \leq r \leq \frac{1}{1-b}$.

(b) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that the structure equations are

$$\begin{aligned} \varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{2}{5}\varepsilon_{h_0} + \frac{2}{5}(1+b)\varepsilon_{h_1} + \frac{2}{5}(1+a)\varepsilon_{h_2}, \\ \varepsilon_{s_1} * \varepsilon_{s_1} &= \varepsilon_{s_2} * \varepsilon_{s_2} = \frac{2}{5}(1-b)\varepsilon_{h_1} + \frac{2}{5}(1-a)\varepsilon_{h_2}. \end{aligned}$$

**Example 4.13.** The case that $H = K(S_3) = \{h_0, h_1, h_2\}$, $L = \mathbb{Z}_2$, $K = H \cup S = \{h_0, h_1, h_2, s_1, s_2\}$.

(1) By Example 4.7, a two-dimensional irreducible signed actions $\alpha$ of $H$ are given by

$$\alpha(h_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha(h_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$. We obtain the structure equations :

$$\begin{aligned} \varepsilon_{s_1} * \varepsilon_{s_2} &= \varepsilon_{h_1}, \\ \varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{1}{3}\varepsilon_{h_0} + \frac{2}{3}\varepsilon_{h_1}, \quad \varepsilon_{s_2} * \varepsilon_{s_2} = \frac{1}{3}\varepsilon_{h_0} + \frac{2}{3}\varepsilon_{h_1}. \end{aligned}$$

(b) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that the structure equations are

$$\varepsilon_{s_1} * \varepsilon_{s_2} = \frac{1}{3}\varepsilon_{h_0} + \frac{2}{3}\varepsilon_{h_1}, \quad \varepsilon_{s_1} * \varepsilon_{s_1} = \varepsilon_{s_2} * \varepsilon_{s_2} = \varepsilon_{h_2}.$$

(2) By Example 4.7, the other two-dimensional irreducible signed actions $\beta^r$ of $H$ are given by

$$\beta^r(h_1) = \begin{pmatrix} -\frac{1}{2} + \frac{3}{2}r & \frac{3}{2}r \\ \frac{3}{2} - \frac{3}{2}r & 1 - \frac{3}{2}r \end{pmatrix}, \ \beta^r(h_2) = \begin{pmatrix} r & r \\ 1-r & 1-r \end{pmatrix}$$

where $\frac{1}{3} \le r \le \frac{2}{3}$.

(a) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$. We obtain the structure equations :

$$\begin{aligned} \varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{1}{2}\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{h_2}, \\ \varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{1}{6r}\varepsilon_{h_0} + \left(\frac{1}{2} - \frac{1}{6r}\right)\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{h_2}, \\ \varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{1}{6(1-r)}\varepsilon_{h_0} + \left(\frac{1}{2} - \frac{1}{6(1-r)}\right)\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{h_2} \end{aligned}$$

where $\frac{1}{3} \le r \le \frac{2}{3}$.

(b) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that the structure equations are

$$\begin{aligned} \varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{1}{3}\varepsilon_{h_0} + \frac{1}{6}\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{h_2}, \\ \varepsilon_{s_1} * \varepsilon_{s_1} &= \varepsilon_{s_2} * \varepsilon_{s_2} = \frac{1}{2}\varepsilon_{h_1} + \frac{1}{2}\varepsilon_{h_2}. \end{aligned}$$

**Example 4.14.** The case that $H = K(\hat{S}_3) = \{h_0, h_1, h_2\}$, $L = \mathbb{Z}_2 = \{\ell_0, \ell_1\}, K = H \cup S = \{h_0, h_1, h_2, s_1, s_2\}$.

(1) By Example 4.8, a two-dimensional irreducible signed actions $\alpha^t$ of $H$ are given by

$$\alpha^t(h_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha^t(h_2) = \begin{pmatrix} -\frac{1}{2} + \frac{3}{2}t & \frac{3}{2}t \\ \frac{3}{2} - \frac{3}{2}t & 1 - \frac{3}{2}t \end{pmatrix}$$

where $\frac{1}{3} \le t \le \frac{2}{3}$.

(a) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$. We obtain the structure equations :

$$\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_2} &= \varepsilon_{h_2}, \\
\varepsilon_{s_1} * \varepsilon_{s_1} &= \frac{1}{6t}\varepsilon_{h_0} + \frac{1}{6t}\varepsilon_{h_1} + \left(1 - \frac{1}{3t}\right)\varepsilon_{h_2}, \\
\varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{1}{6(1-t)}\varepsilon_{h_0} + \frac{1}{6(1-t)}\varepsilon_{h_1} + \left(1 - \frac{1}{3(1-t)}\right)\varepsilon_{h_2}
\end{aligned}$$

where $\frac{1}{3} \leq t \leq \frac{2}{3}$.

(b) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that the structure equations are

$$\varepsilon_{s_1} * \varepsilon_{s_2} = \frac{1}{3}\varepsilon_{h_0} + \frac{1}{3}\varepsilon_{h_1} + \frac{1}{3}\varepsilon_{h_2}, \ \varepsilon_{s_1} * \varepsilon_{s_1} = \varepsilon_{s_2} * \varepsilon_{s_2} = \varepsilon_{h_2}.$$

(2) By Example 4.8, the other two-dimensional irreducible signed actions $\beta$ of $H$ are given by

$$\beta(h_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \beta(h_2) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(a) The case that $K$ is hermitian, namely $s_1^- = s_1$ and $s_2^- = s_2$. We obtain the structure equations :

$$\begin{aligned}
\varepsilon_{s_1} * \varepsilon_{s_2} &= \frac{1}{3}\varepsilon_{h_1} + \frac{2}{3}\varepsilon_{h_2}, \quad \varepsilon_{s_1} * \varepsilon_{s_1} = \frac{1}{3}\varepsilon_{h_0} + \frac{2}{3}\varepsilon_{h_2}, \\
\varepsilon_{s_2} * \varepsilon_{s_2} &= \frac{1}{3}\varepsilon_{h_0} + \frac{2}{3}\varepsilon_{h_2}.
\end{aligned}$$

(b) The case that $K$ is not hermitian, namely $s_1^- = s_2$ and $s_2^- = s_1$. In this case it is easy to see that the structure equations are

$$\varepsilon_{s_1} * \varepsilon_{s_2} = \frac{1}{3}\varepsilon_{h_0} + \frac{2}{3}\varepsilon_{h_2}, \quad \varepsilon_{s_1} * \varepsilon_{s_1} = \varepsilon_{s_2} * \varepsilon_{s_2} = \frac{1}{3}\varepsilon_{h_1} + \frac{2}{3}\varepsilon_{h_2}.$$

## 5. Conditional entropy associated with hypergroups

5.1. **Entropy of hypergroup.** Let $X = \{x_1, x_2, \cdots, x_m\}$ be a finite set. For a probability measure $\mu = a_1\delta_{x_1} + a_2\delta_{x_2} + \cdots + a_m\delta_{x_m}$ on $X$, Shannon's entropy $\mathcal{H}(\mu)$ of $\mu$ is

$$\mathcal{H}(\mu) = \sum_{j=1}^{m} \eta(a_j),$$

where $\eta(x)$ is the *entropy function* i.e.

$$\eta(x) = \begin{cases} -x\log x & 0 < x \leq 1, \\ 0 & x = 0. \end{cases}$$

Let $K$ be a finite signed hypergroup and $X$ be a finite set. For an irreducible signed action $\alpha$ of $K$ on $X$, there exists the unique invariant probability measure $\mu^\alpha$ on $X$ under $\alpha$ by Proposition 4.2. We define the *entropy* $\mathcal{H}(\alpha)$ *of the irreducible signed action* $\alpha$ of $K$ on $X$ by

$$\mathcal{H}(\alpha) := \mathcal{H}(\mu^\alpha).$$

Moreover, we denote the entropy $\mathcal{H}(\rho^K)$ by $\mathcal{H}(K)$ for the regular action $\rho^K$ of $K$.

Let $M$ be a finite commutative $*$-algebra with unit 1 which is generated by minimal projections $e_0, e_1, \cdots, e_n$ such that $\sum_{i=0}^n e_i = 1$. For a state $\phi$ of $M$, the entropy $\mathcal{H}_\phi(M)$ of $\phi$ is given by

$$\mathcal{H}_\phi(M) = \sum_{i=0}^{n} \eta(\phi(e_i)).$$

Let $K = (K, M^b(K))$ be a signed hypergroup. For the canonical state $\phi$ of $M^b(K)$, we denote $\mathcal{H}_\phi(M^b(K))$ by $\mathcal{H}_\phi(K)$.

**Proposition 5.1.** *Let $K = (K, M^b(K))$ be a finite commutative signed hypergroup and $\hat{K}$ be the dual signed hypergroup of $K$. Let $\phi$ and $\hat{\phi}$ be the canonical state of $M^b(K)$ and $M^b(\hat{K})$ respectively.*

*Then, the following formulae hold.*

(1) $\mathcal{H}_\phi(K) = \log w(\hat{K}) - \sum_{\chi\in\hat{K}} \frac{w(\chi)}{w(\hat{K})} \log w(\chi)$,

(2) $\mathcal{H}(K) = \log w(K) - \sum_{c\in K} \frac{w(c)}{w(K)} \log w(c)$,

(3) $\mathcal{H}_\phi(K) = \mathcal{H}(\hat{K}),\ \mathcal{H}(K) = \mathcal{H}_{\hat{\phi}}(\hat{K})$.

*Proof.* (1) Let $\hat{K} = \{\chi_0, \cdots, \chi_n\}$ be the dual signed hypergroup of $K$. We denote the minimal projection by $e_i$ corresponding to each $\chi_i \in \hat{K}$. Since $\phi(e_i) = \frac{w(\chi_i)}{w(K)}$ and $w(\hat{K}) = w(K)$, we have

$$\begin{aligned}\mathcal{H}_\phi(K) &= \sum_{i=0}^{n} \eta(\phi(e_i)) = \sum_{i=0}^{n} \eta\left(\frac{w(\chi_i)}{w(K)}\right) \\ &= \sum_{i=0}^{n} \frac{w(\chi_i)}{w(K)} \log w(K) - \sum_{i=0}^{n} \frac{w(\chi_i)}{w(K)} \log w(\chi_i) \\ &= \log w(\hat{K}) - \sum_{\chi \in \hat{K}} \frac{w(\chi)}{w(\hat{K})} \log w(\chi).\end{aligned}$$

(2) Since the regular action $\rho^K$ of $K$ is irreducible and the $\rho^K$-invariant probability measure $\mu^{\rho^K}$ on $K$ is the normalized Haar measure

$$e_K = \sum_{c \in K} \frac{w(c)}{w(K)} \delta_c$$

of $K$, we have

$$\begin{aligned}\mathcal{H}(K) &= \mathcal{H}(\rho^K) = \mathcal{H}(e_K) = \sum_{c \in K} \eta\left(\frac{w(c)}{w(K)}\right) \\ &= \log w(K) - \sum_{c \in K} \frac{w(c)}{w(K)} \log w(c)\end{aligned}$$

in a similar way to the above.

(3) Applying the formula (1) to $\hat{K}$, one can obtain

$$\mathcal{H}(\hat{K}) = \log w(\hat{K}) - \sum_{\chi \in \hat{K}} \frac{w(\chi)}{w(\hat{K})} \log w(\chi).$$

Hence it is clear that $\mathcal{H}_\phi(K) = \mathcal{H}(\hat{K})$ by the formula (2).

Moreover, we have

$$\mathcal{H}_{\hat{\phi}}(\hat{K}) = \mathcal{H}(\hat{\hat{K}}) = \mathcal{H}(K)$$

by the above equality and the duality $\hat{\hat{K}} \cong K$. □

**Remark.** It is easy to check that

$$\mathcal{H}(K) \leq \log |K|.$$

The entropy $\mathcal{H}(K)$ attains the maximum value $\log |K|$ if and only if $K$ is a group.

**Example 5.2.** Let $K = \{0, 1\}$ be a signed hypergroup of order two with unit 0 where the structure is characterized by a parameter $q$ $(0 < q)$ as follows.

$$\delta_1 \circ \delta_1 = q\delta_0 + (1-q)\delta_1.$$

We often denote this hypergroup $K$ by $\mathbb{Z}_q(2)$. Let $\alpha$ be an $m$-dimensional irreducible signed action of $\mathbb{Z}_q(2)$ on $X = \{x_1, x_2, \cdots, x_m\}$. Then the representing matrix of the action $\alpha$ associated with the basis $\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_m}$ in $M^b(X)$ is given by

$$T_\alpha(\delta_0) = I, \quad T_\alpha(\delta_1) = \begin{pmatrix} (1+q)t_1 - q & (1+q)t_1 & \dots & (1+q)t_1 \\ (1+q)t_2 & (1+q)t_2 - q & \dots & (1+q)t_2 \\ \vdots & \vdots & \ddots & \vdots \\ (1+q)t_m & (1+q)t_m & \dots & (1+q)t_m - q \end{pmatrix}$$

where $0 < t_i < 1$ and $\sum_{i=1}^m t_i = 1$ by Proposition 4.3.

The above action $\alpha$ is determined by the parameters $t := (t_1, t_2, \cdots, t_m)$ so that we denote the action $\alpha$ by $\alpha^t$.

In the case that $K = \mathbb{Z}_q(2)$ is a hypergroup, namely $0 < q \leq 1$, the signed action $\alpha^t$ of $K$ is an action if and only if $\dim \alpha^t = m \leq \frac{1+q}{q}$ and $\frac{q}{1+q} \leq t_i \leq \frac{1}{1+q}$ $(m \geq 2)$.

**Proposition 5.3.** *Let $\alpha^t$ be an $m$-dimensional irreducible action of $\mathbb{Z}_q(2)$ on $X$ where a parameter $t = (t_1, t_2, \cdots, t_m)$ satisfies that $\frac{q}{1+q} \leq t_i \leq \frac{1}{1+q}$ for all $i$ and $\sum_{i=1}^m t_i = 1$.*

*Then the following hold.*

(1) $\mathcal{H}(\alpha^t) = \sum_{i=1}^m \eta(t_i)$.

(2) *$\mathcal{H}(\alpha^t)$ attains the maximum value $\log m$ if and only if $\alpha^t$ is a $*$-action.*

(3) *For a two-dimensional irreducible action $\alpha^t$ of $\mathbb{Z}_q(2)$, $\mathcal{H}(\alpha^t)$ has the minimum value if and only if $\alpha^t$ is equivalent to the regular action of $\mathbb{Z}_q(2)$.*

(4) *For two-dimensional irreducible actions $\alpha^t$ and $\alpha^{t'}$ of $\mathbb{Z}_q(2)$, $\alpha^t$ is equivalent to $\alpha^{t'}$ as actions if and only if $\mathcal{H}(\alpha^t) = \mathcal{H}(\alpha^{t'})$.*

*Proof.* (1) Since the invariant probability measure $\mu^{\alpha^t}$ under the action $\alpha^t$ of $\mathbb{Z}_q(2)$ on $X$ is

$$\mu^{\alpha^t} = t_1\delta_{x_1} + t_2\delta_{x_2} + \cdots + t_m\delta_{x_m},$$

we see that the entropy of $\alpha^t$ is

$$\mathcal{H}(\alpha^t) = \mathcal{H}(\mu^{\alpha^t}) = \sum_{i=1}^m \eta(t_i).$$

(2) It is known that $\mathcal{H}(\alpha^t) \leq \log m$. Moreover $\mathcal{H}(\alpha^t) = \sum_{i=1}^m \eta(t_i) = \log m$ if and only if $t_1 = t_2 = \cdots = t_m = \frac{1}{m}$. This condition is equivalent to

$T_{\alpha^t}(\delta_1)^* = T_{\alpha^t}(\delta_1)$, namely, $\alpha^t$ is a $*$-action of $\mathbb{Z}_q(2)$ in the sense of Sunder-Wildberger [SW].

(3) The two dimensional irreducible action $\alpha^t$ is parameterized by $t = (t, 1-t)$ such that $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$. Under the condition that $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$, it is easy to see that $\mathcal{H}(\alpha^t)$ has the minimum value if and only if $t = \frac{q}{1+q}$ or $t = \frac{1}{1+q}$. This condition implies that $\alpha^t$ is equivalent to the regular action of $\mathbb{Z}_q(2)$.

(4) It is easy to see the statement (4) by the fact that $\alpha^t \cong \alpha^{t'}$ if and only if $t = t'$ or $t = 1 - t'$, by Proposition 4.3. □

**Remark.** Let $\alpha^t$ $(0 < t < 1)$ be a two-dimensional irreducible signed action of $\mathbb{Z}_q(2) = \{0, 1\}$ and $\pi^t$ be the representation of $\mathbb{Z}_q(2)$ associated with the action $\alpha^t$. The representing matrix of $\pi^t(\delta_1)$ is given by

$$T_{\pi^t}(\delta_1) = \begin{pmatrix} (1+q)t - q & (1+q)\sqrt{t}\sqrt{1-t} \\ (1+q)\sqrt{t}\sqrt{1-t} & (1+q)(1-t) - q \end{pmatrix}.$$

Let $u^t$ be the unitary matrix such that

$$(u^t)^* T_{\pi^t}(\delta_1) u^t = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}.$$

Then $u^t$ is given by

$$u^t = \begin{pmatrix} \sqrt{t} & -\sqrt{1-t} \\ \sqrt{1-t} & \sqrt{t} \end{pmatrix}.$$

The entropy $\mathcal{H}(b^t)$ of the unistochastic matrix $b^t$ defined by $u^t$ is

$$\mathcal{H}(b^t) = \eta(t) + \eta(1-t).$$

Let $A^t$ be the maximal abelian $*$-subalgebra of $M_2(\mathbb{C})$ which is generated by $T_{\pi^t}(\delta_0)$ and $T_{\pi^t}(\delta_1)$, and $B$ be the diagonal algebra of $M_2(\mathbb{C})$. Here we note that $B = (u^t)^* A^t u^t$. By the paper [C], M. Choda introduced the conditional entropy $h(A^t|B)$ and showed that $h(A^t|B) = \mathcal{H}(u^t)$ under the above situation. Then we have a remarkable fact :

$$\mathcal{H}(\alpha^t) = \mathcal{H}(u^t) = h(A^t|B).$$

5.2. **Conditional entropy associated with a subhypergroup.** First, we recall the classical conditional entropy. Let $\mu$ be a probability measure of a finite set $X = \{x_0, x_1, \cdots, x_n\}$. For a mapping $\psi$ from $X$ onto $Y = \{y_0, y_1, \cdots, y_m\}$, we have a decomposition $\{B_0, B_1, \cdots, B_m\}$ of $X$ by $B_j = \psi^{-1}(y_j)$ and the conditional probability measure $\mu_j$ on $B_j$ by

$$\mu_j(x) = \frac{\mu(x)}{\mu(B_j)}$$

for $x \in B_j$. Then the conditional entropy of the decomposition of $(X, \mu)$ given by $\psi : X \longrightarrow Y$ is defined by

$$\mathcal{H}_\mu(\psi : X|Y) = \sum_{j=0}^{m} \mu(B_j)\mathcal{H}(\mu_j)$$

where

$$\mathcal{H}(\mu_j) = \sum_{x \in B_j} \eta(\mu_j(x)) = \sum_{x \in B_j} \eta\left(\frac{\mu(x)}{\mu(B_j)}\right).$$

Let $M$ be a finite commutative $*$-algebra with unit 1 such that $M$ consists of linear hulls of the minimal projections $e_0, e_1, \cdots, e_n$ such that $\sum_{i=0}^{n} e_i = 1$. Let $N$ be a $*$-subalgebra of $M$ with the unit 1 of $M$. We denote the minimal projections of $N$ by $f_0, f_1 \cdots, f_m$ such that $\sum_{j=0}^{m} f_j = 1$. For each minimal projection $e_i$ of $M$, there exists the unique minimal projection $f_j$ of $N$ such that $e_i \circ f_j = e_i$. Then, we define a mapping $\sigma$ from $\{0, 1, \cdots, n\}$ onto $\{0, 1, \cdots, m\}$ by $e_i \circ f_{\sigma(i)} = e_i$. We note that $f_j = \sum_{i \in \sigma^{-1}(j)} e_i$. Let $\phi$ be a state of $M$. Then, the conditional entropy of the conditional expectation $E$ from $M$ onto $N$ such that $\phi \circ E = \phi$ is defined by

$$\mathcal{H}_\phi^E(M|N) = \sum_{i=0}^{n} \phi(\eta(E(e_i))) = \sum_{j=0}^{m} \phi(f_j)\mathcal{H}_\phi(\sigma^{-1}(j))$$

where

$$\mathcal{H}_\phi(\sigma^{-1}(j)) := \sum_{i \in \sigma^{-1}(j)} \eta\left(\frac{\phi(e_i)}{\phi(f_j)}\right).$$

Let $H, K, L$ be finite commutative hypergroups. Let $H$ be a subhypergroup of $K$ and $\varphi$ be a hypergroup homomorphism from $K$ onto $L$ such that $\mathrm{Ker}\varphi = H$, namely,

$$1 \longrightarrow H \longrightarrow K \xrightarrow{\varphi} L \longrightarrow 1$$

is exact. Then the hypergroup $K$ is called an extension of $L$ by $H$. Let $e_K$ be the normalized Haar measure of $K$.

Under the above situation, we define the conditional entropy $\mathcal{H}(K|L)$ of the decomposition of $(K, e_K)$ given by $\varphi : K \longrightarrow L$ by

$$\mathcal{H}(K|L) := \mathcal{H}_{e_K}(\varphi : K|L).$$

We denote the conditional entropy $\mathcal{H}_\phi^E(K|H)$ of the conditional expectation $E$ from $M^b(K)$ onto the $*$-subalgebra $M^b(H)$ such that $\phi \circ E = \phi$ for the canonical state $\phi$ of $M^b(K)$ by

$$\mathcal{H}_\phi^E(K|H) := \mathcal{H}_\phi^E(M^b(K)|M^b(H)).$$

**Remark.** In the case that $K, H, L$ are finite commutative signed hypergroups, the above two definitions of conditional entropy are also well-defined.

Let $\hat{H}, \hat{K}, \hat{L}$ be the dual signed hypergroups of $H, K, L$ respectively. Then, we have the dual exact sequence:

$$1 \longrightarrow \hat{L} \longrightarrow \hat{K} \xrightarrow{\hat{\varphi}} \hat{H} \longrightarrow 1.$$

Let $\hat{E}$ be the conditional expectation from $M^b(\hat{K})$ onto $M^b(\hat{L})$ such that $\hat{\phi} \circ \hat{E} = \hat{\phi}$ for the canonical state $\hat{\phi}$ of $M^b(\hat{K})$.

**Theorem 5.4.** *Let $H$ be a subhypergroup of a finite commutative hypergroup $K$ and $L$ be the quotient hypergroup $K/H$ of $K$ by $H$. Under the above situation, the following formulae hold.*

(1) $\mathcal{H}_\phi^E(K|H) = \sum_{\tau \in \hat{H}} \sum_{\chi \in \hat{\varphi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{K})} \log \frac{w(\tau) w(\hat{L})}{w(\chi)} = \mathcal{H}_\phi(K) - \mathcal{H}_\phi(H).$

(2) $\mathcal{H}(K|L) = \sum_{\ell \in L} \sum_{c \in \varphi^{-1}(\ell)} \frac{w(c)}{w(K)} \log \frac{w(\ell) w(H)}{w(c)} = \mathcal{H}(K) - \mathcal{H}(L).$

(3) $\mathcal{H}_\phi^E(K|H) = \mathcal{H}(\hat{K}|\hat{H})$ *and* $\mathcal{H}(K|L) = \mathcal{H}_{\hat{\phi}}^{\hat{E}}(\hat{K}|\hat{L}).$

*Proof.* (1) Let $\hat{K} = \{\chi_0, \cdots, \chi_n\}$ and $\hat{H} = \{\tau_0, \cdots, \tau_m\}$. Then we have minimal projections $\{e_i\}_{i=0}^n$ in $M^b(K)$ and $\{f_j\}_{j=0}^m$ in $M^b(H)$ which satisfy

$$\chi_p(e_i) = \delta_{p,i}, \ \ \tau_q(f_j) = \delta_{q,j}$$

for $\chi_p \in \hat{K}$ and $\tau_q \in \hat{H}$ respectively. We note that $\phi(e_i) = \frac{w(\chi_i)}{w(\hat{K})}$ and $\phi(f_j) = \frac{w(\tau_j)}{w(\hat{H})}$. Let $\sigma$ be the mapping from $\{0, 1, \cdots, n\}$ onto $\{0, 1, \cdots, m\}$ given by $e_i \circ f_{\sigma(i)} = e_i$. Hence,

$$\begin{aligned} \mathcal{H}_\phi^E(K|H) &= \sum_{j=0}^m \phi(f_j) \sum_{i \in \sigma^{-1}(j)} \eta\left(\frac{\phi(e_i)}{\phi(f_j)}\right) = \sum_{j=0}^m \sum_{i \in \sigma^{-1}(j)} \phi(e_i) \log \frac{\phi(f_j)}{\phi(e_i)} \\ &= \sum_{j=0}^m \sum_{i \in \sigma^{-1}(j)} \frac{w(\chi_i)}{w(\hat{K})} \log \left(\frac{w(\tau_j)}{w(\hat{H})} \cdot \frac{w(\hat{K})}{w(\chi_i)}\right). \end{aligned}$$

It is easy to see that $e_i \circ f_j = e_i$ if and only if $\hat{\varphi}(\chi_i) = \tau_j$. This means that $i \in \sigma^{-1}(j)$ if and only if $\chi_i \in \hat{\varphi}^{-1}(\tau_j)$. By the fact that $w(\hat{K}) = w(\hat{H}) w(\hat{L})$ (see [IK2]), we get the desired conclusion.

(2) For each $\ell \in L$, the conditional probability measure $\mu_\ell$ of $e_K$ on $\varphi^{-1}(\ell)$ is given by

$$\mu_\ell = \sum_{c \in \varphi^{-1}(\ell)} \frac{w(c)}{w(\varphi^{-1}(\ell))} \delta_c.$$

Then we have

$$\begin{aligned}\mathcal{H}(K|L) &= \sum_{\ell\in L} e_K(\varphi^{-1}(\ell))\mathcal{H}(\mu_\ell) = \sum_{\ell\in L}\sum_{c\in\varphi^{-1}(\ell)} \frac{w(\varphi^{-1}(\ell))}{w(K)}\eta\left(\frac{w(c)}{w(\varphi^{-1}(\ell))}\right) \\ &= \sum_{\ell\in L}\sum_{c\in\varphi^{-1}(\ell)} \frac{w(c)}{w(K)}\log\frac{w(\varphi^{-1}(\ell))}{w(c)}.\end{aligned}$$

By the fact that $w(\varphi^{-1}(\ell)) = w(\ell)w(H)$ (see [IK2]), we get the desired formula.

(3) Applying the formula (1) to the exact sequence: $1 \longrightarrow \hat{L} \longrightarrow \hat{K} \xrightarrow{\hat{\varphi}} \hat{H} \longrightarrow 1$, one can obtain

$$\mathcal{H}(\hat{K}|\hat{H}) = \sum_{\tau\in\hat{H}}\sum_{\chi\in\hat{\varphi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{K})}\log\frac{w(\tau)w(\hat{L})}{w(\chi)}.$$

Hence it is clear that $\mathcal{H}^E_\phi(K|H) = \mathcal{H}(\hat{K}|\hat{H})$ by the formula (2).

Moreover, we have

$$\mathcal{H}^{\hat{E}}_{\hat{\phi}}(\hat{K}|\hat{L}) = \mathcal{H}(\hat{\hat{K}}|\hat{\hat{L}}) = \mathcal{H}(K|L)$$

by the above formula and the duality. □

**Remark.** (1) In the category of finite commutative signed hypergroups, the above statements are also valid.

(2) For the regular action $\rho^K$ of a finite hypergroup $K$, let $\rho^K_H$ be the action of $K$ which is the restriction of $\rho^K$ to $H$. Then $\rho^K_H$ is decomposed as

$$(\rho^K_H, K) = \sum_{\ell\in L}\oplus(\rho_\ell, \varphi^{-1}(\ell))$$

where $\rho_\ell$ is an irreducible action of $H$ on $\varphi^{-1}(\ell)$ for each $\ell \in L$ and $\rho_{\ell_0} = \rho^H$ because $\varphi^{-1}(\ell_0) = H$ for the unit $\ell_0$ of $L$. Then, we know that the invariant probability measure under the action $\rho_\ell$ on $\varphi^{-1}(\ell)$ is the conditional probability measure of $e_K$ on $\varphi^{-1}(\ell)$. Therefore, the conditional entropy $\mathcal{H}(K|L)$ of the decomposition can be rewritten as

$$\mathcal{H}(K|L) = \sum_{\ell\in L}\frac{w(\ell)}{w(L)}\mathcal{H}(\rho_\ell).$$

**An application and an example for the extension problem.**

We consider the exact sequence

$$1 \longrightarrow H \longrightarrow K \xrightarrow{\varphi} L \longrightarrow 1$$

in the case of $H = \mathbb{Z}_q(2)(0 < q \leq 1)$ and $L = \mathbb{Z}_p(2)(0 < p \leq 1)$ where the order of an extension hypergroup $K$ is four. In Chapter 4, an extension $K = K(t, r)$ is determined by two-dimensional irreducible actions $\rho^t$ and $\rho^r$

of $\mathbb{Z}_q(2)$ and $\mathbb{Z}_p(2)$ which are parameterized by $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$ and $\frac{p}{1+p} \leq r \leq \frac{1}{1+p}$ respectively. Let $\phi$ and $\phi'$ be the canonical states of $M^b(K)$ and $M^b(L)$ respectively. By the formula in Theorem 5.4, we have

$$\mathcal{H}_\phi(H) = \mathcal{H}(H) = \log(1+q) + \frac{1}{1+q}\eta(q),$$

$$\mathcal{H}_{\phi'}(L) = \mathcal{H}(L) = \log(1+p) + \frac{1}{1+p}\eta(p),$$

$$\mathcal{H}_\phi(K) = \mathcal{H}_\phi^E(K|H) + \mathcal{H}(H) = \frac{q}{1+q}\mathcal{H}_{\phi'}(L) + \frac{1}{1+q}(\eta(r)+\eta(1-r)) + \mathcal{H}_\phi(H),$$

$$\mathcal{H}(K) = \mathcal{H}(K|L) + \mathcal{H}(L) = \frac{p}{1+p}\mathcal{H}(H) + \frac{1}{1+p}(\eta(t)+\eta(1-t)) + \mathcal{H}(L).$$

**Proposition 5.5.** *Under the above situation, For two extensions $K_1 = K(t_1, r_1)$ and $K_2 = K(t_2, r_2)$ of $\mathbb{Z}_p(2)$ by $\mathbb{Z}_q(2)$, $K_1$ is equivalent to $K_2$ if and only if $\mathcal{H}_\phi(K_1) = \mathcal{H}_\phi(K_2)$ and $\mathcal{H}(K_1) = \mathcal{H}(K_2)$ hold.*

*Proof.* By the paper [IK1], it is known that $K_1 = K(t_1, r_1)$ is equivalent to $K_2 = K(t_2, r_2)$ if and only if $t_2 = t_1$ or $t_2 = 1 - t_1$, and $r_2 = r_1$ or $r_2 = 1 - r_1$. The latter condition is equivalent to $\mathcal{H}(K_1) = \mathcal{H}(K_2)$ and $\mathcal{H}_\phi(K_1) = \mathcal{H}_\phi(K_2)$. □

**Remark.** Two extensions $K(t)$ and $K(t')$ of $\mathbb{Z}_2$ by $\mathbb{Z}_q(2)$ are equivalent as extensions if and only if $\mathcal{H}(K(t)) = \mathcal{H}(K(t'))$ holds.

5.3. **Conditional entropy associated with a generalized orbital hypergroup.** We modify the definition of a generalized orbital hypergroup in [FK].

**Definition.** Let $K = (K, M^b(K))$ be a finite hypergroup and $\phi$ be the canonical state of $M^b(K)$. Let $N$ be a $*$-subalgebra with the unit of $M^b(K)$. Let $E$ be the conditional expectation from $M^b(K)$ onto $N$ such that $\phi \circ E = \phi$. For a finite hypergroup $K_1 = (K_1, M^b(K_1))$, if $M^b(K_1)$ is isomorphic to $N$ by a $*$-isomorphism $\Psi$ from $M^b(K_1)$ onto $N$ and for $c \in K$ there exists $b \in K_1$ such that $E(c) = \Psi(b)$, then we say $K_1$ a *generalized orbital hypergroup* of $K$ by $E$ and denote $K_1$ by $K^E$.

We note that the above definition of a generalized orbital hypergroup is well-defined for a finite signed hypergroup.

In this Chapter, we identify $N$ with $M^b(K^E)$ hereafter.

**Lemma 5.6.** *Let $\psi$ be a mapping from $K$ onto $K^E$ which is the restriction to $K$ of the conditional expectation $E$. Then we have,*

(1) $w(\psi^{-1}(b)) = w(b)$ *for* $b \in K^E$,
(2) $w(K) = w(K^E)$.

*Proof.* Take the Haar measure $\mu_K = \sum_{c\in K} w(c)\delta_c$ of $K$ and $\mu_{K^E} = \sum_{b\in K^E} w(b)\delta_b$ of $K^E$ respectively. For any $\nu \in M^b(K^E)$, $\nu \circ E(\mu_K) = E(\nu \circ \mu_K) = E(\mu_K)$ holds. Hence one can write $E(\mu_K) = a\mu_{K^E}$ for some $a \geq 0$. Since $\phi(E(\mu_K)) = \phi(\mu_K) = 1$ and $\phi(\mu_{K^E}) = 1$, we get $a = 1$, namely $E(\mu_K) = \mu_{K^E}$. We obtain

$$E(\mu_K) = \sum_{c\in K} w(c)E(\delta_c) = \sum_{b\in K^E} \sum_{c\in\psi^{-1}(b)} w(c)\delta_b,$$

so that we arrive at the equation (1). Moreover, it is easy to see the equality (2) by (1). □

In a similar way to the Section 5.2, two kinds of entropy associated with a generalized orbital hypergroup $K^E$ of $K$ are defined by

$$\mathcal{H}(K|K^E) := \mathcal{H}_{e_K}(\psi : K|K^E) \text{ and } \mathcal{H}_\phi^E(K|K^E) := \mathcal{H}_\phi^E(M^b(K)|M^b(K^E)).$$

Let $\hat{K}$ and $\widehat{K^E}$ be the dual signed hypergroups of $K$ and $K^E$ respectively. Then we have a conditional expectation $\hat{E}$ from $M^b(\hat{K})$ onto $M^b(\widehat{K^E})$ given by $\hat{E}(\chi) = \chi|_{M^b(K^E)}$ for a character $\chi$ of $M^b(K)$ and a mapping $\hat{\psi}$ from $\hat{K}$ onto $\widehat{K^E}$ by the restriction of $\hat{E}$ to $\hat{K}$. We note that $\hat{\phi} \circ \hat{E} = \hat{\phi}$ for the canonical state $\hat{\phi}$ of $M^b(\hat{K})$.

**Theorem 5.7.** *Let $K^E$ be a generalized orbital hypergroup of a finite commutative hypergroup $K$ by the conditional expectation $E$ such that $\phi \circ E = \phi$ for the canonical state $\phi$ of $M^b(K)$. Under the above situation, the following formulae hold.*

(1) $\mathcal{H}_\phi^E(K|K^E) = \displaystyle\sum_{\tau\in\widehat{K^E}} \sum_{\chi\in\hat{\psi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{K})} \log \frac{w(\tau)}{w(\chi)} = \mathcal{H}_\phi(K) - \mathcal{H}_\phi(K^E)$.

(2) $\mathcal{H}(K|K^E) = \displaystyle\sum_{b\in K^E} \sum_{c\in\psi^{-1}(b)} \frac{w(c)}{w(K)} \log \frac{w(b)}{w(c)} = \mathcal{H}(K) - \mathcal{H}(K^E)$.

(3) $\mathcal{H}_\phi^E(K|K^E) = \mathcal{H}(\hat{K}|\widehat{K^E})$ *and* $\mathcal{H}(K|K^E) = \mathcal{H}_{\hat{\phi}}^{\hat{E}}(\hat{K}|\widehat{K^E})$.

*Proof.* (1) Let $\hat{K}$ and $\widehat{K^E}$ be $\hat{K} = \{\chi_0, \cdots, \chi_n\}$ and $\widehat{K^E} = \{\tau_0, \cdots, \tau_m\}$ respectively. Then we have minimal projections $\{e_i\}_{i=0}^n$ in $M^b(K)$ and $\{f_j\}_{j=0}^m$ in $M^b(K^E)$ which satisfy

$$\chi_p(e_i) = \delta_{pi}, \ \tau_q(f_j) = \delta_{qj}$$

for $\chi_p \in \hat{K}$ and $\tau_q \in \widehat{K^E}$ respectively. We note that $\phi(e_i) = \frac{w(\chi_i)}{w(\hat{K})}$ and $\phi(f_j) = \frac{w(\tau_j)}{w(\widehat{K^E})}$. Let $\sigma$ be the mapping from $\{0, 1, \cdots, n\}$ onto $\{0, 1, \cdots, m\}$ given by $e_i \circ f_{\sigma(i)} = e_i$. Hence,

$$\begin{aligned}\mathcal{H}_\phi^E(K|K^E) &= \sum_{j=0}^{m} \phi(f_j) \sum_{i\in\sigma^{-1}(j)} \eta\left(\frac{\phi(e_i)}{\phi(f_j)}\right) = \sum_{j=0}^{m} \sum_{i\in\sigma^{-1}(j)} \phi(e_i) \log \frac{\phi(f_j)}{\phi(e_i)} \\ &= \sum_{j=0}^{m} \sum_{i\in\sigma^{-1}(j)} \frac{w(\chi_i)}{w(\hat{K})} \log \left( \frac{w(\tau_j)}{w(\widehat{K^E})} \cdot \frac{w(\hat{K})}{w(\chi_i)} \right).\end{aligned}$$

It is easy to see that $e_i \circ f_j = e_i$ if and only if $\hat{\psi}(\chi_i) = \tau_j$. This means that $i \in \sigma^{-1}(j)$ if and only if $\chi_i \in \hat{\psi}^{-1}(\tau_j)$. Since $w(K) = w(K^E)$ by (2) of Lemma 5.6, we get the desired conclusion.

(2) For each $b \in K^E$, the conditional probability measure $\mu_b$ of $e_K$ on $\psi^{-1}(b)$ is given by

$$\mu_b = \sum_{c\in\psi^{-1}(b)} \frac{w(c)}{w(\psi^{-1}(b))} \delta_c.$$

Then we have

$$\begin{aligned}\mathcal{H}(K|K^E) &= \sum_{b\in K^E} e_K(\psi^{-1}(b))\mathcal{H}(\mu_b) = \sum_{b\in K^E} \sum_{c\in\psi^{-1}(b)} \frac{w(\psi^{-1}(b))}{w(K)} \eta\left(\frac{w(c)}{w(\psi^{-1}(b))}\right) \\ &= \sum_{b\in K^E} \sum_{c\in\psi^{-1}(b)} \frac{w(c)}{w(K)} \log \frac{w(\psi^{-1}(b))}{w(c)}.\end{aligned}$$

Since $w(\psi^{-1}(b)) = w(b)$ by (1) of Lemma 5.6, we get the desired formula.

(3) We can show that $\widehat{K^E} = \hat{K}^{\hat{E}}$ holds. Applying the formula (1) to $\hat{\psi} : \hat{K} \to \hat{K}^{\hat{E}}$, one can obtain

$$\mathcal{H}(\hat{K}|\widehat{K^E}) = \mathcal{H}(\hat{K}|\hat{K}^{\hat{E}}) = \sum_{\tau\in\hat{K}^{\hat{E}}} \sum_{\chi\in\hat{\psi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{K})} \log \frac{w(\tau)}{w(\chi)}.$$

Hence it is clear that $\mathcal{H}_\phi^E(K|K^E) = \mathcal{H}(\hat{K}|\widehat{K^E})$ by the formula (2).

Moreover, we have

$$\mathcal{H}_{\hat{\phi}}^{\hat{E}}(\hat{K}|\widehat{K^E}) = \mathcal{H}_{\hat{\phi}}^{\hat{E}}(\hat{K}|\hat{K}^{\hat{E}}) = \mathcal{H}(\hat{\hat{K}}|\widehat{\hat{K}^{\hat{E}}}) = \mathcal{H}(\hat{\hat{K}}|\widehat{\widehat{K^E}}) = \mathcal{H}(K|K^E)$$

by the above equality and the duality $\hat{\hat{K}} \cong K$ and $\widehat{\widehat{K^E}} \cong K^E$. □

**Remark.** Let $K^\alpha = \{b_0, b_1, \cdots, b_m\}$ be the orbital hypergroup by an action $\alpha$ of a finite group $G$ on a finite commutative hypergroup $K$. Let $\hat{\alpha}$ be the action of $G$ on the dual signed hypergroup $\hat{K}$ defined by $\hat{\alpha}_g(\chi)(c) := \chi(\alpha_{g^{-1}}(c))$ for $g \in G, \chi \in \hat{K}$ and $c \in K$. We denote by $O_j$ $\alpha$-orbit corresponding to $b_j \in K^\alpha$. Let $\psi$ be a mapping from $K$ onto $K^\alpha$ such that $\psi^{-1}(b_j) = O_j$

and $E$ be the conditional expectation from $M^b(K)$ onto $M^b(K^\alpha)$ such that $E|_K = \psi$ and $\phi \circ E = \phi$ for the canonical state $\phi$ of $M^b(K)$. We note that $M^b(K^\alpha)$ is equal to the fixed point algebra $M^b(K)^\alpha$ of $M^b(K)$ by $\alpha$. Let $O'_j$ be the $\hat{\alpha}-$orbit in $\hat{K}$ corresponding to $\tau_j \in \widehat{K^\alpha}$. We denote $|O_j|$ and $|O'_j|$ by $d_j$ and $d'_j$ respectively.

Then we remark the following.

(1) $\mathcal{H}^E_\phi(K|K^\alpha) = \displaystyle\sum_{j=0}^{m} \frac{w(\chi^{(j)})}{w(K)} d'_j \log d'_j$, where $\chi^{(j)} \in O'_j$.

(2) $\mathcal{H}(K|K^\alpha) = \displaystyle\sum_{j=0}^{m} \frac{w(c^{(j)})}{w(K)} d_j \log d_j$, where $c^{(j)} \in O_j$.

## Acknowledgement

I would like to express my thanks to Professor Herbert Heyer, University Tübingen, whose comments and suggestions were exceedingly valuable throughout the course of my study. It is my pleasure to thank to Professor Takeshi Hirai, Kyoto University, and to Professor Marie Choda, Osaka Kyoiku University, who give many invaluable information and warm encouragement. I also express my sense of gratitude to Professor Moto O'uchi, Osaka Prefecture University whose information has helped me very much for understanding hypergroup theory. I am also thankful for Professor Satoshi Kawakami, Nara University of Education, who introduces the hypergroup theory and gives insightful comments and suggestion throughout my study.

# References


[BH] W. Bloom and H. Heyer : *Harmonic Analysis of Probability Measures on Hypergroups*, Walter de Gruyter, de Gruyter Studies in Mathematics 20(1995).

[C] M. Choda : *Relative entropy for maximal abelian subalgebras of matrices and the entropy of unistochastic matrices*, Internat. J. Math. 19(2008), No. 7, 767–776.

[D] C. Dunkl : *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. 179(1973), 331–348.

[FK] Y. Funakoshi and S. Kawakami : *Entropy of probability measures on finite commutative hypergroups*, Bull. Nara Univ. Educ., 57(2008), No. 2, 17–20.

[HJKK] H. Heyer, T. Jimbo, S. Kawakami and K. Kawasaki : *Finite commutative hypergroups associated with actions of finite abelian groups*, Bull. Nara Univ. Educ., 54(2005), No. 2, 23–29.

[HK1] H. Heyer and S. Kawakami : *Extensions of Pontryagin hypergroups*, Probab. Math. Statist., 26(2006), Fasc. 2, 245–260.

[HK2] H. Heyer and S. Kawakami : *A cohomology approach to the extension problem for commutative hypergroups*, Semigroup Forum, 83(2011), No. 3, 371–394.

[HK3] H. Heyer and S. Kawakami : *Imprimitivity theorem for covariant representations of hypergroups*, in preparations.

[HKKK] H. Heyer, Y. Katayama, S. Kawakami and K. Kawasaki : *Extensions of finite commutative hypergroups*, Sci. Math. Jpn., 65(2007), No. 3, 373–385.

[I] R. Ichihara : *Order structures of hypergroup extensions with respect to subhypergroups and their quotioents*, Doctor's thesis of chiba University, March 2011.

[IK1] R. Ichihara and S. Kawakami : *Strong hypergroups of order four arising from extensions*, Sci. Math. Jpn., 68(2008), No. 3, 371–381.

[IK2] R. Ichihara and S. Kawakami : *Estimation of the orders of hypergroup extensions*, Sci. Math. Jpn., 72(2010), No. 1, 307–317.

[IKS] R. Ichihara, S. Kawakami and M. Sakao : *Hypergroup extensions of finite abelian groups by hypergroups of order two*, Nihonkai Math. J., 21(2010), No. 1, 47–71.

[J] R. Jewett : *Spaces with an abstract convolution of measures*, Adv. Math., 18(1975), 1–101.

[K] S. Kawakami : *Extensions of commutative hypergroups*, Infinite Dimensional Harmonic Analysis, World Scientific, 2008, 135–148.

[KI] S. Kawakami and W. Ito : *Crossed products of commutative finite hypergroups*, Bull. Nara Univ. Educ., 48(1999), No. 2, 1–6.

[KKY] S. Kawakami, K. Kawasaki and S. Yamanaka : *Extensions of the Golden hypergroup by finite abelian groups*, Bull. Nara Univ. Educ., 57(2008), No. 2, 1–10.

[KM] S. Kawakami and I. Mikami : *Hypergroup extensions of the group of order two by the Golden hypergroup*, Bull. Nara Univ. Educ., 57(2008), No. 2, 11–16.

[KMTY] S. Kawakami, I. Mikami, T. Tsurii and S. Yamanaka : *Actions of finite hypergroups and applications to extension problem*, Bull. Nara Univ. Educ. 60(2011), No. 2, 19–28.

[KST] S. Kawakami, M. Sakao and T. Takeuchi : *Non-splitting extensions of hypergroups of order two*, Bull. Nara Univ. Educ., 56(2007), No. 2, 7–13.

[KSTY] S. Kawakami, M. Sakao, T. Tsurii and S. Yamanaka : *Signed actions of finite hypergroups and the extension problem*, preprint, 2012.

[KSY] S. Kawakami, M. Sakao and S. Yamanaka : *Extensions of hypergroups of order two by locally compact abelian groups*, Bull. Nara Univ. Educ., 59(2010), No. 2., 1–6.

[KY] S. Kawakami and S. Yamanaka : *Extensions of the Golden hypergroup by locally compact abelian groups*, Acta Sci. Math. (Szeged), 78(2012), 351–368.

[S] R. Spector : *Measures invariants sur les hypergroups*, Trans. Amer. Math. Soc., 239(1978), 147–165.

[SW] V. Sunder and N. Wildberger : *Actions of finite hypergroups*, J. Algebraic Combin., 18(2003), 135–151.

[V] M. Voit : *Hypergroups on two tori*, Semigroup Forum, 76(2008), 192–203.

[W1] N. Wildberger : *Finite commutative hypergroups and applications from group theory to conformal field theory*, Applications of Hypergroups and Related Measure Algebras, Amer. Math. Soc., Providence, 1995, 413–434.

[W2] N. Wildberger : *Strong hypergroups of order three*, J. Pure and Applied Algebra, 174(2002), 95–115.

[Y] S. Yamanaka : *Duality of conditional entropy associated with commutative hypergroups*, Sci. Math. Jpn., 75(2012), 163–176.

[Z] H. Zeuner : *One-dimensional hypergroups*, Adv. Math., 76(1989), 1–18.