# Some Examples of Galois Coverings over the Complex Projective Plane

# SOME EXAMPLES OF GALOIS COVERINGS OVER THE COMPLEX PROJECTIVE PLANE

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**Abstract**

In this short note, we study finite Galois coverings of the complex projective plane $\mathbf{P}^2(\mathbf{C})$ which branch along several lines. We give some examples of Galois branched coverings from $\mathbf{P}^2(\mathbf{C})$ to itself.

**Key Words** : Projective space, Branched covering, Galois group

## 1 Introduction

In [N2] Namba gave a following problem :
**Problem.** For $n \geq 2$, determine the equivalence classes of finite Galois coverings $\pi : \mathbf{P}^n(\mathbf{C}) \to \mathbf{P}^n(\mathbf{C})$.

For one dimensional case, it is known that , for suitable choice of homogeneous coordinates of $\mathbf{P}^1(\mathbf{C})$, a finite Galois covering $\pi : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$ of the complex projective line $\mathbf{P}^1(\mathbf{C})$ can be given as follows [Ho][K][N1][N2][S-G]:

(I)$\pi([X_0 : X_1]) = [X_0^m : X_1^m] \quad (m = 1, 2, \dots)$
(II)$\pi([X_0 : X_1]) =$
$[-(X_0 - X_1)^m : 4X_0^m X_1^m] \quad (m = 1, 2, \dots)$
(III)$\pi([X_0 : X_1]) =$
$[(X_0^4 + 2\sqrt{3}X_0^2X_1^2 - X_1^4)^3 : (X_0^4 - 2\sqrt{3}X_0^2X_1^2 - X_1^4)^3]$
(IV)$\pi([X_0 : X_1]) =$
$[(X_0^8 + 14X_0^4X_1^4 + X_1^8)^3 : 108X_1^4(X_1^4 - X_0^4)^4X_0^4]$
(V)$\pi([X_0 : X_1]) =$
$[(X_0^20 - 228X_0^{15}X_1^5 + 49X_0^{10}X_1^{10} + 228X_0^5X_1^{15})^3 :$
$-1728X_0^5(X_0^{10} + 11X_0^5X_1^5 - X_1^{10})^5X_1^5]$.

Here $\pi$ branches at
$D = m(\infty) + m(0)$ for (I),
$D = m(\infty) + 2(0) + 2(1)$ for (II),
$D = 3(\infty) + 3(0) + 2(1)$ for (III),
$D = 4(\infty) + 3(0) + 2(1)$ for (IV),
$D = 5(\infty) + 3(0) + 2(1)$ for (V).

And it is also well known that the covering transformation group (i.e., Galois group) of above finite Galois covering $\pi : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$ is one of the following groups :

$Z_m$: Cyclic group of order $m$ for (I),
$D_m$: Dihedral group of order $2m$ for (II),
$A_4$: 4-th alternative group for (III),
$S_4$: 4-th symmetric group for (IV),
$A_5$:5-th alternative group for (V).

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In this short note, we study coverings of the complex projective plane $\mathbf{P}^2(\mathbf{C})$ which branch along several lines and we give some examples of finite Galois branched coverings from $\mathbf{P}^2(\mathbf{C})$ to itself.

## 2 Preliminaries

A branched covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ of $\mathbf{P}^2(\mathbf{C})$ is, by definition, a normal irreducible complex surface $X$ together with a proper finite holomorphic mapping $\pi$. The ramification locus $R_\pi$ of $\pi$ is the set of points $x \in X$ such that $\pi$ is not biholomorphic locally around $x$. The branch locus $B_\pi$ of $\pi$ is the image $\pi(R_\pi)$ under $\pi$. It is clear that the restriction $\pi : X - R_\pi \to \mathbf{P}^2(\mathbf{C})$ of $\mathbf{P}^2(\mathbf{C}) - B_\pi$ is a topological covering.

**Definition 2.1.** For a branched covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$, if the covering transformation group acts transitively on every fiber of $\pi$, then $\pi$ is said to be Galois.

Suppose that irreducible curves $C_1, \dots, C_k$ of $\mathbf{P}^2(\mathbf{C})$ are given. Put $B = C_1 \cup \cdots \cup C_k$. Suppose also that positive integers $e_1, \dots, e_k$ are given. Consider the positive divisor $D = e_1C_1 + \dots + e_kC_k$ on $\mathbf{P}^2(\mathbf{C})$. A finite branched cvering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ is said to branch at $D$ if $B_\pi = B$ and, for every $j$ and for every irreducible component $R_{j,l}$ of $\pi^{-1}(C_j)$, the ramification index is $e_j$.

Here we recall Bertini's theorem :

**Theorem 2.1.** *The generic element of a linear system is smooth away from the base locus of the system.*

*Proof.* For the proof, see, for example, [G-H].

Now we assume that $X$ is non-singular and $\pi : X \to \mathbf{P}^2(\mathbf{C})$ is a finite Galois covering which branches at $D = e_1C_1 + \ldots + e_kC_k$. Let $L$ be a line of $\mathbf{P}^2(\mathbf{C})$ and $\hat{L} = \pi^{-1}(L)$. From Bertini's theorem above, if we take $L$ in general position, $\hat{L}$ is non-singular and irreducible. Self-intersection number $\hat{L}\cdot\hat{L}$ of $\hat{L}$ is equal to the degree of $\pi$. From adjunction formula, we have :

$$2g(\hat{L}) - 2 = \hat{L}\cdot\hat{L} + \hat{L}\cdot K_X,$$

where $g(\hat{L})$ is a genus of $\hat{L}$ and $K_X$ is a canonical divisor of $X$. The restriction $\pi_{|\hat{L}} : \hat{L} \to L$ is a finite branched covering of $L$ ($\cong \mathbf{P}^1(\mathbf{C})$) with its degree $\hat{L}\cdot\hat{L}$.From Riemann-Hurwitz formula, we have :

$$2g(\hat{L}) - 2 = -2\hat{L}\cdot\hat{L} + \sum(R_{j,l}\cdot\hat{L})(e_j - 1),$$

where $R_{j,l}\cdot\hat{L}$ is a intersection number of $R_{j,l}$ and $\hat{L}$. Combining above two equations, we have :

$$\sum(R_{j,l}\cdot\hat{L})(e_j - 1) = 3\hat{L}\cdot\hat{L} + \hat{L}\cdot K_X.$$

*Remark.* The degree of $\pi$ $deg\pi$ is a sqare of some integer $d$. Because $\pi$ is given as

$$\pi([X_0 : X_1 : X_2]) = [F_0(X) : F_1(X) : F_2(X)],$$

where $X = [X_0 : X_1 : X_2]$ is a homogeneous coordinates and $F_0$, $F_1$ and $F_2$ are homogeneous polynomials of same degree $d$.

Hree we assume $X = \mathbf{P}^2(\mathbf{C})$. Put $b_j = degC_j$ ($= C_j\cdot L$), $deg(\pi) = d^2$ and $b = \sum_{j=1}^k b_j$. Since we assume $\pi$ is Galois, we have :

$$\sum_{j=1}^k b_j(1 - \frac{1}{e_j}) = 3 + \frac{\hat{L}\cdot K_X}{\hat{L}\cdot\hat{L}}.$$

Then we have :

**Lemma 2.2.** *If there exists a finite Galois covering $\pi : \mathbf{P}^2(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$, then . $b \leq 5$.*

*Proof.* For $K_{\mathbf{P}^2(\mathbf{C})} = [-3H]$, where $H$ is a hyperplane of $\mathbf{P}^2(\mathbf{C})$, we have :

$$\sum_{j=1}^k b_j(1 - \frac{1}{e_j}) = 3 - \frac{3}{d}.$$

If $b \geq 6$, $\sum_{j=1}^k b_j(1 - \frac{1}{e_j}) \geq \sum_{j=1}^k b_j\cdot\frac{1}{2} = \frac{b}{2} \geq 3$, while $3 - \frac{3}{d} < 3$. It's a contradiction. *q.e.d.* .

# 3 Main theorem

Now we consider the line confuigurations of $\mathbf{P}^2(\mathbf{C})$ which consists of $k$ lines $B = L_1 \cup \cdots \cup L_k$ . Arrangements of lines are discussed in [Hi]. Let $t_r$ be the number of $r$-fold points of lines. The following equation holds [Hi] :

$$\frac{k(k-1)}{2} = \sum t_r \cdot \frac{r(r-1)}{2}.$$

So if $k \leq 5$, the following only 12 cases may occur.

(1) The case $k = 1$.
(2) The case $k = 2$ and $t_2 = 1$.
(3) The case $k = 3$ and $t_3 = 1$.
(4) The case $k = 3$ and $t_2 = 3$.
(5) The case $k = 4$ and $t_4 = 1$.
(6) The case $k = 4$, $t_3 = 1$ and $t_2 = 3$.
(7) The case $k = 4$ and $t_2 = 6$.
(8) The case $k = 5$ and $t_5 = 1$.
(9) The case $k = 5$, $t_4 = 1$ and $t_2 = 4$.
(10) The case $k = 5$, $t_3 = 2$ and $t_2 = 4$.
(11) The case $k = 5$, $t_3 = 1$ and $t_2 = 7$.
(12) The case $k = 5$ and $t_2 = 10$.

Among above 12 cases, we study the case (3) and (6) in this note.

For the case (3), it is known that there is a Kummer covering :

$$\pi([X_0 : X_1 : X_2]) = [X_0^m : X_1^m : X_2^m] \quad (m = 1, 2, \ldots).$$

Galois group of $\pi$ is $Z_m \times Z_m$ in this case.

Next we study the case (6). See *Figure*1 below.

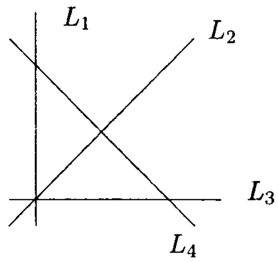


*Figure*1

We may take $L_1 = \{X_0 = 0\}$, $L_2 = \{X_0 - X_1 = 0\}$, $L_3 = \{X_1 = 0\}$ and $L_4 = \{X_2 = 0\}$, because, If we take another 4 lines of type (6), these 4 lines will be mapped to $L_1, \ldots, L_4$ by a projective linear transfomation. Take $p_0 \in \mathbf{P}^2(\mathbf{C}) \setminus (L_1 \cup \cdots \cup L_4)$ and fix it. Then fundamental group of the complement $\pi_1(\mathbf{P}^2(\mathbf{C}) \setminus (L_1 \cup \cdots \cup L_4),\ p_0) =< \gamma_1, \gamma_2, \gamma_3, \delta \,|\, \delta = \gamma_3\gamma_2\gamma_1,\ \delta\gamma_j = \gamma_j\delta$ (for $j = 1, 2, 3$) $>$ . For calculation of the fundamental group, see [M] for example.

**Lemma 3.1.** *If $G$ is one of $Z_m$, $D_m$, $A_4$, $S_4$, $A_5$, there is a central extension by cyclic group $Z_d$:*

$$1 \to Z_d \to Z_d \rtimes G \to G \to 1 \ (exact),$$

*where $d$ is a order of $G$.*

*Proof.* We show the existence of a cyclic extension by giving permutations directly. For the case $G = D_3$, let $A = (a\ b)(c\ d)(e\ f)$, $B = (a\ c\ e)(b\ f\ d)$ and $C = (a\ d)(b\ e)(c\ f)$. We denote by $< A, B, C >$ the group generated by three permutations of six letters $A$, $B$ and $C$. $D_3$ is generated by these permutations and so $D_3 \cong < A, B, C >$. The order of $D_3$ is 6. Put

$$\hat{F} = (a_1\ \ldots\ a_6)(b_1\ \ldots b_6)\ldots(f_1\ \ldots\ f_6),$$

$$\hat{A} = \begin{array}{c} \begin{pmatrix} \cdots & a_j & \cdots & b_j & \cdots \\ \cdots & b_{j+2} & \cdots & a_{j+2} & \cdots \end{pmatrix} \\ \begin{pmatrix} \cdots & c_j & \cdots & d_j & \cdots \\ \cdots & d_{j+2} & \cdots & c_{j+2} & \cdots \end{pmatrix} \\ \begin{pmatrix} \cdots & e_j & \cdots & f_j & \cdots \\ \cdots & f_{j+2} & \cdots & e_{j+2} & \cdots \end{pmatrix} \end{array},$$

$$\hat{B} = \begin{array}{c} \begin{pmatrix} \cdots & a_j & \cdots & c_j & \cdots & e_j & \cdots \\ \cdots & c_{j+3} & \cdots & e_{j+3} & \cdots & a_{j+3} & \cdots \end{pmatrix} \\ \begin{pmatrix} \cdots & b_j & \cdots & f_j & \cdots & d_j & \cdots \\ \cdots & f_{j+3} & \cdots & d_{j+3} & \cdots & b_{j+3} & \cdots \end{pmatrix} \end{array},$$

$$\hat{C} = \begin{array}{c} \begin{pmatrix} \cdots & a_j & \cdots & d_j & \cdots \\ \cdots & d_{j+2} & \cdots & a_{j+2} & \cdots \end{pmatrix} \\ \begin{pmatrix} \cdots & b_j & \cdots & e_j & \cdots \\ \cdots & e_{j+2} & \cdots & b_{j+2} & \cdots \end{pmatrix} \\ \begin{pmatrix} \cdots & c_j & \cdots & f_j & \cdots \\ \cdots & f_{j+2} & \cdots & c_{j+2} & \cdots \end{pmatrix} \end{array}.$$

Here each index of letters should be thought to coincide cyclically with a number in the set $\{1, \ldots, 6\}$. Let $\hat{G} =< \hat{A}, \hat{B}, \hat{C}\hat{F}, >$ be a group generated by $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{F}$ in $S_{36}$. Then it is easy to see that $\hat{F} = \hat{C}\hat{B}\hat{A}$ and the group $< \hat{F} >$ is the center of $\hat{G}$. We can define a surjective homomorphism $\Psi : \hat{G} \to G$ as follows :

$$\Psi(\hat{A}) = A, \Psi(\hat{B}) = B, \Psi(\hat{C}) = C.$$

The kernel of $\Psi$ is $< \hat{F} > \cong Z_6$. Then $\hat{G} \cong Z_6 \rtimes G$. For the other cases, the proof is similar and is omitted. *q.e.d.*

Let $(e_1, e_2, e_3, e_4)$ be one of the followings : (I) $(m, 0, m, m)$, (II) $(2, 2, m, 2m)$, (III) $(3, 2, 3, 12)$, (IV) $(3, 2, 4, 24)$, (V) $(3, 2, 2, 60)$.

Then we have the following theorem:

**Theorem 3.2.** *There exists a finite Galois covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ which branches at $D = e_1L_1 + e_2L_2 + e_3L_3 + e_4L_4$ and the Glois group of $\pi$ is one of $\hat{G}$ : (I)$Z_m \times Z_m$, (II)$Z_{2m} \rtimes D_m$, (III)$Z_{12} \rtimes A_4$,(IV)$Z_{24} \rtimes S_4$* or (V)$Z_{60} \rtimes A_5$.

*Proof.* It is given as Kummer coverings for the case (I). For cases (II), (III), (IV) and (V), from Lemma3.1, it is easy to see that there is a surjective homomorphism :

$$\Phi : \pi_1(\mathbf{P}^2(\mathbf{C}) \setminus (L_1 \cup \cdots \cup L_4),\ p_0) \to \hat{G},$$

defined by $\Phi(\gamma_1) = \hat{A}$, $\Phi(\gamma_2) = \hat{B}$, $\Phi(\gamma_3) = \hat{C}$, $\Phi(\delta) = \hat{F}$.

Here $\hat{G}$ is the same as is defined in the proof of Lemma3.1. Corresponding to the kernel $Ker(\Psi)$ of $\Psi$, there exists a finite Galois coverings $\pi : X \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$. *q.e.d.*

In above cases, by direct calculations, $X$ is nonsingular and $K_X \cdot K_X = 9$ and Euler number $e(X) = 3$. And comparing the monodoromy, we have :

**Theorem 3.3.** *There exists a Galois covering $\pi$ : $\mathbf{P}^2(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$ and the Galois group of $\pi$ is one of (I)$Z_m \times Z_m$, (II)$Z_{2m} \rtimes D_m$, (III)$Z_{12} \rtimes A_4$, (IV)$Z_{24} \rtimes S_4$ or (V)$Z_{60} \rtimes A_5$. The mappings are given as follows :*

(I)$\pi([X_0 : X_1 : X_2]) = [X_0^m : X_1^m : X_2^m]$ $\quad (m = 1, 2, \ldots)$

(II)$\pi([X_0 : X_1 : X_2]) =$
$[-(X_0 - X_1)^m : 4X_0^m X_1^m : X_2^{2m}]$ $\quad (m = 1, 2, \ldots)$

(III)$\pi([X_0 : X_1 : X_2]) =$
$[(X_0^4 + 2\sqrt{3}X_0^2X_1^2 - X_1^4)^3 : (X_0^4 - 2\sqrt{3}X_0^2X_1^2 - X_1^4)^3 : X_2^{12}]$

(IV)$\pi([X_0 : X_1 : X_2]) =$
$[(X_0^8 + 14X_0^4X_1^4 + X_1^8)^3 : 108X_1^4(X_1^4 - X_0^4)^4X_0^4 : X_2^{24}]$

(V)$\pi([X_0 : X_1 : X_2]) =$
$[(X_0^20 - 228X_0^{15}X_1^5 + 49X_0^{10}X_1^{10} + 228X_0^5X_1^{15})^3 : -1728X_0^5(X_0^{10} + 11X_0^5X_1^5 - X_1^{10})^5X_1^5 : X_2^{60}]$.

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