Some examples of S_R(H)-blocks 2

Some examples of $S_R(H)$-blocks II

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Abstract

$S_R(H)$-blocks (of the irreducible characters of G) and their defect groups are defined by G.R.Robinson [R2]. In this note we give some $S_R(H)$-blocks for the symmetric group of degree 6.

Key Words : $S_R(H)$-block, Hecke algebra

1 Introduction

Let G be a finite group, p a prime number which divides the order of G and (K, R, k) a p-modular system, i.e., R is a complete discrete valuation ring with maximal ideal (π), K is the quotient field of R of characteristic 0 and $k(:= R/(\pi))$ is the residue field of R of characteristic p. Moreover, we assume that K contains the $|G|$th roots of unity.

For a subset X of G, $\widehat{X}$ denotes the sum of all elements of X in the group algebra $\mathfrak{o}G$, where $\mathfrak{o}$ is R, K or k.

In this paper we consider the Hecke algebra $S_{\mathfrak{o}}(H) := \mathrm{End}_{\mathfrak{o}G}(\widehat{H}\mathfrak{o}G)$ for a subgroup H of G.

As $e_H := \widehat{H}/|H|$ is an idempotent of KG, $S_K(H) = e_H KG e_H$. For $\chi \in \mathrm{Irr}(G)$, let e_χ be the central primitive idempotent of KG corresponding to χ and put $\Phi_H^G := \{\chi \in \mathrm{Irr}(G); (\chi|_H, 1_H)_H \neq 0\}$. Then we have that $\{e_\chi e_H; \chi \in \Phi_H^G\}$ is the set of all central primitive idempotents of $S_K(H)$ (see [C-R, (11.26) Corollary]).

As $S_K(H) = K \otimes_R S_R(H)$, for a central idempotent ε of $S_R(H)$, there exists a non-empty subset β of Φ_H^G such that $\varepsilon = \sum_{\chi\in\beta} e_\chi e_H$. Then the element of this form is denoted by ε_β and if ε_β is a centrally primitive, β (or $\varepsilon_\beta S_R(H)$) is called an $S_R(H)$-block.

On the other hand, the multiplication induces the K-algebra homomorphism $\phi : Z(KG) \longrightarrow Z(S_K(H))$. Using the map ϕ, G.R.Robinson [R2] has proved that $Z(S_R(H)) \simeq \mathrm{End}_{R[G\times G]}(RG\widehat{H}RG)$ as R-algebras and so each $S_R(H)$-block corresponds to a unique indecomposable direct summand M_β of $RG\widehat{H}RG$. Therefore we can define a defect group for an $S_R(H)$-block β (i.e., a vertex of M_β) in $G \times G$.

Recall that for any $S_R(H)$-block β there exists the unique p-block B such that $\beta \subset \mathrm{Irr}(B)$ (See [R2]). Also if e_B is a block idempotent of RG with the condition $\phi(e_B) \neq 0$, then $\phi(e_B) = \sum_{\beta\in\mathfrak{B}} \varepsilon_\beta$, where $\mathfrak{B}$ is the suitable non-empty subset of $S_R(H)$-blocks. So $\mathrm{Irr}(B) \cap \Phi_H^G$ is a (disjoint) union of $S_R(H)$-blocks.

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Moreover, we have $S_R(H)/\pi S_R(H) \simeq S_k(H)$ as $\widehat{H}RG$ is a permutation module. Hence the set of $S_R(H)$-blocks corresponds bijectively to the set of $S_k(H)$-blocks.

In [H1]$\sim$[H4] and [H-T] we gave some examples of $S_R(H)$-blocks for the symmetric and alternating groups of degree $3, 4, 5$. So in this note we show some $S_R(H)$-blocks for the symmetric group of degree 6.

The notation is almost standard. Concerning some basic facts and terminologies used here, we refer to [C-R] and [N-T] for example.

2 Preliminaries

First we recall the next proposition which tells us some relations between p-blocks and $S_R(H)$-blocks.

Proposition 2.1. ([R2, Remark of Lemma 2.1]) *If $H = \{1\}$, then $\mathrm{Irr}(B)$ is an $S_R(\{1\})$-block for any p-block B of G. Moreover, a defect group of an $S_R(\{1\})$-block $\mathrm{Irr}(B)$ is the diagonal subgroup $\delta(B)^\Delta := \{(x, x) \in G \times G; x \in \delta(B)\}$, where $\delta(B)$ is a (usual) defect group of B.*

For any $\chi \in \Phi_H^G$ there exists a unique $S_R(H)$-block β such that $\chi \in \beta$. In particular, the trivial character 1_G of G is always in Φ_H^G for any subgroup H of G. So it lies in a unique $S_R(H)$-block of G, which we denote by β_0 and call the *principal $S_R(H)$-block*.

Now we shall exhibit some results on $S_R(H)$-blocks.

Proposition 2.2. ([R2, Lemma 2.1,Lemma 2.3(i)-(iv),Corollary 2.4]) *Let β be an $S_R(H)$-block of G, $\delta_H(\beta)$ a defect group of β and $C := \bigcap_{g\in G} H^g$. Then*

(1) For any $x, y \in G$,

$$\frac{|\delta_H(\beta)|}{|C_G(x)||C_G(y)|}\sum_{\chi\in\beta}\chi(x)\chi(y) \in R.$$

In particular, $\dfrac{|\delta_H(\beta)|}{|G\times G|}\sum_{\chi\in\beta}\chi(1)^2 \in R.$

(2) *If $\widetilde{H}$ is a subgroup of H, then Φ_H^G is a subset of $\Phi_{\widetilde{H}}^G$ and β is contained in a single $S_R(\widetilde{H})$-block $\widetilde{\beta}$ of G.*

In particular, β is contained in a single p-block B of G in the usual sence, and if B has a defect group D, then $\delta_H(\beta)$ is contained (up to conjugacy) in $D \times D$.

(3) *There is a bijection between the set of $S_R(H)$-blocks of G and the set of $S_R(H/C)$-blocks of G/C.*

In particular, if H is normal in G (i.e., $C = H$), then the $S_R(H)$-blocks of G are precisely the p-blocks of $R[G/H]$.

(4) *For the principal $S_R(H)$-block β_0, $\beta_0 = \{1_G\}$ if and only if H contains a Sylow p-subgroup of G.*

(5) *If $\chi \in \Phi_H^G$ is in p-block of defect zero of G/C in the usual sence, then $\{\chi\}$ is an $S_R(H)$-block of G.*

Corollary 2.3. ([H3, Corollary 2.4]) *The following hold.*

(1) *If $\sum_{\chi\in\beta}\chi(1)^2 (= \mathrm{rank}_R M_\beta)$ is prime to p for an $S_R(H)$-block β, then a defect group of β is a Sylow p-subgroup of $G \times G$. In particular, if H contains a Sylow p-subgroup of G, then a defect group of β_0 is a Sylow p-subgroup of $G \times G$.*

(2) *If $\chi \in \Phi_H^G$ is in p-block B of defect zero, then $\{\chi\} = \mathrm{Irr}(B)$ is an $S_R(H)$-block and its defect group is the trivial subgroup $\{(1,1)\}$ of $G \times G$.*

Let B be a p-block of RG. If $\phi(e_B) \neq 0$, there is a non-empty subset $\mathfrak{B}$ of $S_R(H)$-blocks such that $\phi(e_B) = \sum_{\beta\in\mathfrak{B}} \varepsilon_\beta$. Hence $\mathrm{Irr}(B) \cap \Phi_H^G = \bigcup_{\beta\in\mathfrak{B}} \beta$. So we write $\mathfrak{B} := \{\beta_1, \beta_2, \cdots, \beta_t\}$.

In the rest of this section we assume that <u>H is a p'-subgroup of G</u> and consider only those <u>blocks such that $\phi(e_B) \neq 0$</u>.

In this case $e_H \in RG$, i.e., $\widehat{H}RG = e_H RG$ is a projective RG-module and kH is a semisimple k-algebra. Now for any $\varphi \in \mathrm{IBr}(G)$, let S_φ (resp. P_φ) be a simple kG-module (resp. an indecomposable projective RG-module) corresponding to φ and $\Psi_H^G := \{\varphi \in \mathrm{IBr}(G); k_H | S_{\varphi\downarrow H}\}$. Here we have that $\Psi_H^G = \{\varphi \in \mathrm{IBr}(G); P_\varphi | e_H RG\}$ by Robinson's reciprocity ([R1, Theorem 3]).

Concerning $S_R(H)$-blocks for p'-subgroup H, the next proposition is fundamental.

Proposition 2.4. ([H-T]) *The decomposition matrix D_B of B has the following form :*

$$D_B = \left(\begin{array}{cccc|c} D_{\beta_1} & 0 & \cdots & 0 & * \\ 0 & D_{\beta_2} & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & D_{\beta_t} & * \\ \hline 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \cdots & \vdots & * \\ 0 & 0 & \cdots & 0 & * \end{array}\right) = (D_B' | D_B''), \tag{2.1}$$

where D_B' denotes the set of the first $|\mathrm{IBr}(B) \cap \Psi_H^G|$ columns of D_B and D_B'' the rest.

For $\chi, \chi' \in \Phi_H^G$, we denote $\chi -_H \chi'$ if there exists $\varphi \in \Psi_H^G$ such that $d_{\chi\varphi} \neq 0 \neq d_{\chi'\varphi}$. Moreover, if there exists a finite sequence $\chi = \chi_1 -_H \chi_2 -_H \cdots -_H \chi_m = \chi'$ in Φ_H^G, we denote $\chi \sim_H \chi'$.

Using these notations, we have the following by the form (2.1).

Corollary 2.5. ([H-T]) *$\chi, \chi' \in \Phi_H^G$ are in the same $S_R(H)$-block if and only if $\chi \sim_H \chi'$.*

Also we know the following orthogonality relation for the $S_R(H)$-block.

Theorem 2.6. ([H-T, Theorem 5]) *Let β be an $S_R(H)$-block. Then we have $\sum_{\chi\in\beta} \chi(xe_H)\chi(y) = 0$ for any $y \in G - G_{p'}$ and $x \in G_{p'}$ such that $\langle x, H\rangle$ is a p'-subgroup.*

Moreover, Y. Tsushima [T] proved the following theorem (cf. [H-T, Proposition 10]).

Theorem 2.7. ([T, Theorem 1]) *Let G be the symmetric group on n letters and suppose that H is a subgroup of order 2. If $p > 2$, then $\{e_B e_H; B \in \mathrm{Bl}_p(G)\}$ is the set of all central primitive idempotents of $S_R(H) = e_H RG e_H$.*

3 Some examples of $S_R(H)$-blocks for the symmetric group of degree 6

Let n be a natural number and $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$) denote the symmetric (resp. alternating) group of degree n. Also we denote the ordinary irreducible character of $\mathfrak{S}_n$ corresponding to $\lambda \in P(n)$, the sets of the partitions of n, by the same notation of the Young diagram

$[\lambda]$ corresponding to λ (for example $[n] := [(n)]$ means the trivial character $1_{\mathfrak{S}_n}$).

In this section let G be the symmetric group of degree 6 $\mathfrak{S}_6$ and B_0 the principal p-block of G.

We know that G has the following p-blocks :
$p = 2 : \mathrm{Irr}(B_0) = \{[6], [5,1], [4,2], [4,1^2], [3^2], [3,1^3], [2^3], [2^2,1^2], [2,1^4], [1^6]\}, \mathrm{Irr}(B_1) = \{[3,2,1]\}$.

$p = 3 : \mathrm{Irr}(B_0) = \{[6], [5,1], [4,1^2], [3^2], [3,2,1], [3,1^3], [2^3], [2,1^4], [1^6]\}, \mathrm{Irr}(B_1) = \{[4,2]\}$,
$\mathrm{Irr}(B_2) = \{[2^2,1^2]\}$.

$p = 5 : \mathrm{Irr}(B_0) = \{[6], [4,2], [3,2,1], [2^2,1^2], [1^6]\}$,
$\mathrm{Irr}(B_1) = \{[5,1]\}, \mathrm{Irr}(B_2) = \{[4,1^2]\}$,
$\mathrm{Irr}(B_3) = \{[3^2]\}, \mathrm{Irr}(B_4) = \{[3,1^3]\}$,
$\mathrm{Irr}(B_5) = \{[2^3]\}, \mathrm{Irr}(B_6) = \{[2,1^4]\}$.

We mainly consider the principal $S_R(H)$-blocks by Corollary 2.3(2).

First we can immediately show the following two examples.

Example 1. (See [H4, Example 3])
Let $H := \langle (1\ 2\ 6), (2\ 3\ 6), (3\ 4\ 6), (4\ 5\ 6)\rangle\ (= \mathfrak{A}_6)$.

(1) $\Phi_H^G = \{[6], [1^6]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \Phi_H^G$.
(b) If $p = 3$ or 5, then $\Phi_H^G = \beta_0 \cup \beta_1 \subset \mathrm{Irr}(B_0)$, where $\beta_0 = \{[6]\}$ and $\beta_1 = \{[1^6]\}$.

Example 2. (See [H1, Example 12])
Let $H := \langle (1\ 5), (2\ 5), (3\ 5), (4\ 5)\rangle\ (= \mathfrak{S}_5)$.

(1) $\Phi_H^G = \{[6], [5,1]\}$.

(2) (a) If $p = 2$ or 3, then $\beta_0 = \Phi_H^G$.
(b) If $p = 5$, then $\Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[6]\}$ and $\beta_1 = \mathrm{Irr}(B_1)$.

Moreover, we shall give the following examples by using the facts in section 2, the Branching Theorem ([J, Theorem 9.2]) and the value of characters.

Example 3.
Let $H := \langle (1\ 2\ 5), (2\ 3\ 5), (3\ 4\ 5)\rangle\ (= \mathfrak{A}_5)$.

(1) $\Phi_H^G = \{[6], [5,1], [2,1^4], [1^6]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \Phi_H^G$.
(b) If $p = 3$, then $\Phi_H^G = \beta_0 \cup \beta_1 \subset \mathrm{Irr}(B_0)$, where $\beta_0 = \{[6], [5,1]\}$ and $\beta_1 = \{[2,1^4], [1^6]\}$.
(c) If $p = 5$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \{[6]\}$ and $\beta_1 = \{[1^6]\}$.

Example 4. Let $H := \langle (1\ 4), (2\ 4), (3\ 4)\rangle\ (= \mathfrak{S}_4)$.

(1) $\Phi_H^G = \{[6], [5,1], [4,2], [4,1^2]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \Phi_H^G$.
(b) If $p = 3$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[6], [5,1], [4,1^2]\}$.
(c) If $p = 5$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[6], [4,2]\}$.

Example 5. Let $H := \langle (1\ 2\ 4), (2\ 3\ 4)\rangle\ (= \mathfrak{A}_4)$.

(1) $\Phi_H^G = \{[6], [5,1], [4,2], [4,1^2], [3,1^3], [2^2,1^2], [2,1^4], [1^6]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \Phi_H^G$.
(b) If $p = 3$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \{[6], [5,1], [4,1^2]\}$ and $\beta_1 = \{[3,1^3], [2,1^4], [1^6]\}$.
(c) If $p = 5$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \{[6], [4,2]\}$ and $\beta_1 = \{[2^2,1^2], [1^6]\}$.

Example 6.
Let $H := \langle (1\ 4), (2\ 4), (3\ 4), (5\ 6)\rangle\ (\simeq \mathfrak{S}_4 \times \mathfrak{S}_2)$.

(1) $\Phi_H^G = \{[6], [5,1], [4,2]\}$.

(2) (a) If $p = 2$, then $\Phi_H^G = \beta_0 \cup \beta_1 \subset \mathrm{Irr}(B_0)$, where $\beta_0 = \{[6]\}$ and $\beta_1 = \{[5,1], [4,2]\}$.
(b) If $p = 3$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[6], [5,1]\}$.
(c) If $p = 5$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[6], [4,2]\}$.

Example 7.
Let $H := \langle (1\ 2), (1\ 3), (4\ 5), (4\ 6)\rangle\ (\simeq \mathfrak{S}_3 \times \mathfrak{S}_3)$.

(1) $\Phi_H^G = \{[6], [5,1], [4,2], [3^2]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \Phi_H^G$.
(b) If $p = 3$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \{[6]\}$ and $\beta_1 = \{[5,1], [3^2]\}$.
(c) If $p = 5$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \{[6], [4,2]\}$.

Example 8. Let $H := \langle (1\ 3), (2\ 3)\rangle\ (= \mathfrak{S}_3)$.

(1) $\Phi_H^G = \{[6], [5,1], [4,2], [4,1^2], [3^2], [3,2,1], [3,1^3]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \Phi_H^G \setminus \{[3,2,1]\}$.
(b) If $p = 3$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G = \Phi_H^G \setminus \{[4,2]\}$.

(c) If $p = 5$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G$
$= \{[6], [4, 2], [3, 2, 1]\}$.

Example 9.
Let $H := \langle (1\ 2\ 3) \rangle (= \mathfrak{A}_3)$.

(1) $\Phi_H^G = \mathrm{Irr}(G)$.

(2) (a) If $p = 2$, then $\Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_j = \mathrm{Irr}(B_j)$ for $j = 0, 1$.

(b) If $p = 3$, then $\Phi_H^G = \bigcup_{i=0}^{2} \beta_i$, where $\beta_j = \mathrm{Irr}(B_j)$ for $j = 0, 1, 2$.

(c) If $p = 5$, then $\Phi_H^G = \bigcup_{i=0}^{6} \beta_i$, where $\beta_j = \mathrm{Irr}(B_j)$ for $j = 0, 1, 2, \cdots, 6$.

Example 10.
Let $H := \langle (1\ 2), (3\ 4), (1\ 3)(2\ 4), (5\ 6) \rangle (\in Syl_2(G))$.

(1) $\Phi_H^G = \{[6], [5, 1], [4, 2], [3, 2, 1], [2^3]\}$.

(2) (a) If $p = 2$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$,where $\beta_0 = \{[6]\}$ and $\beta_1 = \{[5, 1], [4, 2], [2^3]\}$.

(b) If $p = 3$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G$
$= \{[6], [5, 1], [3, 2, 1], [2^3]\}$.

(c) If $p = 5$, then $\beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G$
$= \{[6], [4, 2], [3, 2, 1]\}$.

Example 11.
Let $H := \langle (1\ 2\ 3), (4\ 5\ 6) \rangle (\in Syl_3(G))$.

(1) $\Phi_H^G = \mathrm{Irr}(G) \setminus \{[3, 2, 1]\}$.

(2) (a) If $p = 2$, then $\beta_0 = \Phi_H^G = \mathrm{Irr}(B_0)$.

(b) If $p = 3$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \bigcup_{i=0}^{4} \beta_i$, where $\beta_0 = \{[6]\}, \beta_1 = \{[5, 1], [3^2]\}$, $\beta_2 = \{[4, 1^2], [3, 1^3]\}, \beta_3 = \{[2^3], [2, 1^4]\}$ and $\beta_4 = \{[1^6]\}$.

(c) If $p = 5$, then $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \{[6], [4, 2]\}, \beta_1 = \{[2^2, 1^2], [1^6]\}$.

Example 12.
Let $H := \langle (1\ 2\ 3\ 4\ 5) \rangle (\in Syl_5(G))$.

(1) $\Phi_H^G = \mathrm{Irr}(G)$.

(2) (a) If $p = 2$, then $\Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_j = \mathrm{Irr}(B_j)$ for $j = 0, 1$.

(b) If $p = 3$, then $\Phi_H^G = \bigcup_{i=0}^{2} \beta_i$, where $\beta_j = \mathrm{Irr}(B_j)$ for $j = 0, 1, 2$.

(c) If $p = 5$, then $\mathrm{Irr}(B_0) = \bigcup_{i=0}^{2} \beta_i$, where $\beta_0 = \{[6]\}, \beta_1 = \{[5, 1], [3, 2, 1], [2^2, 1^2]\}$ and $\beta_2 = \{[1^6]\}$.

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