Central Potential Dynamical Systems with Closed Orbit Property in a Conformally Flat Space

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ABSTRACT

In classical dynamical systems, the dynamical systems with central potentials have been fully investigated. Of these dynamical systems the Kepler motion and the harmonic oscillator are known to have celebrated properties. One of these is that Bertrand's theorem is valid, namely any bounded orbit is closed. As a generalization of the Kepler motion, MIC-Kepler motion and multifold Kepler motion have been found and their symmetries discussed. This article deals with 'conformally extended' central potential dynamical systems, which have nonstandard kinetic energy and contain two undetermined functions. From the viewpoint of dynamical symmetries, these two functions are determined so that any bounded orbit may be closed. As a result, we have found three kinds of dynamical systems. One is regarded as a generalization of the Kepler motion on spaces of constant curvature. The second system is also considered as an extended harmonic oscillator on constant curvature spaces. The third system corresponds to a generalized Khare dynamical system.

Key Words: dynamical symmetry, closed orbit, Bertrand's theorem, Kepler motion, harmonic oscillator, multifold Kepler motion, constant curvature space, conformally flat space, Khare's system.

1. Introduction

In classical mechanics as well as quantum mechanics, dynamical systems with central potentials have been fully investigated. Of these dynamical systems, the Kepler motion and the harmonic oscillator are well known as celebrated dynamical systems because of their fruitful dynamical symmetries. One of their properties is that Bertrand's theorem is valid, namely any bounded orbit is closed [1,2,3].

From the viewpoint of dynamical symmetries, many generalized dynamical systems have been found. As an example, MIC-Kepler motion is considered as a Kepler motion in a magnetic monopole field [4,5,6]. The present author and T.Iwai have generalized MIC-Kepler motion to find multifold Kepler motion which contains a rational parameter α [7,8]. For the multifold Kepler system any bounded motion is closed. And for a suitable value of α, the multifold Kepler system corresponds with any dynamical system with closed orbit property mentioned above.

On the other hand, from the geometrical points of view, the Kepler motion and the harmonic oscillator have been generalized on constant curvature spaces [9,10,11].

In this short article, we consider the central potential dynamical systems in a conformally flat space. Our dynamical system has two undetermined functions, one is connected with kinetic energy and the other is central potential function. By making use of modified Bertrand's method, these two functions are determined so that any bounded orbit may be closed. As a result, we can get three kinds of dynamical systems. The first system is a generalization of the Kepler motion on spaces of constant curvature. The second one is considered as an extension of the

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harmonic oscillator on constant curvature spaces. The third system corresponds to a generalized Khare dynamical system, which is known to admit local degeneracy [12].

The contents of this short article are summarized as follows. In Sec. 2, central potential dynamical systems in 'a conformally flat space' are defined. In Sec.3, with modified Bertrand's method, two undetermined functions are decided so that any bounded orbit may be closed. Concluding remarks and further discussions are given in Sec.4.

2. Central potential dynamical systems in a conformally flat space

In this section, we define a central potential dynamical system in a conformally flat space. And the general system is shown to contain many dynamical systems with symmetries.

With Cartesian coordinates $x^i (i = 1,2,3)$, we may introduce a conformally flat 3-dimensional metric defined below:

$$ds^2 = f(r)\sum_{i-1}^{3}(dx^i)^2 \tag{2.1}$$

where

$$r = \sqrt{\sum_{i=1}^{3}(x^i)^2} \tag{2.2}$$

is a radius and $f(r)$ is a positive-valued C^∞-function of the variable r. The kinetic energy T associated with the metric (2.1) is given by

$$T = \frac{1}{2}f(r)\left((\frac{dx^1}{dt})^2 + (\frac{dx^2}{dt})^2 + (\frac{dx^3}{dt})^2\right). \tag{2.3}$$

Here we will introduce polar coordinates (r,θ,φ), which are related with x^i by

$$\begin{aligned} x^1 &= r\sin(\theta)\cos(\varphi), \\ x^2 &= r\sin(\theta)\sin(\varphi), \\ x^3 &= r\cos(\theta). \end{aligned} \tag{2.4}$$

With central potential function $U(r)$ which is also a real-valued C^∞-function of the variable r, the Lagrangian L takes the following form,

$$\begin{aligned} L &= T - U(r) \\ &= \frac{1}{2}f(r)\left((\frac{dr}{dt})^2 + r^2(\frac{d\theta}{dt})^2 + r^2\sin^2\theta(\frac{d\varphi}{dt})^2\right) \\ &\quad - U(r). \end{aligned} \tag{2.5}$$

We may call the above L given in (2.5) Lagrange's function for central potential dynamical systems in the conformally flat space under consideration. Some dynamical systems having closed orbit property are contained in this general system in (2.5).

If we take with constant k

$$f(r) = 1, \qquad U(r) = \frac{-k}{r} \quad , \tag{2.6}$$

then the Lagrangian (2.5) becomes that of the Kepler motion. As a second example, if the functions $f(r)$ and $U(r)$ are given by ,

$$f(r) = 1, \qquad U(r) = \frac{1}{2}kr^2 \quad , \tag{2.7}$$

then we can obtain the harmonic oscillator from (2.5). As a third example, if we have with constants K and a,

$$f(r) = \left(1 + \frac{K}{4}r^2\right)^{-2}, U(r) = \frac{-a}{r}(1 - \frac{K}{4}r^2) \tag{2.8}$$

then the system (2.5) becomes the Kepler motion on constant curvature spaces. We will show further two examples which are contained in (2.5). If the functions $f(r)$ and $U(r)$ are taken to be

$$f(r) = \left(1 + \frac{K}{4}r^2\right)^{-2}, U(r) = ar^2(1 - \frac{K}{4}r^2)^{-2} \tag{2.9}$$

then the system (2.5) corresponds with the harmonic oscillator on constant curvature spaces.

The last example is that with constants a,b,c,d and a rational number α,

$$f(r)=r^{\alpha-2}(a+br^{\alpha}) \ , \qquad (2.10)$$

and

$$U(r)=\frac{c+dr^{\alpha}}{a+br^{\alpha}} \qquad (2.11)$$

give a planar-multifold Kepler motion. It is to be noted that multifold Kepler motion is defined as a modified Kepler motion in a magnetic monopole field. So the trajectories for the multifold Kepler motion are shown to lie on a cone. But the planar multifold Kepler motion can be gotten from the multifold Kepler motion through BBCEL transformation [13]. The orbits on a cone are transformed into the ones on a plane. These dynamical systems mentioned above have closed orbit property. In other words, any bounded orbit is closed.

Before ending this section, we will summarize the relations among these systems. The planar multifold Kepler motion for $a=0, b=1, \ c=-k, \alpha=1$ becomes the usual Kepler motion (2.6) and for $a=1, b=0$ $, d=k/2, \alpha=2$ the usual harmonic oscillator (2.7). But the planar multifold Kepler system cannot produce the systems (2.8) and (2.9). So it does not seem meaningless to get the general form of two undetermined functions $f(r)$ and $U(r)$ from periodicity of the orbits.

3. Application of Bertrand's method and three kinds of dynamical systems

In this section, we apply Bertrand's method to the central potential systems in the conformally flat space defined in the last section. We will determine the two functions $f(r)$ and $U(r)$ so that all bounded orbit may be closed.

In classical mechanics, Bertrand proved that the Kepler motion and the harmonic oscillator are the only central potential systems for all the bounded motions to be closed [1,2]. The method which he used in the proof is called Bertrand's method.

With Cartesian coordinates $x^i (i=1,2,3)$, Lagrange's function yields equations of motion which are given by

$$f(r)\frac{d^2x^i}{dt^2}+\frac{df}{dr}\left(\sum_{j=1}^{3}x^j\frac{dx^j}{dt}\right)\frac{1}{r}\frac{dx^i}{dt} =-\frac{dU}{dr}\frac{x^i}{r}+\frac{1}{2}\left(\sum_{j=1}^{3}\left(\frac{dx^j}{dt}\right)^2\right)\frac{df}{dr}\frac{x^i}{r}, \qquad (3.1)$$

$$(i=1,2,3).$$

But the polar coordinates are very useful in our system. Namely, one can easily see that the variable φ is a cyclic one. So the angular momentum

$$J=f(r)r^2\sin^2(\theta)\left(\frac{d\varphi}{dt}\right) \qquad (3.2)$$

is conserved. With position vector $x=(x^1, x^2, x^3)$ and velocity vector $dx/dt=(dx^1/dt, dx^2/dt, dx^3/dt)$, angular momentum vector $f(r)x\times dx/dt$ is conserved and the value of (3.2) is also shown to be

$$J=f(r)\left|x\times\frac{dx}{dt}\right| \ , \qquad (3.3)$$

where $|x|$ denotes the absolute value of vector x and $\times$ stands for vector product. With this rotational invariance, one can easily get

$$\theta=\frac{\pi}{2}. \qquad (3.4)$$

With the notation of vector analysis, we can also show

$$\left(f(r)\left(x\times\frac{dx}{dt}\right)\bullet x\right)=0, \qquad (3.5)$$

where center dot $\bullet$ denotes the inner product of two vectors. Without loss of generality, one can take $x \times \frac{dx}{dt}$ axis in the direction of x^3 axis. This fact ensures that (3.4) is satisfied and the orbit lies on the plane which is perpendicular to the vector $x \times (dx/dt)$. Then the system is reduced to two- degrees-of-freedom system.

With r and φ, the total energy E and angular momentum J are expressed as

$$E = \frac{1}{2} f(r)\left((\frac{dr}{dt})^2 + r^2 (\frac{d\varphi}{dt})^2 \right) + U(r) \qquad (3.6)$$

$$J = f(r) r^2 (\frac{d\varphi}{dt}) \qquad (3.7)$$

With these conserved quantities, one can get an equation of orbit as follows:

$$d\varphi = \frac{J}{f(r)r^2} \frac{f(r)r}{\sqrt{(2E - 2U(r)) f(r) r^2 - J^2}} dr$$
$$= \frac{J}{r} \frac{1}{\sqrt{2(E - U(r)) r^2 f(r) - J^2}} dr . \qquad (3.8)$$

If the trajectory $r = r(\varphi)$ is closed, r should take a maximum and a minimum. Let r_1 be a minimum and r_2 the following maximum. Since $dr/d\varphi = 0$ for the values, one has from (3.8)

$$2(E - 2U(r_k)) r_k{}^2 f(r_k) - J^2 = 0, \qquad k = 1,2 . \qquad (3.9)$$

The increment of the angle, $\Delta\varphi$, during the motion from $r = r_1$ the following $r = r_2$ is therefore given by

$$\Delta\varphi = \int_{r_1}^{r_2} \frac{J}{\sqrt{2(E - U(r)) r^4 f(r) - r^2 J^2}} dr . \qquad (3.10)$$

We assume further that no critical values of r exist between r_1 and r_2. Then a necessary and sufficient condition for the trajectory to be closed is that

$\Delta\varphi = m\pi$ for some rational number m. (3.11)

We can here assume that $m > 0$ without loss of generality. In fact, if $m < 0$ the integration in (3.10) is performed over the range from r_2 to r_1 in the inverse direction.

We now introduce a variable u by

$$u = \left(- \int \frac{dr}{r^2 \sqrt{f(r)}} \right)^{1/m} \qquad (3.12)$$

where we have modified Bertrand's method. In ordinary treatment, m is set to be one with $f(r) = 1$;

$$u = 1/r . \qquad (3.13)$$

In the case of constant curvature space, $f(r) = \left(1 + \frac{K}{4} r^2 \right)^{-2}$, the new variable u in (3.12) with $m = 1$ becomes

$$u = \frac{1}{r} \left(1 - \frac{K}{4} r^2 \right) \qquad (3.14)$$

which was already treated in [9]. In this stage, the new variable u can not be expressed explicitly with r such as (3.13) and (3.14). So the following discussions are taken to be formal.

The right-hand side of (3.10) is then rewritten as

$$-m \int_{u_1}^{u_2} \frac{J du}{\sqrt{2u^{2-2m}(E - U) - u^{2-2m} J^2 g(u)}} , \qquad (3.15)$$

where $g(u) = 1/(r^2 f(r))$ and $u_k (k = 1,2)$ corresponds to $r_k (k = 1,2)$ through (3.12). On setting

$$V(u) = E(1 - u^{2-2m}) - J^2 u^2 / 2 + J^2 g(u) u^{2-2m} / 2 + u^{2-2m} U(r(u)) , \qquad (3.16)$$

and on putting equations (3.10) to (3.16) together, the closed trajectory condition (3.11) is brought into the form

$$\int_{u_2}^{u_1} \frac{J du}{\sqrt{2(E - V(u)) - J^2 u^2}} = \pi . \qquad (3.17)$$

This equation is the same as the one which Bertrand treated in the ordinary central potential problem [1]. See also Greenberg [3] for this equation. For $V(u)$, the condition (3.9) is expressed as

$$2(E - V(u_k)) - J^2 {u_k}^2 = 0, \qquad k = 1,2 \qquad (3.18)$$

Following Bertrand [1], we can find from (3.17) that the function $V(u)$ has to take the form

$$V(u) = \xi_0 u + \xi_1, \qquad (3.19)$$

where ξ_0 and ξ_1 are constants. In addition to (3.19),

$$V(u) = \xi_2 u^{-2} + \xi_3 \ , \quad (\xi_2 \ and \ \xi_3 \ ; consts) \qquad (3.20)$$

also gives

$$\int_{u_2}^{u_1} \frac{J du}{\sqrt{2(E - V(u)) - J^2 u^2}} = \frac{\pi}{2} \ . \qquad (3.21)$$

From (3.16) and (3.19), as the total energy E is an arbitrary constant, we have three cases.

CASE 1. $E \neq 0$ and $m = 1$.

In this case, one can take with constants $\eta_1, \eta_2, \eta_3, \eta_4$,

$$g(u) = u^2 + \eta_1 u + \eta_2 = \frac{1}{r^2 f(r)}, \qquad (3.22)$$

and

$$U(r(u)) = \eta_3 u + \eta_4 . \qquad (3.23)$$

The variables u and r are related by (3.12). By putting

$$F(r) = \left(-\int \frac{dr}{r^2 \sqrt{f(r)}} \right) , \qquad (3.24)$$

we have a differential equation

$$r^2 (\frac{dF(r)}{dr})^2 = F(r)^2 + \eta_1 F(r) + \eta_2 \qquad (3.25)$$

This equation can be easily solved to get

$$F(r) = c_0 r + c_1 + c_2 r^{-1}. \qquad (3.26)$$

where c_0, c_1 and c_2 are constants.

Going back to the expression of the original variable r, we have

$$f(r) = \frac{1}{(c_0 r^2 - c_2)^2} \ , \qquad (3.27)$$

$$U(r) = \alpha (c_0 r + c_2 r^{-1}) \ . \ \alpha : const.$$

This system can be regarded as a generalization of the Kepler motion on spaces of constant curvature given in (2.8).

CASE 2. $E \neq 0$ and $m = 1/2$.

In the second case, with constants $\eta_1, \eta_2,$, η_3, η_4 we have

$$g(u) = u + \eta_1 + \frac{\eta_2}{u} = \frac{1}{r^2 f(r)} \qquad (3.28)$$

and

$$U(r(u)) = \eta_3 + \frac{1}{u} \eta_4 . \qquad (3.29)$$

According to the Case 1, with (3.24) and $u^2 = F(r)$ we have a differential equation

$$r^2 (\frac{dF(r)}{dr})^2 = F(r)^2 + \eta_2 F(r)^{-2} + \eta_1 \qquad (3.30)$$

With $G(r) = F(r)^2$, (3.30) can be transformed into

$$r^2 (\frac{dG(r)}{dr})^2 = 4(G(r)^2 + \eta_1 G(r) + \eta_2) \qquad (3.31)$$

One can solve this differential equation easily. So we get the following solution;

$$G(r) = c_0 r^2 + c_1 + c_2 r^{-2} , \qquad (3.32)$$

where c_0, c_1 and c_2 are constants. As a result, we obtain

$$f(r)=\frac{c_0 r^4+c_1 r^2+c_2}{(c_0 r^4-c_2)^2}, \tag{3.33}$$

$$U(r)=\frac{\alpha}{c_0 r^2+c_1+c_2 r^{-2}} \quad . \ \alpha : const.$$

This system can be considered as a generalization of the harmonic oscillator on constant curvature spaces given in (2.9).

CASE 3 $E=0$.

In the last Case, we can make the same discussions in the former cases. Consequently, one may get the following functions.

$$f(r)=\frac{(c_0 r^{2/m}+c_1 r^{1/m}+c_2)^{2-2m}}{(c_0 r^{2/m}-c_2)^2}, \tag{3.34}$$

$$U(r)=(c_0 r^{1/m}+c_1+c_2 r^{-1/m})^{2m-2}$$
$$(\alpha(c_0 r^{1/m}+c_1+c_2 r^{-1/m})+\beta)$$

$$\alpha \ and \ \beta : consts.$$

Here if one take $c_0=c_1=0, c_2=1$ and $\alpha=-B, \beta=A$, then the system (3.34) is seen to coincide with Khare"s system [12];

$$f(r)=1, \qquad U(r)=Ar^{2/m-2}-Br^{1/m-2}. \tag{3.35}$$

So this system is said to be a generalization of Khare's system.

Conversely, in these systems (3.27), (3.33) and (3.34) with $E=0$, we can show that any bounded orbit is closed under suitable constraints among the constants $c_0, c_{1,}, c_2, \alpha$ and β.

Before ending this section, it is to be noted that we could not find a generalization of the planar multifold Kepler motion defined by (2.10) and (2.11) in this approach. This fact is due to the variable transformation (3.12). In [7], the multifold Kepler motion could be obtained through $u=\left(r^{-1}\right)^{1/m}$.

4. Concluding remarks

We have found three kinds of dynamical systems associated with a conformally flat metric from the viewpoint of closed orbit property. In this research, we have only found forms of the functions. So, further investigations concerning the found systems should be made. For the systems in Case 1 and Case 2, we should study first integrals and clear up its dynamical symmetry. On the other hand, the third system in Case 3 is also expected to admit local degeneracy. These further investigations will be made in a forthcoming paper.

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