Dynamical Systems Associated with Eguchi-Hanson Metric

Dynamical Systems Associated with Eguchi-Hanson Metric

Noriaki KATAYAMA*

ABSTRACT

The Eguchi-Hanson metric is well known as a self-dual and Einstein four-dimensional metric. In this short article, dynamical systems in the Eguchi-Hanson space are considered. The central potential systems in the Eguchi-Hanson space can be reduced with U(1) symmetry. The reduced dynamical system is of three-degrees-of-freedom. Then, Bertrand's method is applied to determine the central potential in order that all bounded orbits for the reduced system are closed. Finally, Kepler-type and harmonic-oscillator-type systems are found.

Key Words: Eguchi-Hanson metric, symmetry, reduction, Kepler-type system, harmonic-oscillator-type system, closed orbit.

1. Introduction

The Eguchi-Hanson metric as well as the Taub-NUT metric is known as a self-dual and Einstein four-dimensional metric [1]. In a series of papers [2-5], the Taub-NUT metric has been generalized from the view point of dynamical symmetry. In these researches, the geodesic flow system in the generalized Taub-NUT space is reduced with U(1) symmetry. The reduced dynamical system is of three-degrees-of-freedom and discussed on the closeness of bounded orbits.

On the other hand, the geodesic flow system in the Eguchi-Hanson space was studied in [6]. For classical and quantum motion in the Eguchi-Hanson space, the equations for the problems are separable in both cases, but analytic solutions are difficult to obtain.

In this short article, we consider the central potential dynamical systems in Eguchi-Hanson space. The aim of this paper is to find some dynamical systems with symmetry associated with Eguchi-Hanson metric. As well as Taub-NUT space, the dynamical system is easily seen to be reduced to a lower dimensional system with U(1) symmetry. For the reduced system, we are able to apply Bertrand's method to determine the central potential for all bounded orbits to be closed. As a result, we have found two dynamical systems. One is Kepler-type system and the other is harmonic-oscillator-type system. The Kepler problem and the harmonic oscillator are well known to be celebrated examples to admit dynamical symmetry in classical and quantum mechanics. For this reason, many researches for these systems have been made. However, for the two systems found in this paper, total energy can not take any value. Therefore, we may say that these systems admit "local degeneracy".

The contents of this short article are summarized as follows. In Sec. 2, the central potential systems in Eguchi-Hanson space are defined and the reduction procedure can be applied to our systems. In Sec. 3, by applying Bertrand's method, the central potential function is determined so that the bounded motion may be periodic and local degeneracy is proved for the Kepler-type and harmonic-oscillator-type system. Concluding remarks and further discussions are given in Sec. 4.

(Received April 11, 2001)

* Department of Systems and Control Eng.

2. Central potential systems in Eguchi-Hanson space and reduced dynamical system

In this section, we define the central potential systems in Eguchi-Hanson space. And the dynamical system is shown to be reduced to one with U(1) symmetry.

Using Cartesian coordinates $x^i(i=1,2,3,4)$, we introduce curvilinear coordinates (r,θ,φ,ψ), which are related with Cartesian coordinates $x^i(i=1,2,3,4)$ as follows:

$$
\begin{aligned}
x^1 &= \sqrt{r}\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi+\varphi}{2}\right)\\
x^2 &= \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi+\varphi}{2}\right)\\
x^3 &= \sqrt{r}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi-\varphi}{2}\right)\\
x^4 &= \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi-\varphi}{2}\right).
\end{aligned}
\tag{2.1}
$$

Then, the line element with these curvilinear coordinates in Eguchi-Hanson space is defined by

$$
\begin{aligned}
ds^2 &= \frac{r}{4(r^2-a^4)}dr^2+\frac{r}{4}(d\theta^2+\sin^2\theta d\varphi^2)\\
&+\frac{r^2-a^4}{4r}(d\psi+\cos\theta d\varphi)^2,
\end{aligned}
\tag{2.2}
$$

where a is a constant. From the view point of differential geometry, Eguchi-Hanson metric is known to be a self-dual, Einstein and Ricci-flat metric. The line element (2.2) with $x^i(i=1,2,3,4)$ takes the following form:

$$
\begin{aligned}
ds^2 &= \frac{a^4}{r(r^2-a^4)}(\sum_{i=1}^{4}x^idx^i)^2+\sum_{i-1}^{4}(dx^i)^2\\
&-\frac{a^4}{r^3}(x^1dx^2-x^2dx^1+x^3dx^4-x^4dx^3)^2.
\end{aligned}
\tag{2.3}
$$

Then, we can get the following Hamiltonian (2.4) from Eq. (2.3) for the central potential system in Eguchi–Hanson space with central potential function $V(r)$.

$$
\begin{aligned}
H &= \frac{1}{2}\sum_{i=1}^{4}(y_i)^2-\frac{a^4}{2r^3}(\sum_{i-1}^{4}x^iy_i)^2\\
&+\frac{a^4}{2r(r^2-a^4)}(x^1y_2-x^2y_1+x^3y_4-x^4y_3)^2+V(r),
\end{aligned}
\tag{2.4}
$$

where y_i denotes the generalized momentum conjugate to x^i. The classical dynamical system governed by the Hamiltonian (2.4) is easily seen to admit U(1) symmetry. In $R^4-\{0\}$ we can define $U(1)\cong SO(2)$ action as follows:

$$
\begin{aligned}
&(x^1+ix^2,x^3+ix^4)\\
&\quad\to\exp(iu/2)(x^1+ix^2,x^3+ix^4),
\end{aligned}
\tag{2.5}
$$

where u is a parameter and $i=\sqrt{-1}$. This action can be lifted naturally on cotangent bundle $T^*(R^4-\{0\})$ to get a symplectic action.

Therefore, we can apply a reduction procedure to our Hamiltonian dynamical system $(T^*(R^4-\{0\}), dy\wedge dx, H)$. As for the reduction of dynamical systems with symmetry, one can refer the papers [2-5] in detail. As a result, we have a reduced Hamiltonian dynamical system on $T^*(R^3-\{0\})$, whose symplectic form ω_μ and Hamiltonian H_μ are given by Eqs. (2.6) and (2.7), respectively.

$$
\omega_\mu=\sum_{k=1}^{3} dp_k\wedge dq^k-\frac{\mu}{2r^3}\sum \varepsilon_{ijk}q^idq^j\wedge dq^k \tag{2.6}
$$

$$
\begin{aligned}
H_\mu &= 2r\sum_{k=1}^{3}p_k+\frac{2r\mu^2}{r^2-a^4}\\
&-\frac{2a^4}{r^3}(\sum_{k=1}^{3}q^kp_k)^2+V(r),
\end{aligned}
\tag{2.7}
$$

where $q=(q^1,q^2,q^3)$ and $p=(p_1,p_2,p_3)$ are three-dimensional vectors, namely $(q,p)\in T^*(R^3-\{0\})\cong(R^3-\{0\})\times R^3$ and constant μ is a value of momentum map Φ defined by

$$
\Phi(x,y)=\frac{1}{2}(x^1y_2-x^2y_1+x^3y_4-x^4y_3)\ . \tag{2.8}
$$

Thus, we have gotten the reduced dynamical system $(T^*(R^3-\{0\}),\omega_\mu,H_\mu)$ which is of three-degrees-of-freedom. Before ending this section, it is to be noted that three dimensional vector $q=(q^1,q^2,q^3)$ and the curvilinear coordinates (r,θ,φ,ψ) are satisfied with the following equalities:

$$\begin{aligned} q^1 &= r\sin(\theta)\cos(\varphi), \\ q^2 &= r\sin(\theta)\sin(\varphi), \\ q^3 &= r\cos(\theta). \end{aligned} \tag{2.9}$$

In other words, curvilinear coordinates (r,θ,φ) become spherical coordinates in the reduced configuration space.

3. Application of Bertrand's method and two dynamical systems

In this section, we apply Bertrand's method to the reduced system derived in the last section. One can determine the potential function $V(r)$ so that all bounded orbit may be closed.

In classical mechanics, it is well known that the Kepler motion and the harmonic oscillator are the only central potential systems for all the bounded motions to be closed. This fact was proved by Bertrand [7.8]. The method which he used in the proof is called Bertrand's method.

For the reduced system, the equations of motion take the following form for the sake of modified symplectic form (2.6):

$$\frac{dq^j}{dt} = \frac{\partial H_\mu}{\partial p_j}$$

$$\frac{dp_j}{dt} = -\frac{\partial H_\mu}{\partial q^j} + \frac{\mu}{r^3}\sum_{l,k} \varepsilon_{lkj} q^l \frac{dq^k}{dt} \qquad (j=1,2,3) \tag{3.1}$$

After a long calculation, one can easily show that

$$J = q\times p + \frac{\mu}{r}q \tag{3.2}$$

is conserved. In Eq. (3.2), $q\times p$ stands for vector product of q and p. Conserved vector J is called angular momentum vector. From Eq. (3.2), we have

$$(J\bullet\frac{q}{r}) = \mu \quad , \tag{3.3}$$

where center dot $\bullet$ denotes inner product of two vectors. Without loss of generality, one can take J axis in the direction of q^3 axis. Therefore, Eq. (3.3) yields

$$\cos(\theta) = \frac{\mu}{|J|} \quad . \tag{3.4}$$

Thus, one can say that orbit lies on the cone whose axis is vector J. This fact corresponds to the planar motion for the usual central potential system. As the angular coordinate θ is conserved, the orbit is seen to be described with the rest two coordinates (r,φ).

As a second step in this section, we will try to show the equations of orbit. From Eqs. (3.2) and (3.4), the absolute value of angular momentum vector J becomes

$$|J| = \frac{r}{4}\frac{d\varphi}{dt}. \tag{3.5}$$

In addition to the angular momentum vector, total energy E is also conserved. After a long and tedious calculation, total energy (2.7) becomes

$$E = \frac{r}{8(r^2-a^4)}\left(\frac{dr}{dt}\right)^2 + \frac{2}{r}\left(|J|^2-\mu^2\right) + \frac{2r\mu^2}{r^2-a^4} + V(r). \tag{3.6}$$

Thus, we have gotten the equation of the orbit from Eqs. (3.5) and (3.6) as follows:

$$\begin{aligned} \varphi &= \int \left(\frac{d\varphi}{dt}\Big/\frac{dr}{dt}\right)dr \\ &= \int \frac{|J|dr}{\sqrt{\frac{r(r^2-a^4)(E-V(r))}{2} - |J|^2r^2 + a^4(|J|^2-\mu^2)}} \end{aligned} \tag{3.7}$$

Here, we can apply the Bertrand's method to find dynamical systems with periodic orbits [3, 4, 5, 7, 8]. For bounded orbits with $r_1 \le r \le r_2$, at the points $r = r_1$ and $r = r_2$, we have $dr/dt = 0$. So the increment of the angle φ is defined by

$$\Delta\varphi = \int_{r_1}^{r_2} (\frac{d\varphi}{dr}) dr . \tag{3.8}$$

In short,

$$\Delta\varphi = m\pi, \quad (m: \text{a rational number}) \tag{3.9}$$

is equivalent to the statement that the bounded motion is closed. By Bertrand's method, it is a necessary and sufficient condition for all the bounded orbits to be closed that the denominator in Eq. (3.7) takes the following forms with constants c_0 and c_1:

$$\begin{aligned} &\frac{r(r^2-a^4)(E-V(r))}{2} - |J|^2 r^2 + a^4(|J|^2 - \mu^2) \\ &\qquad = -|J|^2 r^2 + c_0 + c_1 r, \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} &\frac{r(r^2-a^4)(E-V(r))}{2} - |J|^2 r^2 + a^4(|J|^2 - \mu^2) \\ &\qquad = -|J|^2 r^2 + c_0 + c_1 r^{-2}. \end{aligned} \tag{3.11}$$

From Eqs. (3.10) and (3.11) with constraint $E = 0$, the potential function $V(r)$ is given by

$$V(r) = \frac{\xi_0 + \xi_1 r}{r(r^2 - a^4)}, \tag{3.12}$$

or

$$V(r) = \frac{\xi_0 + \xi_1 r^{-2}}{r(r^2 - a^4)}, \tag{3.13}$$

respectively, where ξ_0 and ξ_1 are constants.

For Eq. (3.12), we get from Eq. (3.8)

$$\Delta\varphi = \pi . \tag{3.14}$$

So, we call the potential (3.12) Kepler-type potential in the reduced system $(T^*(R^3 - \{0\}), \omega_\mu, H_\mu)$ with constraint $E = 0$.

On the other hand, in case of Eq. (3.13), we have

$$\Delta\varphi = \frac{\pi}{2}. \tag{3.15}$$

Therefore, one may call the potential (3.13) harmonic-oscillator-type potential in the reduced system $(T^*(R^3 - \{0\}), \omega_\mu, H_\mu)$ with constraint $E = 0$.

It is to be noted that for the usual Kepler problem and harmonic oscillator total energy can take any negative value or any positive value, respectively, for bounded motions. The Kepler-type system and harmonic-oscillator-type system found in this paper are said to admit local degeneracy because of the constraint $E = 0$.

4. Concluding remarks

We have found two dynamical systems associated with Eguchi-Hanson metric. Although these systems do not admit "full" dynamical symmetry, the bounded orbits of these systems are closed only for a particular value of total energy. One can say that these systems admit "local degeneracy". Further investigation for local degeneracy will be made in a forthcoming paper. And some attentions concerning singular sphere $r = a^2$ will be also given in the future investigation.

References

[1] T. Eguchi, P. B. Gillkey and A. J. Hanson, Phys. Rep. 66(1980), 213-393.
[2] T. Iwai and N. Katayama, J. Geom. Phys. 12 (1993), 55-75.
[3] T. Iwai and N. Katayama, J. Phys. A Math. Gen. 27 (1994), 3179-3190.
[4] T. Iwai and N. Katayama, J. Math. Phys. 35 (1994) 2914-2933.
[5] T. Iwai and N. Katayama, J. Math. Phys. 36 (1995) 1790-1811.
[6] S. Mignemi, J. Math. Phys. 32(1991) 3047-3054.
[7] J. Bertrand, C. R. Acad. Sci. Paris 77(1873), 849-853
[8] H. Goldstein, "Classical Mechanics", 2nd ed., Addison-Wesley, Reading MA, 1980.