Some examples of S_R (H)-blocks

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Abstract

G.R.Robinson introduced the $S_R(H)$-block (he called it$A_R(H)$-block) of the irreducible characters of G and its defect group. But we don't see enough examples of $S_R(H)$-blocks and their defect groups in his paper. So in this paper we give some examples of $S_R(H)$-blocks.

Key Words: $S_R(H)$-block, Hecke algebra

1 Introduction

Let G be a finite group, p a prime number which divides the order of G and (K, R, k) a p-modular system, i.e., R is a complete discrete valuation ring with maximal ideal (π), K is the quotient field of R of characteristic 0 and $k(:= R/(\pi))$ is the residue field of R of characteristic p. Moreover, we assume that K contains the $|G|$th roots of unity.

For a subset X of G, $\widehat{X}$ denotes the sum of all elements of X in the group algebra $\mathfrak{o}G$, where $\mathfrak{o}$ is R, K or k.

We consider the Hecke algebra (or the Schur algebra) $S_{\mathfrak{o}}(H) := \mathrm{End}_{\mathfrak{o}G}(\widehat{H}\mathfrak{o}G)$ for a subgroup H of G. As $e_H := \widehat{H}/|H|$ is an idempotent of KG, $S_K(H) = e_H KG e_H$. For $\chi \in \mathrm{Irr}(G)$, let e_χ be the central primitive idempotent of KG corresponding to χ and put $\Phi_H^G := \{\chi \in \mathrm{Irr}(G); (\chi|_H, 1_H)_H \neq 0\}$. Then we have that $\{e_\chi e_H; \chi \in \Phi_H^G\}$ is the set of all central primitive idempotents of $S_K(H)$ (see [1, (11.26) Corollary]).

As $S_K(H) = K \otimes_R S_R(H)$, for a central idempotent ε of $S_R(H)$, there exists a non-empty subset β of Φ_H^G such that $\varepsilon = \sum_{\chi\in\beta} e_\chi e_H$. Then the element of this form is denoted by ε_β and if ε_β is a centrally primitive, β (or $\varepsilon_\beta S_R(H)$) is called an $S_R(H)$-block. On the other hand, the multiplication induces the R-algebra homomorphism $\phi : Z(RG) \longrightarrow Z(S_R(H))$. Using the map ϕ, G.R.Robinson [6] has proved that $Z(S_R(H)) \simeq \mathrm{End}_{R[G\times G]}(RG\widehat{H}RG)$ as R-algebras and so each $S_R(H)$-block corresponds to a unique indecomposable direct summand M_β of $RG\widehat{H}RG$.

Therefore we can define a defect group for an $S_R(H)$-block β (i.e., a vertex of M_β) in $G \times G$. (See Definition 1.)

Recall that for any $S_R(H)$-block β there exists the unique p-block B such that $\beta \subset \mathrm{Irr}(B)$ (See [6]). Also if e_B is a block idempotent of RG with the condition $\phi(e_B) \neq 0$, then $\phi(e_B) = \sum_{\beta\in\Lambda} \varepsilon_\beta$, where Λ is the suitable non-empty subset of $S_R(H)$-blocks. So $\mathrm{Irr}(B) \cap \Phi_H^G$ is a (disjoint) union of $S_R(H)$-blocks.

Moreover, we have $S_R(H)/\pi S_R(H) \simeq S_k(H)$ as $\widehat{H}RG$ is a permutation module. Hence the set of $S_R(H)$-blocks corresponds bijectively to the set of $S_k(H)$-blocks.

In this paper we show some examples of $S_R(H)$-blocks and their defect groups.

The notation is almost standard. Concerning some basic facts and terminologies used here, we refer to [1] and [5] for example.

2 Preliminaries

For later use, we shall exhibit some results on $S_R(H)$-blocks, which are almost proved in [3] or [6].

Definition 1. (See [6]) For any $S_R(H)$-block β, there exists a minimal subgroup D of $G \times G$ and $\lambda \in \mathrm{Inv}_D E$ with the following condition : $\mathrm{Tr}_D^{G\times G}(\lambda) = \varepsilon_\beta$, where E is the endomorphism ring $\mathrm{End}_R(RG\widehat{H}RG)$.

Then D is called a defect group of β and denoted by $\delta_H(\beta)$.

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Remark 1-1. By the theory of G-algebra a defect group $\delta_H(\beta)$ of $S_R(H)$-block β is a p-subgroup of $G \times G$ and unique up to $G \times G$-conjugacy.

Remark 1-2. ([6, Remark of Proposition 2.2]) If $H = \{1\}$, then $\mathrm{Irr}(B)$ is a $S_R(\{1\})$-block for any p-block B of G. Moreover, a defect group of an $S_R(\{1\})$-block $\mathrm{Irr}(B)$ is the diagonal subgroup $\delta(B)^\Delta := \{(x,x) \in G \times G; x \in \delta(B)\}$, where $\delta(B)$ is a (usual) defect group of B.

Proposition 2. ([6, Lemma 2.1]) *(i) For any $S_R(H)$-block β and $x, y \in G$,* $\frac{|\delta_H(\beta)|}{|C_G(x)||C_G(y)|}\sum_{\chi\in\beta}\chi(x)\chi(y) \in R$.

In particular, $\frac{|\delta_H(\beta)|}{|G\times G|}\sum_{\chi\in\beta}\chi(1)^2 \in R$.

(ii) β is contained in a single p-block B of G in the usual sence, and if B has a defect group D, then $\delta_H(\beta)$ is contained (up to conjugacy) in $D \times D$.

Corollary 3. *If* $\sum_{\chi\in\beta}\chi(1)^2$ *(i.e.,* $\mathrm{rank}_R M_\beta$*) is prime to p for an $S_R(H)$-block β, then a defect group of β is a Sylow p-subgroup of $G \times G$.*

(Proof) By Proposition 2(i). □

As the trivial character 1_G is always in Φ_H^G for any subgroup H of G. Then there exists the $S_R(H)$-block, which has the trivial character. So we call it the principal $S_R(H)$-block and denote it β_0.

Proposition 4. (See [6, Lemma 2.3(iii)]) *For the principal $S_R(H)$-block β_0, $\beta_0 = \{1_G\}$ if and only if H containes a Sylow p-subgroup of G.*

Moreover, in a such case a defect group of β_0 is a Sylow p-subgroup of $G \times G$.

(Proof) The first half is [6, Lemma 2.3(iii)] and the later half follows from Proposition 2(i). □

Proposition 5. ([6, Corollary 2.4]) *If H is normal in G, then the $S_R(H)$-blocks of G are precisely the p-block of $R[G/H]$.*

In the rest of this section we assume that H is a p'-subgroup of G and consider only those blocks such that $\phi(e_B) \neq 0$.

In this case $e_H \in RG$, i.e., $\widehat{H}RG = e_H RG$ is a projective RG-module and kH is a semisimple k-algebra.

Now for any $\varphi \in \mathrm{IBr}(G)$, let S_φ (resp. P_φ) be an irreducible kG-module (resp. an indecomposable projective RG-module) corresponding to φ. Also, we let $\Psi_H^G := \{\varphi \in \mathrm{IBr}(G); k_H | S_{\varphi\downarrow H}\}$. Note that $\Psi_H^G = \{\varphi \in \mathrm{IBr}(G); P_\varphi | e_H RG\}$. So we can define $\beta^* := \{\varphi \in \mathrm{IBr}(B); P_\varphi | \varepsilon_\beta(e_H RG)\}$.

Therefore the decomposition matrix D_B of B has the following form : (See [3])

$$D_B = \left(\begin{array}{cccc|c} D_{\beta_0} & 0 & \cdots & 0 & * \\ 0 & D_{\beta_1} & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & D_{\beta_t} & * \\ \hline 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \cdots & \vdots & * \\ 0 & 0 & \cdots & 0 & * \end{array}\right). \tag{2.1}$$

From the form of the decomposition matrix (2.1), we get the following orthogonality relation for the $S_R(H)$-block.

Theorem 6. ([3, Theorem 5]) *Let H be a p'-subgroup of G and β an $S_R(H)$-block. Then we have*

$$\sum_{\chi\in\beta}\chi(xe_H)\chi(y) = 0$$

for any $y \in G - G_{p'}$ and $x \in G_{p'}$ such that $\langle x, H\rangle$ is a p'-subgroup.

Remark 6-1. (See [3, Remark 12]) Let $G := \mathfrak{S}_5$ be the symmetric group of degree 5, $H := \langle (1,2)(3,4), (1,3)(2,4)\rangle$, the Klein four group and $char\, k = 5$. If we take $x := (4,5) \in G_{5'}$,then $Hx \subset HxH \subset G_{5'}$ and $\langle H, x\rangle \not\subset G_{5'}$. And we take $y := (1,2,3,4,5) \in G - G_{5'}$, then

$$\sum_{\chi\in\beta_0}\chi(xe_H)\chi(y) = \frac{5}{4} \neq 0.$$

So we can consider that it is necessary in Theorem 6 to assume that $\langle x, H\rangle$ is a p'-subgroup.

3 Some examples of $S_R(H)$-blocks

In this section we show some $S_R(H)$-blocks and their defect groups. Now $\mathfrak{S}_n$ denotes the symmetric group of degree n and B_0 the principal p-block of G.

Now recall $\mathfrak{S}_3$ has the following character table :

		[3]	[2, 1]	[1^3]
(4.1)	1_G	1	1	1
	χ	2	0	−1
	sgn	1	−1	1

and the following p-blocks :
if $p = 2$, then $\mathrm{Irr}(B_0) = \{1_G, \mathrm{sgn}\}$,
$\mathrm{Irr}(B_1) = \{\chi\}$
if $p = 3$, then $\mathrm{Irr}(B_0) = \{1_G, \chi, \mathrm{sgn}\}(= \mathrm{Irr}(G))$.

Then we get the following Example 7 and 8.

Example 7. $G := \mathfrak{S}_3, H := \langle(1, 2)\rangle$

(1) $\Phi_H^G = \{1_G, \chi\}$.

(2) (a) If $p = 2$, then $\Phi_H^G = \beta_0 \cup \beta_1$, where $\beta_0 = \{1_G\}, \beta_1 = \{\chi\}(= \mathrm{Irr}(B_1))$ and $\delta_H(\beta_0) \in Syl_2(G \times G)$, $\delta_H(\beta_1) = \{1\} \times \{1\}$

(b) If $p = 3$, then $\Phi_H^G = \beta_0 = \mathrm{Irr}(B_0) \cap \Phi_H^G$ and $\delta_H(\beta_0) \in Syl_3(G \times G)$.

(Proof) (1) From the character table (4.1).
(2) (a) holds by Proposition 2 and 4.
(b) follows from Proposition 4 and Corollary 3. □

Remark 7-1. (cf. Remark 1-2) The above example $G := \mathfrak{S}_3, H := \langle(1, 2)\rangle$ and $p = 3$ tells us $\delta_H(\beta) \neq_{G\times G} \delta(B)^\Delta$ though $\beta = \mathrm{Irr}(B) \cap \Phi_H^G$.

Example 8. $G := \mathfrak{S}_3, H := \langle(1, 2, 3)\rangle$

(1) $\Phi_H^G = \{1_G, \mathrm{sgn}\}$.

(2) (a) If $p = 2$, then $\Phi_H^G = \beta_0 = \mathrm{Irr}(B_0)$ and $\delta_H(\beta_0) =_{G\times G} \delta(B_0)^\Delta$.

(b) If $p = 3$, then $\Phi_H^G = \beta_0 \cup \beta_0'$, where $\beta_0 = \{1_G\}, \beta_0' = \{\mathrm{sgn}\}$ and $\delta_H(\beta_0)$, $\delta_H(\beta_0') \in Syl_3(G \times G)$.

(Proof) (1) From the character table (4.1).
(2) (a) holds by Proposition 4 and 5.
(b) follows from Proposition 4 and Corollary 3. □

Example 9. $G := \mathfrak{A}_4$, the alternating group of degree 4, $H := \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle$ and $p := 2$.

We denote $\mathrm{Irr}(G) = \{1_G, \chi_2, \chi_3, \chi_4\}$, where $\chi_2(1) = 3, \chi_3(1) = \chi_4(1) = 1$. Now $\mathrm{Irr}(B_0) = \mathrm{Irr}(G)$.

(1) $\Phi_H^G = \mathrm{Irr}(G) \setminus \{\chi_2\}$.

(2) $\Phi_H^G = \beta_0 \cup \beta_0' \cup \beta_0''$ with $\beta_0 = \{1_G\}$, $\beta_0' = \{\chi_3\}, \beta_0'' = \{\chi_4\}$.

(3) $\delta_H(\beta) \in Syl_2(G\times G)$ for any $S_R(H)$-block β.

(Proof) (1) As we can check the character values, the assertion holds.
(2) follows from Proposition 5.
(3) holds from Proposition 4 and Corollary 3. □

Recall the case $G := \mathfrak{S}_5, H := \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle$. Then we get the next example.

Example 10. (See [3, Remark 12]) $G := \mathfrak{S}_5, H := \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle, p := 5$

We know $\mathrm{Irr}(G) = \{[5], [4, 1], [3, 2], [3, 1^2], [2^2, 1], [2, 1^3], [1^5]\}$ and $\mathrm{Irr}(B_0) = \{[5], [4, 1], [3, 1^2], [2, 1^3], [1^5]\}, \mathrm{Irr}(B_1) = \{[3, 2]\}, \mathrm{Irr}(B_2) = \{[2^2, 1]\}$, where we denote irreducible characters the same notations corresponding to the Young diagrams. (For example [5] means the trivial character.) Then

(1) $\Phi_H^G = \mathrm{Irr}(G) \setminus \{[3, 1^2]\}$.

(2) $\mathrm{Irr}(B_0) \cap \Phi_H^G = \beta_0 \cup \beta_0'$ with $\beta_0 = \{[5], [4, 1]\}$ and $\beta_0' = \{[2, 1^3], [1^5]\}$, $\beta_1 = \{[3, 2]\}(= \mathrm{Irr}(B_1)), \beta_2 = \{[2^2, 1]\}(= \mathrm{Irr}(B_2))$

(3) $\delta_H(\beta_0), \delta_H(\beta_0') \in Syl_5(G \times G), \delta_H(\beta_i) = \{1\} \times \{1\}$ $(i = 1, 2)$.

(Proof) (1),(2) See [3, Remark 12] and check the character values.
(3) follows from Proposition 2 and Corollary 3. □

For the principal $S_R(H)$-block the following Example 11 and 12 hold.

Example 11. ([2, example 6]) $G := \mathfrak{S}_p$, $H := \mathfrak{S}_t$ $(1 \leq t \leq p)$, char$k = p$.

(1) $\beta_0 = \mathrm{Irr}(B_0) = \{\chi_i; 0 \leq i \leq p - t\}$, where χ_i corresponds to the Young diagram $[p - i, 1^i]$.

(2) $\delta_H(\beta_0) =_{G\times G} \begin{cases} P^\Delta & t = 1 \\ P \times P & 2 \leq t \leq p \end{cases}$, where P is a Sylow p-subgroup of G.

(Proof) We may assume that $p > 2$.
(1) See [2, Proposition 11].
(2) Since $\chi(1) \equiv \pm 1 (\mathrm{mod}\ \ p)$ for any $\chi \in \mathrm{Irr}(B_0)$,

$\sum_{\chi\in\beta_0}\chi(1)^2 \equiv p-t+1 (\mathrm{mod}\ p)$. Hence the assertion follows from Remark 1-2 and Proposition 2. □

Example 12. $G := \mathfrak{S}_n, H := \mathfrak{S}_{n-1}$.

(1) $\Phi_H^G = \{[n], [n-1,1]\}$.

(2) (a) If p does not divide n, then $\beta_0 = \{[n]\}$ and $\delta_H(\beta_0) \in Syl_p(G\times G)$.

(b) If p divides n, then $\beta_0 = \Phi_H^G$. In particular, if p is odd prime, then $\delta_H(\beta_0) \in Syl_p(G\times G)$.

(Proof) (1) By the Branching theorem [4, Theorem 9.2].
(2) (a) follows from Proposition 4.
(b) The first half holds from Proposition 4 and the later half follows from Corollary 3. □

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