The Problem of Fenchel about F-Groups

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Abstract

The problem of Fenchel about F-groups is solved by S. Bundgaard, J.Nielsen and R.H.Fox. But in this paper, we give an another proof by direct construction of permutation monodromies.

Key Words: branched covering, Fenchel's problem

1 Introduction

The part of the problem of Fenchel about F-groups that R.H.Fox solved in 1952 (see Fox [4]) is stated as follows: Let $n_1, \ldots, n_d$ be integers greater than 1. Let F be a group generated by d elements $S_1, \ldots, S_d$ with the defining system of relations

$$\prod_{i=1}^{d} S_i = 1 \quad S_i^{n_i} = 1 \ (i = 1, \ldots, d).$$

It is known that the only elements of finite order in F are the elements

$$S_i^k \ (k \geq 0, \ i = 1, \ldots, d)$$

and their conjugates (see [5]).

Fenchel's problem *Does F possess a normal subgroup N with finite index in F and containing no element of finite order other than unity ?*

If $d = 1$ or $d = 2$ and $n_1 = n_2$, then the group F is cyclic with finite order, so that $N = \{1\}$ has the required properties. Fox gave the affirmative answer of this problem for the case of $d \geq 3$ by proving the following lemma in his paper [4].

Lemma 1 (Fox [4]) *For any integers a, b and c greater than 1, there are permutations A and B of order a and b respectively such that AB has the order c.*

Using this lemma, Fox solved the part of Fenchel's problem as follows: We can find a permutation A_i of order n_{i-1} and a permutation B_i of order n_i such that the permutation A_iB_i has the order n_{i+1} for each $i = 2, \ldots, d-1$. Let G_i be the group generated by A_i and B_i.

Let Φ_i be the homomorphism from F to G_i defined by $\Phi_i(S_j) = 1$ *for* $(j \neq i-1, \ i, \ i+1)\Phi_i(S_{i-1}) = A_i$, $\Phi_i(S_i) = B_i$, $\Phi_i(S_{i+1}) = (A_i \ B_i)^{-1}$. Let N_i be the kernel of Φ_i. Put $N = N_2 \cap \ldots \cap N_{d-1}$. Then N has the required properties.

Let $D = n_1P_1 + \ldots + n_dP_d$ be the effective divisor of the complex projective line P^1. The truth of this part of the problem implies that there is a finite Galois covering $\pi : X \rightarrow P^1$ which branches at D if and only if $(i)d = 2$ and $n_1 = n_2$ or $(ii)d \geq 3$ (see Namba [8]).In this paper we shall prove:

Theorem 1 *Suppose $d \geq 3$. For any integers n_j $(j = 1, \ldots, d)$ greater than 1, there can be found permutations A_j $(j = 1, \ldots, d)$ of order n_j respectively such that $A_1A_2 \ldots A_d = 1$ and the group G generated by A_j $(j = 1, \ldots, d)$ is a transitive subgroup of the symmetric group of n letters or $2n$ letters, where n is the maximum number of $\{n_j\}$.*

From Theorem 1 we have:

Theorem 2 *Let $\pi_1(P^1 - \{P_1, \ldots, P_d\}) = < \gamma_1, \ldots, \gamma_d \mid \gamma_1 \ldots \gamma_d = 1 >$ be the fundamental group of $P^1 - \{P_1, \ldots, P_d\}$. Let $\Psi : \pi_1(P^1 - \{P_1, \ldots, P_d\}) \rightarrow G$ be the homomorphism defined by $\Psi(\gamma_j) = A_j$, where A_j are permutations and G is the group given in Theorem 1. Then for the effective divisor $D = n_1P_1 + \ldots n_dP_d$ $(d \geq 3)$ there exists a finite Galois covering $\pi : X \rightarrow P^1$ which branches at D and is the Galois closure of a covering over P^1 of degree either n or $2n$ with the monodromy representation Ψ, where n is the maximum number of $\{n_j\}$.*

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Here, a Galois covering $\pi : X \to P^1$ is called the Galois closure of a covering $\mu : Y \to P^1$ if (a) there is a covering $\nu : X \to Y$ such that $\mu \circ \nu = \pi$ and (b) for any Galois covering $\pi' : X' \to P^1$ and a covering $\nu' : X' \to Y$ such that $\mu \circ \nu' = \pi'$, there is a covering $\varphi : X' \to X$ such that $\nu' = \nu \circ \varphi$.

Remark 1. *Note that from Theorem 1 we have a rather direct proof for Fenchel's problem than those proofs which have been known (see* [2][3][4][7]). *In fact, let $\Phi : F \to G$ be the homomorphism defined by $\Phi(S_j) = A_j$, where A_j are the permutations in Theorem 1. Let N be the kernel of Φ. Then N is a normal subgroup of F which has the required properties of Fenchel's problem. Note also that we can calculate practically the group F/N which is the Galois group of the covering $\pi : X \to P^1$ by using a computer while we cannot do it by other's method of proving Fenchel's problem.*

Furthermore we have as a corollaly to the proof of Theorem 1:

Theorem 3 *Suppose $d \geq 4$. Let $D = n_1P_1 + \dots + n_dP_d$ be any effective divisor with $n_j \geq 2$ of the complex projective line P^1. Then, there are infintely many (non isomorphic) finite Galois coverings $\pi : X \to P^1$ over P^1 which branch at D.*

Though Theorem 3 can be proved by considering of characteristic subgroups of the fundamental group $\pi_1(X)$ of X (see Macbeath [6]), we will give another proof of Theorem 3 by constructing permutations explicitly in this paper.

In the final section, we also give another proof of the part of Fenchel's problem that S.Bundgaard and J.Nielsen solved in 1951 (see [1]).

2 Proof of the Theorem 1

We may assume that $n_1 = \max\{n_j | 1 \leq j \leq d\}$. First we assume $n_1 \geq 3$. We will give a concrete construction of $\{A_j\}$ inductively. The construction will be divided into four cases: (I) n_1 is odd and n_2 is odd; (II) n_1 is odd and n_2 is even; (III) n_1 is even and n_2 is odd; (IV) n_1 is even and n_2 is even. (I) If n_1 is odd and n_2 is odd, then we take A_1 and A_2 as follows: $A_1 = (u_{n_1}\ u_{n_1-1}\ \dots\ u_1)$ and $A_2 = (u_{n_2}\ u_{n_2-1}\ \dots\ u_1)$. Then the product A_1A_2 must be $A_1A_2 = (u_{n_1}\ u_{n_1-1}\ \dots\ u_{n_2+1}\ u_{n_2-1}, u_{n_2-3}\ \dots\ u_2\ u_{n_2}\ \dots\ u_3\ u_1)$ $(if\ n_1 > n_2) A_1A_2 = (u_{n_1}\ u_{n_1-2}\ \dots\ u_1\ u_{n_1-1}\ \dots\ u_2)$ $(if\ n_1 = n_2)$ A_1A_2 has order n_1.

(II) If n_1 is odd and n_2 is even, then we take A_1 and A_2 as follows: $A_1 = (w_{n_1-n_2}\ \dots\ w_1\ u_{n_2-2}\ u_{n_2-4} \dots\ u_2\ v_1\ u_{n_2-1}\ u_{n_2-3}\ \dots\ u_1) and\ A_2 = (u_1\ u_2\ \dots\ u_{n_2-1}\ v_1)$. Then the product A_1A_2 must be $A_1A_2 = (w_{n_1-n_2}\ \dots\ w_2\ w_1\ u_{n_2-1}\ \dots\ u_2\ u_1)$. A_1A_2 is a single cycle of order $n_1 - 1$.

If n_1 is odd, n_2 is even and n_3 is even, then we change the symbols $\{u_1\ \dots\ u_{n_2-1}\ w_1\ \dots\ w_{n_1-n_2}\}$ into $\{x_1, \dots, x_{n_3-1}, y_1\ \dots\ y_{n_1-n_3}\}$ such that $(w_{n_1-n_2}\ \dots\ w_2\ w_1\ u_{n_2-1}\ \dots\ u_2\ u_1) = (x_1\ \dots\ x_{n_3-1}\ y_1\ \dots\ y_{n_1-n_3})$. We take A_3 as follows: $A_3 = (x_1\ x_2\ \dots\ x_{n_3-1}\ z_1)$. Then the product $A_1\ A_2\ A_3$ must be $A_1\ A_2\ A_3 = (x_1\ x_3\ \dots\ x_{n_3-1}\ y_1\ \dots\ y_{n_1-n_3}\ x_2\ x_4\ \dots\ x_{n_3-2}\ z_1)$. $A_1\ A_2\ A_3$ is a single cycle of order n_1.

If n_1 is odd, n_2 is even, n_3 is odd and $n_3 < n_1$, then we change the symbols $\{u_1, \dots, u_{n_2-1}, w_1, \dots, w_{n_1-n_2}\}$ into $\{x_1, \dots, x_{n_3}, y_1, \dots, y_{n_1-n_3-1}\}$ such that $(w_{n_1-n_2}\ \dots\ w_2\ w_1\ u_{n_2-1}\ u_{n_2-2} \dots\ u_2\ u_1) = (x_1\ \dots\ x_{n_3}\ y_1\ \dots\ y_{n_1-n_3-1})$. We take A_3 as follows: $A_3 = (x_1\ \dots\ x_{n_3})$. Then the product $A_1\ A_2\ A_3$ must be $A_1\ A_2\ A_3 = (x_1\ x_3\ \dots\ x_{n_3}\ y_1\ y_2\ \dots\ y_{n_1-n_3-1}\ x_2\ x_4\ \dots\ x_{n_3-1})$. $A_1\ A_2\ A_3$ is a single cycle of order $n_1 - 1$.

If n_1 is odd, n_2 is even and $n_3 = n_1$, then we change the symbols $\{u_1, \dots, u_{n_2-1}, w_1, \dots, w_{n_1-n_2}\}$ into $\{x_1, x_2, \dots, x_{n_1-2}, y_1\}$ such that $(w_{n_1-n_2}\ \dots\ w_2\ w_1\ u_{n_2-1}\ u_{n_2-2}\ \dots\ u_2\ u_1) = (y_1\ x_{n_1-2}\ \dots\ x_1)$. We take A_3 as follows: $A_3 = (x_1\ x_3\ \dots\ x_{n_1-2}\ y_1\ x_2\ x_4\ \dots\ x_{n_1-3}\ z_1)$. Then the product $A_1\ A_2\ A_3$ must be $A_1\ A_2\ A_3 = (x_1\ x_2\ \dots\ x_{n_1-2}\ z_1)$. $A_1\ A_2\ A_3$ is a single cycle of order $n_1 - 1$.

If n_1 is odd, n_2 is even, n_3 is odd, $n_3 < n_1$ and n_4 is even, then we change the symbols $\{x_1, \dots, x_{n_3}, y_1, \dots, y_{n_1-n_3-1}\}$ into $\{X_1, \dots, X_{n_4-1}, Y_1, \dots, Y_{n_1-n_4}\}$ such that $(x_1\ x_3\ \dots\ x_{n_3}\ y_1\ y_2\ \dots\ y_{n_1-n_3-1}\ x_2\ x_4\ \dots\ x_{n_3-1}) = (X_1\ X_2\ \dots\ X_{n_4-1}\ Y_1\ Y_2\ \dots\ Y_{n_1-n_4})$. We take A_4 as follows: $A_4 = (X_1\ \dots\ X_{n_4-1}\ Z_1)$. Then the product $A_1\ A_2\ A_3\ A_4$ must be $A_1\ A_2\ A_3\ A_4 = (X_1\ X_3\ \dots\ X_{n_4-1}\ Y_1\ \dots\ Y_{n_1-n_4}\ X_2\ X_4\ \dots\ X_{n_4-2}\ Z_1)$. $A_1\ A_2\ A_3\ A_4$ is a single cycle of order n_1.

If n_1 is odd, n_2 is even and $n_3 = n_1$ and n_4 is even, then we change the symbols $\{x_1, x_2, \dots, x_{n_1-2}, z_1\}$ into $\{X_1, \dots, X_{n_4-1}, Y_1, \dots, Y_{n_1-n_4}\}$ such that $(x_1\ x_2\ \dots\ x_{n_1-2}\ z_1) = (X_1\ X_2\ \dots\ X_{n_4-1}\ Y_1\ Y_2\ \dots\ Y_{n_1-n_4})$. We take A_4 as follows: $A_4 = (X_1\ \dots\ X_{n_4-1}\ Z_1)$. Then the product $A_1\ A_2\ A_3\ A_4$ must be $A_1\ A_2\ A_3\ A_4 = (X_1\ X_3\ \dots\ X_{n_4-1}\ Y_1\ \dots\ Y_{n_1-n_4}\ X_2\ X_4\ \dots\ X_{n_4-2}\ Z_1)$. $A_1\ A_2\ A_3\ A_4$ is a single cycle of order n_1.

(III) If n_1 is even and n_2 is odd, then we take A_1 and A_2 as follows: $A_1 = (u_1 \ \dots \ u_{n_2} \ v_1 \ \dots \ v_{n_1-n_2})$ *and* $A_2 = (u_1 \ \dots \ u_{n_2})$. Then the product A_1A_2 must be $A_1 \ A_2 = (u_1 \ u_3 \ \dots \ u_{n_2} \ v_1 \ \dots \ v_{n_1-n_2} \ u_2 \ u_4 \ \dots \ u_{n_2-1})$. A_1A_2 is a single cycle of order n_1.

(IV) If n_1 is even and n_2 is even, then we take A_1 and A_2 as follows: $A_1 = (w_{n_1-n_2} \ \dots \ w_1 \ u_{n_2-2} \ u_{n_2-4} \ \dots \ u_2 \ v \ u_{n_2-1} \ u_{n_2-3} \ \dots \ u_1)$ *and* $A_2 = (u_1 \ \dots \ u_{n_2-1} \ v)$. Then the product $A_1 \ A_2$ must be $A_1 \ A_2 = (w_{n_1-n_2} \ \dots \ w_2 \ w_1 \ u_{n_2-1} \ u_{n_2-2} \dots \ u_2 \ u_1)$. $A_1 \ A_2$ is a single cycle of order $n_1 - 1$.

If n_1 is even, n_2 is even and n_3 is odd, then we change the symbols $\{u_1, \dots, u_{n_2-1}, w_1, \dots, w_{n_1-n_2}\}$ into $\{x_1, \dots, x_{n_1-1}\}$ such that $(w_{n_1-n_2} \ \dots \ w_2 \ w_1 \ u_{n_2-1} \ u_{n_2-2} \dots \ u_2 \ u_1) = (x_1 \ \dots \ x_{n_1-1})$. We take A_3 as follows: $A_3 = (x_1 \ \dots \ x_{n_3})$. Then the product $A_1 \ A_2 \ A_3$ must be $A_1 \ A_2 \ A_3 = (x_1 \ x_3 \ \dots \ x_{n_3} \ x_{n_3+1} \ \dots \ x_{n_1-1} \ x_2 \ x_4 \ , \dots \ x_{n_3-1})$. $A_1 \ A_2 \ A_3$ is a single cycle of order $n_1 - 1$.

If n_1 is even, n_2 is even, n_3 is odd and n_4 is even, then we change the symbols $\{x_1, \dots, x_{n_1-1}\}$ into $\{X_1, \dots, X_{n_4-1}, Y_1, \dots, Y_{n_1-n_4}\}$ such that $(x_1 \ x_3 \ \dots \ x_{n_3} \ x_{n_3+1} \ x_{n_1-1} \ x_2 \ x_4 \ \dots \ x_{n_3-1}) = (X_1 \ \dots \ X_{n_4-1} \ Y_1 \ \dots, Y_{n_1-n_4})$. We take A_4 as follows: $A_4 = (X_1 \ \dots \ X_{n_4-1} \ Z_1)$. Then the product $A_1 \ A_2 \ A_3 \ A_4$ must be $A_1 \ A_2 \ A_3 \ A_4 = (X_1 \ X_3 \ \dots \ X_{n_4-1} \ Y_1 \ \dots \ Y_{n_1-n_4} \ X_2 \ X_4 \ \dots \ X_{n_4-2} \ Z_1)$. $A_1 \ A_2 \ A_3 \ A_4$ is a single cycle of order n_1.

If n_1 is even, n_2 is even and n_3 is even, then we change the symbols $\{u_1, \dots, u_{n_2-1}, w_1, \dots, w_{n_1-n_2}\}$ into $\{x_1, \dots, x_{n_3-1}, y_1, \dots, y_{n_1-n_3}\}$ such that $(w_{n_1-n_2} \ \dots \ w_2 \ w_1 \ u_{n_2-1} \ u_{n_2-2} \dots \ u_2 \ u_1) = (x_1 \ \dots \ x_{n_3-1} \ y_1 \ \dots \ y_{n_1-n_3})$. We take A_3 as follows: $A_3 = (x_1 \ \dots \ x_{n_3-1} \ z_1)$. Then the product $A_1 \ A_2 \ A_3$ must be $A_1A_2A_3 = (x_1 \ x_3 \ \dots \ x_{n_3-1} \ y_1 \ \dots \ y_{n_1-n_3} \ x_2 \ x_4 \ \dots \ x_{n_3-2} \ z_1)$. $A_1 \ A_2 \ A_3$ is a single cycle of order n_1.

In this way, we can inductively find permutations $A_1, \dots, A_{d-2}$ such that each A_j is a single cycle of order n_j and the product $A_1 \dots A_{d-2}$ is also a single cycle of order n_1 or $n_1 - 1$. By the following lemma which is only a little different from the original lemma of Fox we can find permutations A_{d-1} and A_d of order n_{d-1} and n_d respectively such that $(A_{d-1} \ A_d)^{-1}$ is either a single cycle of order n_1 or $n_1 - 1$ or a product of two single cycles with no common symbol of order n_1 or $n_1 - 1$ and that the group generated by A_{d-1} and A_d is a transitive subgroup.

Lemma 2 *Given any three integers $a > 1$, $b > 1$ and $c \geq \max\{a, b\}$, there can be found a permutation A of order a and a permutation B of order b such that AB is a single cycle of order c or a product of two single cycles with no common symbol of order c and that the group generated by A and B is a transitive subgroup of the symmetric group of c symbols or $2c$ symbols.*

Proof of Lemma 2. By the proof of Fox (see [2]), the case that has not been proved is only the case $a = b = c$. If $a = b = c$ is odd, then we put $A = (u_1 \ u_2 \ \dots \ u_a)$ and $B = (u_1 \ u_2 \ \dots \ u_a)$. Then the product AB must be $AB = (u_1 \ u_3 \ \dots \ u_a \ u_2 \ u_4 \ \dots \ u_{a-1})$. AB is a single cycle of order a. If $a = b = c$ is even, then we put $A = (u_1 \ u_2 \ \dots \ u_a)(v_1 \ v_2 \ \dots \ v_a)$ and $B = (u_1 \ u_2 \ \dots \ u_{a-1} \ v_a)(v_1 \ v_2 \ \dots \ v_{a-1} \ u_a)$. Then the product AB must be $AB = (u_1 \ u_3 \ \dots \ u_{a-1} \ v_1 \ v_3 \ \dots \ v_{a-1})(u_2 \ u_4 \ \dots \ u_{a-2} \ v_a \ v_2 \ v_4 \ \dots \ v_{a-2} \ u_a)$. AB is a product of two single cycles of order a. *q.e.d.*

If the order of the product $A_1A_2 \dots A_{d-2}$ is n_1 (resp. $n_1 - 1$) and we can find A_{d-1} and A_d such that $(A_{d-1}A_d)^{-1}$ is a single cycle of order n_1 (resp. $n_1 - 1$), we can change the symbols suitably such that $A_1A_2 \dots A_{d-2} \ A_{d-1}A_d = 1$. If the order of the product $A_1A_2 \dots A_{d-2}$ is n_1 (resp. $n_1 - 1$) and we can find A_{d-1} and A_d such that $(A_{d-1} \ A_d)^{-1}$ is a product of two single cycles of order n_1 (resp. $n_1 - 1$), we can find permutations $\{A'_1, \dots, A'_{d-2}\}$ such that A'_j is the same type as A_j and $\{A'_1, \dots, A'_{d-2}\}$ have no common letter with $\{A_1, \dots, A_{d-2}\}$. Put $A''_j = A_j \ A'_j$ $(j = 1, \dots d-2)$. Then we can change the symbols suitably such that

$$A''_1A''_2 \dots A''_{d-2}A_{d-1}A_d = 1.$$

Note that $A_jA'_k = A'_kA_j$ for $j, k = 1, \dots, d-2$. $A''_1, A''_2, \dots, A''_{d-2}, A_{d-1}, A_d$ have required properties.

Next we assume $n_1 = 2$, then $n_1 = n_2 = \dots = n_d = 2$. In this case if d is even, it is trivial that we can find permutations $A_1, \dots, A_d$ which have required conditions. In fact put $A_j = (u_1 \ u_2)$ for every j. If d is odd, then we find A_j inductively as follows. Put $A_1 = (u_1 \ u_2) \ (v_1 \ v_2)$ and $A_2 = (u_1 \ v_2) \ (v_1 \ u_2)$. Then $A_1 \ A_2 = (u_1 \ v_1) \ (u_2 \ v_2)$. So if the product $A_1 \ \dots \ A_{d-1} = (x \ y) \ (z \ w)$, put $A_d = (x \ y) \ (z \ w)$. This completes the proof of Theorem 1.

Now we give the proof of Theorem 3.

Proof of Theorem 3. If the product n_1n_2 is odd, then we take L as L is a odd integer greater than n_1. If the product n_1n_2 is even and either n_1 or n_2 is odd, then we take L as L is a even integer greater than n_1. If n_1 and n_2 is even, then we take L such as L is a odd integer greater than n_1. From the proof of Fox of Lemma 1, we can find permutations A_1 and A_2 of order n_1 and n_2 such that the product A_1A_2 is a single cycle of order L. Then we think L as n_1. By the similar way above we can find permutations $A_3, \ldots, A_d$ which have the required conditions. Note that we can take L in an infinitely many way. *q.e.d.*

3 Another proof of Fenchel's problem

In a simmilar way, we can give another proof of the part of Fenchel's problem that S.Bundgaard and J.Nielsen solved in 1951 (see [1]). Let M be a compact Riemann surface of genus g. Let $\pi_1(M - \{q_1, \ldots, q_d\}) = < \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_d | \prod[\alpha_k, \beta_k]\gamma_1 \ldots \gamma_d = 1 >$ be the fundamental group of $M - \{q_1, \ldots, q_d\}$. From Lemma 2, for a given integer m greater than 1, we can find a permutation A of order 2 and a permutation B of order 2 such that the product AB is a single cycle of order $2m$. Then $ABA^{-1}B^{-1} = ABAB = (AB)^2$ is a product of two single cycles of order m. Now it is easy to see that for given integers m_j $(j = 1, \ldots, d)$ greater than 1 there exists a permutation A of order 2 and a permutation B of order 2 and permutations $\{C_j\}$ of order m_j $(j = 1, \ldots, d)$ respectively such that $ABA^{-1}B^{-1}C_1 \ldots C_d = 1$ and that the group G generated by $\{A, B, C_j$ $(j = 1, \ldots, d)\}$ is a transitive subgroup. Let $\Phi : \pi_1(M - \{q_1, \ldots, q_d\}) \to G$ be the homomorphism defined by $\Phi(\alpha_1) = A$, $\Phi(\beta_1) = B$, $\Phi(\alpha_l) = 1$ $(l = 2, \ldots, g)$, $\Phi(\beta_l) = 1$ $(l = 2, \ldots, g)$, $\Phi(\gamma_j) = C_j$ $(j = 1, \ldots, d)$. Then $Ker(\Phi)$ has the required properties.

References

[1] S.Bundgaard & J.Nielsen, *On normal subgroups with finite index in F-groups*, Mat. Tidsskrift, vol B (1951) 56-58

[2] A.L.Edomonds & J.H.Ewing & R.S. Kulkarni, *Torsion free subgroups of Fuchsian groups and Tessellations of surfaces*, Invent. Math. vol 69 (1982) 331-346

[3] R.Feuer, *Torsion free subgroups of triangle groups*, Proc. Amer. Math. Soc. vol 30 (1971) 235-240

[4] R.H.Fox, *On Fenchel's conjecture about F-groups*, Mat. Tidsskrift, vol B (1952) 61-65

[5] R.Fricke & F.Klein, *Vollesungen ueber die Theorie der automorphen Functionen*, Teubner, Leipzig , (1897) 186-187

[6] A.Macbeath, *On a theorem of Hurwitz*, Proc. Glasgow Math. Assoc. (1961) vol 5 90-96

[7] J.Mennicke, *Eine Bemerkung über Fuchssche Gruppen*, Invent. Math. vol 2 (1967) 301-305

[8] M.Namba, *Branched coverings and algebraic functions*, Research Notes in Math. (1987) vol 161 Pitman-Longman