C^rG vector bundles and Nash G manifolds

$C^r G$ vector bundles and Nash G manifolds

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abstract

We prove that any $C^r G$ $(r>2)$ vector bundle over a compact $C^\infty G$ (resp. $C^\omega G$) manifold has a unique $C^\infty G$ (resp. $C^\omega G$) vector bundle structure when **G** is a compact Lie group. We consider a $C^\infty G$ imbedding of a nonaffine Nash **G** manifold, existence of compactifications of affine Nash **G** manifolds and uniqueness of them up to Nash **G** diffeomorphism.

Key Words: Nash manifolds, Nash G vector bundles, Imbeddings, Compactifications.

1. Introduction

In this paper we consider the following two topics. One is the existence of a $C^\infty G$ (resp. $C^\omega G$) vector bundle structure of a $C^r G$ $(r>1)$ vector bundle over a $C^\infty G$ (resp. $C^\omega G$) manifold when **G** is a compact Lie group. The other is imbeddings and compactifications of Nash G manifolds when **G** is a Nash group.

It is known [3] that any $C^r G$ $(r>1)$ vector bundle η over an affine Nash **G** manifold has a unique Nash **G** vector bundle structure up to Nash **G** vector bundle isomorphism. Namely, there exist a Nash **G** vector bundle ζ over **X** such that η and ζ are $C^r G$ vector bundle isomorphic and that ζ is unique up to Nash **G** vector bundle isomorphism.

Theorem 1.1 Let **G** be a compact Lie group and let **X** be a compact or compactifiable $C^\infty G$ (resp. $C^\omega G$) manifold. Any $C^r G$ $(r>2)$ vector bundle over **X** has a unique $C^\infty G$ (resp. $C^\omega G$) vector bundle structure up to $C^\infty G$ (resp. $C^\omega G$) vector bundle isomorphism.

A $C^r G$ manifold is not always compactifiable as a $C^r G$ manifold when **G** is a compact Lie group, but every affine Nash **G** manifold is compactifiable as a $C^\infty G$ manifold when **G** is a compact affine Nash group [4]. In addition, if **G** is a finite group then it is compactifiable as a Nash **G** manifold because an analogous proof of [9] works in this case, and the resulting Nash **G** manifold is affine.

We can conjecture that Theorem 1.1 holds true when r= 0 or 1, but our proof does not work.

Regarding a $C^\infty G$ imbedding of a Nash **G** manifold, it is possible if it is compact [3]. We have the following result as a partial generalization [3].

Theorem 1.2 Every nonaffine Nash manifold is C^∞ imbeddable into some Euclidean space as an affine Nash manifold.

Clearly we cannot replace "C^∞" by "Nash". Moreover we can conjecture that Theorem 1.2 remains valid in the equivariant category, but we do not know the proof. We get the next result as an equivariant generalization [7, Theorem 2](See Remark 3.2).

Theorem 1.3 Let **G** be a finite group. Let **X**, **Y** be affine Nash **G** manifolds and **X'**, **Y'** its affine compactifications as Nash **G** manifolds, respectively. Then the following three conditions are equivalent.

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(1) **X** and **Y** are Nash **G** diffeomorphic.
(2) **X'** and **Y'** are Nash **G** diffeomorphic.
(3) **X'** and **Y'** are C^1**G** diffeomorphic.

We cannot add the condition that "**X** and **Y** are C^∞ **G** diffeomorphic" to the above conditions even if **G** is a trivial group. There exist two non-compact affine Nash manifolds such that they are C^∞ diffeomorphic but not Nash diffeomorphic [7, section 5]. In particular a compactification of Nash manifold as a C^∞ manifold is not necessarily unique up to Nash diffeomorphism. But it is known that a compactification of an affine Nash manifold as an affine Nash manifold is unique up to Nash diffeomorphism [9]. For two compact affine Nash **G** manifolds, they are C^∞ **G** diffeomorphic if and only if they are Nash **G** diffeomorphic when **G** is a compact affine Nash group [4]. Therefore, in this case, if an affine Nash **G** manifold is compactifiable as an affine Nash **G** manifold then this compactification is unique up to Nash **G** diffeomorphism.

This paper is organized as follows. We prove Theorem 1.1 in section 2. In section 3, we show Theorem 1.2 and 1.3. Unless otherwise stated, Nash manifolds and Nash **G** manifolds are of class C^ω Nash.

2. C^rG vector bundles

First of all, recall the definitions we need. In this section **G** denotes a compact Lie group. A **G** vector bundle η = (**E, p, X**) is called a C^r**G** ($0 \leq r \leq \omega$) **vector bundle** if the total space **E**, the base space **X** are C^r**G** manifolds and the projection **p** is of class C^r. The following three results are important to prove Theorem 1.1.

Lemma 2.1 Let **X** be a C^∞ **G** (resp. C^ω **G**) manifold. Suppose that η and ζ are C^∞ **G** (resp. C^ω **G**) vector bundles over **X**. Then $\eta \oplus \zeta$, $\eta \otimes \zeta$, η^* (the dual of η) and Hom(η, ζ) are C^∞ **G** (resp. C^ω **G**) vector bundles over **X**. □

Proposition 2.2 Every C^0**G** section of a C^∞ **G** (resp. C^ω **G**) vector bundle over a compact C^∞ **G** (resp. C^ω **G**) manifold can be approximated by a C^∞ **G** (resp. C^ω **G**) one. □

Theorem 2.3 For any two C^∞ **G** (resp. C^ω **G**) vector bundle over a compact C^∞ **G** (resp. C^ω **G**) manifold, they are C^0**G** vector bundle isomorphic if and only if they are C^∞ **G** (resp. C^ω **G**) vector isomorphic. □

Theorem 2.3 shows uniqueness of C^∞ **G** (resp. C^ω **G**) vector bundle structure when **X** is compact. To prove Theorem 1.1 the case when **X** is a non-compact C^r**G** manifold, we have to take the double of a C^r**G** manifold with boundary. The next fact guarantees to do so.

Fact 2.4 (Existence of a collar) A compact C^r**G** manifold **X** ($0 < r \leq \omega$) with boundary has a C^r**G** collar (i.e. there exist a C^r**G** imbedding

$$\rho : X \times [0, 1] \longrightarrow X$$

such that $\rho|(X \times 0) = id$, where the action on **[0, 1]** is trivial). □

We quote the next two results proved by A. G. Wassermann [10].

Theorem 2.5 Let **G** be a compact Lie group. Suppose that η = (**E**, π, **M** × **[0, 1]**) is a C^r**G** ($1 < r \leq \omega$) vector bundle. Then there exists a C^r**G** vector bundle isomorphism

$$(E|(M \times 0)) \times [0, 1] \longrightarrow E.$$

Here the action on **[0, 1]** is trivial. □

Corollary 2.6 If η = (**E**, π, **X**) is a C^r**G** vector bundle ($1 < r \leq \omega$) and **f**, **h**:**Y**⟶**X** are C^r **G** homotopic then $f^*(\eta)$ and $h^*(\eta)$ are C^r**G** vector bundle isomorphic. □

We are in position to prove Theorem 1.1.

Proof of Theorem 1.1.

Let η be a C^rG vector bundle over X. Without loss of generality, we can assume that η has the same rank n. Since X is compact, the total space E of η has only finitely many orbit types. So by [5], E can be C^rG imbeddable into some representation Ω of G. We identify E with the image. Take the classifying map $h:X \longrightarrow G(\Omega, n)$ of the normal bundle of X in $E \subset \Omega$. The map h is $C^{r-1}G$ map because E is a C^rG manifold. We now approximate h by a $C^\infty G$ (resp. $C^\omega G$) map. We regard h as a $C^{r-1}G$ map from X to $M_k(R)$, where k denotes the dimension of Ω and the action on $M_k(R)$ is determined by Ω. Since G is compact, we may assume that h is approximated by a polynomial G map [2]. On the other hand, one can find an equivariant Nash tubular neighborhood (U, q) of $G(\Omega, n)$ in $M_k(R)$ [3, 4]. If the approximation is close enough then the image of q is in U. Set $f = q \circ h$. Then f is a $C^\infty G$ (resp. $C^\omega G$) map and $C^{r-1}G$ homotopic to h, namely, there exists a $C^{r-1}G$ map

$$F: X \times [0, 1] \longrightarrow X$$

so that $F(x, 0) = f(x)$, $F(x, 1) = h(x)$ for any $x \in X$, where the action on $[0, 1]$ is trivial. By assumption, $r>1$. Using Corollary 2.6, we have that η and $f^*(\gamma(\Omega, n))$ are $C^{r-1}G$ vector bundle isomorphic. Thus it remains to show that η and $f^*(\gamma(\Omega, n))$ are C^rG vector bundle isomorphic. By Theorem 2.3, the result is proved when X is compact. We now prove when the base space is not compact but compactifiable. Let X be a compactifiable $C^\infty G$ (resp. $C^\omega G$) manifold. Let X' be the compactification of X as a $C^\infty G$ (resp. $C^\omega G$) manifold. By the construction of X', η is naturally extensible over X'. Because of Fact 2.4, we can take the double of a $C^\infty G$ (resp. $C^\omega G$) manifold with boundary. Let Y be the double of X' and η' the double of η. Then Y is a compact $C^\infty G$ (resp. $C^\omega G$) manifold without boundary (closed G manifold) and η' is a $C^\infty G$ (resp. $C^\omega G$) vector bundle over Y. Applying the compact case, we have a $C^\infty G$ (resp. $C^\omega G$) vector bundle η'' over Y which is $C^\infty G$ (resp. $C^\omega G$) vector bundle isomorphic to η'. Hence $\eta''|X$ is the desired one. □

3. Nonaffine Nash G manifolds

In this section G denotes a finite group unless otherwise stated.

We recall definitions of Nash G manifolds and affine Nash G manifolds.

Definition 3.1 Let G be an affine Nash group.

(1) A Nash manifold is called a **Nash G manifold** if it has a G action whose action map $G \times X \longrightarrow X$ is a Nash map.

(2) A Nash G manifold X is said to be **affine** if *there exist some representation* Ω *and* a Nash G submanifold Y of Ω such that X is Nash G diffeomorphic to Y.

(3) Let X, Y be Nash G manifolds. We say that X and Y are **Nash G diffeomorphic** if there exist Nash G maps $f:X \longrightarrow Y$ and $h:Y \longrightarrow X$ so that $f \circ h = id$, $h \circ f = id$.

We now prove Theorem 1.2

Proof of Theorem 1.2.

Let X be a nonaffine Nash manifold. We can compactify X as a C^∞ manifold [6], and let X' be a compactification of X. Then X' is a compact C^∞ manifold with boundary so that X is C^∞ diffeomorphic to the interior of X'. Let Y be the double of X'. We apply a relative Nash theorem [1] to a pair $(Y, \partial X')$. Thus we have a pair (Z, Z') of nonsingular algebraic sets which is pairwise C^∞ diffeomorphic to the pair $(Y, \partial X')$. Therefore some connected components of $Z-Z'$ are C^∞ diffeomorphic to X. Since a union of connected components of a nonsingular algebraic set is an affine Nash manifold, X is C^∞ diffeomorphic to an affine Nash manifold. □

Remark 3.2 In the equivariant category, we do not know whether a relative Nash theorem is true or not. But in this case we can also make the double Y by Lemma 3.2, and Y is C^∞ G imbeddable into some representation of G as a Nash G submanifold Z [3]. However $Z-Z'$ may not be a Nash G manifold.

We next prove Theorem 1.3

Proof of Theorem 1.3.
Since the implications $(2)\longrightarrow(3)$, $(2)\longrightarrow(1)$ are clear, we have only to show $(3)\longrightarrow(2)$ and $(1)\longrightarrow(3)$. First we show that $(3)\longrightarrow(2)$. Let X'' and Y'' be the double of X' and Y', respectively. To show the implication $(3)\longrightarrow(2)$, we need the following lemma.

Lemma 3.3 Let G be a finite group. Let X_1 and X_2 be compact affine Nash G manifolds, X_2 and Y_2 its Nash G submanifolds, respectively. Suppose that $f:(X_1, X_2)\longrightarrow(Y_1, Y_2)$ is a C^rG map $(2<r<\infty)$ so that the restriction on X_1 is of class C^ω Nash. Then f is approximated in the C^r topology by a Nash G map

$$h:(X_1, X_2)\longrightarrow(Y_1, Y_2)$$

with $f|X = h|X$.

Proof of Lemma 3.3. The above result is obtained by applying the averaging operator because non-equivariant cases are already known [8] □.

We continue the proof of Theorem 1.3. Since X' and Y' are C^1G diffeomorphic, there exists a C^1G diffeomorphism $f:X''\longrightarrow Y''$ so that $f(X')=\partial Y'$. Hence we can approximate

$$f:(X'', \partial X')\longrightarrow(Y'', \partial Y')$$

by a Nash G map

$$h:(X'', \partial X')\longrightarrow(Y'', \partial Y')$$

because of Lemma 3.3 Since X'' is compact, h is a diffeomorphism. Thus we have a Nash G diffeomorphism $h:(X'', \partial X)\longrightarrow(Y'', \partial Y)$. The restriction $h|X$ is the required one □□.

Remark 3.4 The statement $(3)\longrightarrow(2)$ can be generalized the following form. Let G be a finite group. Let $L_1\subset L_2$, $L_1'\subset L_2'$ be the compact affine G manifolds possibly with boundary and compact Nash G manifolds with

$$\partial L_1\cap L_2 = \partial L_1'\cap L_2' = \emptyset.$$

If there is a C^rG diffeomorphism $(0<r<\infty)$, from (L_1, L_2) to (L_1', L_2'), we can approximate it by a Nash one in the C^r topology □.

We return to the proof of Theorem 1.3. We now show that (1) implies (3). Let X_1 and Y_1 be the doubles of X' and Y', respectively. By the construction of the compactification, there exists a non-negative proper Nash G map

$$f: X\longrightarrow R \quad (\text{resp. } h:Y\longrightarrow R)$$

such that $f^{-1}([0, n])$ (resp. $h^{-1}([0, m])$) is Nash G diffeomorphic to X' (resp. Y'), where n (resp. m) is a upper bound of the set of critical values of f (resp. h). By hypothesis and Fact 2.4, there exists a C^∞ G diffeomorphism

$$l:f^{-1}([0, n])\longrightarrow h^{-1}([0, m]).$$

Therefore X' is C^∞ G (C^1G) diffeomorphic to Y'. □

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