# Semialgebraic G vector bundles over a semialgebraic G set with free action

# Semialgebraic G vector bundles over a semialgebraic G set with free action

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**abstract**

We show that the set of isomorphism classes of semialgebraic **G** vector bundles over a semialgebraic **G** set **X** are in one-to-one correspondence to that of semialgebraic vector bundles over **X/G** when **G** acts freely on **X**.

Key Words: Semialgebraic **G** vector bundles, Semialgebraic **G** sets, Free actions.

## 1. Introduction.

Let **G** be a compact Lie group. It is well-known that the set of all isomorphism classes of $C^0G$ (resp. $C^\infty G$) vector bundle over a $C^0G$ space (resp. $C^\infty$ **G** manifold) with free action corresponds bijectively to the set of all isomorphism classes of $C^0G$ (resp. $C^\infty G$) vector bundles over the orbit space [1].

In this paper, we consider a semialgebraic version of this fact. We refer to [2] and [6] the notion of semialgebraic vector bundles.

Let **G** be *an algebraic group. A* **G** *invariant* subset of a representation of **G** is called a **semialgebraic G set** if it is semialgebraic. By the aid of [4], one can show that the orbit space of a semialgebraic **G** set has a unique semialgebraic structure such that the projection $X \longrightarrow X/G$ is semialgebraic. It is known that one can find an explicit description of defining inequalities [5].

We have the following result.

**Theorem** Let **G** be a real algebraic group and let **X** be a semialgebraic **G** set. If **G** acts on **X** freely, then the set of all isomorphism classes of semialgebraic **G** vector bundles over **X** is isomorphic to the set of all isomorphism classes of semialgebraic vector bundles over **X/G** as sets.

**Corollary** Suppose the action of **G** on **X** is free. Any semialgebraic **G** vector bundle $\zeta$ has a classifying map in the semialgebraic category, namely, there exist a universal **G** vector bundle $\eta=(E(n, m), p, G(n, m))$ and a semialgebraic **G** map $f: X \longrightarrow G(n, m)$ so that $\zeta$ is *isomorphic to* $f^*(\eta)$ *as semialgebraic G vector* bundles.

This corollary is an extension of the fact that any semialgebraic vector bundle over a semialgebraic set has a classifying map in the semialgebraic category [3, Corollary 12.7.5].

In the present paper **G** (resp. **X**) stands for a real algebraic group (resp. a semialgebraic **G** set) unless otherwise stated.

## 2. Semialgebraic G vector bundles.

We make basic definitions. Let **X** and **Y** be semialgebraic **G** sets. A semialgebraic map (resp. semialgebraic isomorphism ) $f: X \longrightarrow Y$ is said to be a **semialgebraic G map** (resp.

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**semialgebraic G isomorphism)** if it is a G map.

**Definition 2.1.** (1)A semialgebraic vector bundle $\eta=(E, \pi, X)$ is said to be a **semialgebraic G vector bundle** if the following three conditions are satisfied:

[1] The total space $E$ and the base space $X$ are semialgebraic G sets.

[2] The projection $\pi$ is a semialgebraic G map.

[3] For any $x \in X$, $g \in G$, the map

$$\pi^{-1}(x) \longrightarrow \pi^{-1}(gx)$$

is linear.

(2)Two semialgebraic G vector bundles are called **isomorphic as semialgebraic G vector bundles** if there exists a semialgebraic vector bundle isomorphism such that it is a G map.

## 3. Proof of Theorem and Corollary.

We show that for every semialgebraic G vector bundle $\eta=(E, \pi, X)$, the quotient bundle

$$\eta/G=(E/G, \pi/G, X/G)$$

is a semialgebraic vector bundle, where $E/G$ and $X/G$ are the quotient spaces of $E$ and $X$, respectively, and that $\pi/G:E/G \longrightarrow X/G$ is the induced map from $\pi:E \longrightarrow X$.

To show this we prepare two lemmata.

**Lemma 3.1** Let $X \subset R^n$, $Y \subset R^m$ be semialgebraic sets. If $f:X \longrightarrow Y$ is a surjective semialgebraic map, then there exists a semialgebraic map $h:Y \longrightarrow X$ so that $f \circ h = id_Y$.

**Proof.**

Replacing $X \subset R^n$ by the graph of $f \subset R^n \times R^m$, we can assume that $f$ is a restriction of the natural projection $R^n \times R^m \longrightarrow R^m$. We now show when $n=1$. By a partition of semialgebraic set (cf. [2, Theorem 2.2.1]), there exists a finite partition $\{A_i | i \in I\}$ of $R^m$ into connected semialgebraic sets satisfying the following two conditions:

(1)For any $A_i$ there exist finitely many semialgebraic continuous functions

$$f_{i,k}: A_i \longrightarrow R \ (0 \leq k \leq s(i))$$

such that

$$f_{i,k}(x) < f_{i,k+1}(x) \ (0 \leq k \leq s(i))$$

for any $x \in A_i$.

(2)$X$ is a disjoint union of the set of the form:

$$\{(t,x) \in R^n \times R^m | \ x \in A_i, \ t < f_{i,0}(x)\},$$

$$\{(t,x) \in R^n \times R^m | \ x \in A_i, \ f_{i,k}(x) < t < f_{i,k+1}(x)\} \quad (0 \leq k \leq s(i)-1)$$

$$\{(t,x) \in R^n \times R^m | x \in A_i, \ t > f_{i,s(i)}(x)\}, \text{ or}$$

$$\{(t,x) \in R^n \times R^m | x \in A_i, \ t=f_{i,k}(x)\} \quad (0 \leq k \leq s_i).$$

Therefore one can easily construct a semialgebraic map $h$ satisfying $f \circ h=id_Y$. In the general case, we have the required map because the projection $R^n \times R^m \longrightarrow R^m$ is the composition of $R^n \times R^m \longrightarrow R^{n-1} \times R^m \longrightarrow \ldots \longrightarrow R^m$. □

**Lemma 3.2** Let $X, Y$ and $Z$ be semialgebraic sets. Let $s:X \longrightarrow Z$, $t:X \longrightarrow Y$ be semialgebraic maps and let $u:Y \longrightarrow Z$ be a map. If $t$ is surjective and $s=u \circ t$ then $u$ is a semialgebraic map.

**Proof.**

By Lemma 3.1, there exists a semialgebraic map $q:Y \longrightarrow X$ such that $t \circ q=id_Y$. Therefore

$$u=u \circ id_Y=u \circ t \circ q=s \circ q$$

is a semialgebraic map. □

**Proof of Theorem**

Let $\eta=(E, \pi, X)$ be a semialgebraic G vector bundle over $X$. Let $q_X$ and $q_E$ denote the orbit maps of $X \longrightarrow X/G$ and $E \longrightarrow E/G$, respectively. Set $p=q_X \circ \pi$. Then $p=\pi/G \circ q_E$. By Lemma 3.2,

$$\pi/G : E/G \longrightarrow X/G$$

is a semialgebraic map, thus it is a semialgebraic continuous map. Let $\{U_i, \pi_i\}$ be a semialgebraic trivialization of $\eta$. Since any fiber of $\eta$ has the trivial action, one can easily check that $\{q_X(U_i), \pi'_i\}$ is a semialgebraic trivialization of $\eta/G$, where $\pi'_i$ is the map which makes the following diagram commute.

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\pi_i} & U_i \times \mathbf{R}^k \\ \downarrow {\scriptstyle q_E|\pi^{-1}(U_i)} & & \downarrow {\scriptstyle (q_X|U_i)\times id} \\ q_E(\pi^{-1}(U_i)) = (\pi/G)^{-1}(q_i(U_i)) & \xrightarrow[\pi'_i]{} & q(U_i) \times \mathbf{R}^k \end{array}$$

Hence $\eta/G=(E/G, \pi/G, X/G)$ is a semialgebraic vector bundle over $X/G$. Therefore we can define the map

$$F: VEC(X) \longrightarrow VEC(X/G)$$

by $F(\eta)=\eta/G$, where $VEC(X)$ (resp. $VEC(X/G)$) denotes the set of all isomorphism classes of semialgebraic $G$ vector bundles (resp. the set of all isomorphism classes of semialgebraic vector bundles) over $X$ (resp. $X/G$). We define the map

$$K: VEC(X/G) \longrightarrow VEC(X)$$

by $K(\zeta)=q_X^*(\zeta)$.
Then $F \circ K = id$, $K \circ F = id$. This completes the proof. □

**Proof of Corollary.**
By Theorem, for any semialgebraic $G$ vector bundle $\eta$ over $X$, there exists a semialgebraic vector bundle $\eta/G$ over $X/G$ so that

$$\eta \cong \pi^*(\eta/G).$$

In terms of [3, Corolary 12.7.5], $\eta/G$ has a classifying map $k: X/G \longrightarrow G(n, m)$, where $G(n, m)$ denotes some Grassmannian variety. Hence

$$k \circ \pi \longrightarrow G(n, m)$$

is the required classifying map of $\eta$. □

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