On an Extension of Perelomov Transformation

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Noriaki KATAYAMA*

ABSTRACT

In the previous paper [1], it has been shown that generalized Ermakov system is related to Calogero system through two transformations, one is canonical transformation and the other is Perelomov transformation. In this short note, the possibility of an extension of Perelomov transformation is discussed and the time dependent angular frequency $\Omega(t)$ of added harmonic type potential function in Calogero system (see (26) in [1]) is determined in order that Perelomov transformation can be extended.

Key Words: Dynamical Systems, Perelomov Transformation, Calogero System.

This note is a sequel of [1]. In the previous paper, it has been shown that generalized Ermakov system is related to Calogero system through two transformations, one is canonical transformation and the other is Perelomov transformation. Ermakov system is known as harmonic oscillator having the time dependent angular frequency [2, 3, 4] and the Calogero system is known as a completely integrable system in one dimension [5].

In this short note, we start with giving Calogero system having harmonic type potential function of two degrees of freedom. It is to be noted that the notation is the same as that of the previous paper.

Definition 1.

Calogero system having harmonic type potential function of two degrees of freedom is defined by the following Hamiltonian H_{Ch}

$$H_{Ch} = \frac{\tilde{p}_1^2}{2} + \frac{\tilde{p}_2^2}{2} + \frac{\Omega(t)^2}{2}\{(\tilde{q}^1)^2 + (\tilde{q}^2)^2\} + \frac{1}{(\tilde{q}^1 - \tilde{q}^2)^2}, \tag{1}$$

where $\tilde{p}_i$ $(i = 1, 2)$ is a generalized momentum and $\tilde{q}^i$ $(i = 1, 2)$ is a generalized coordinate $(\tilde{p}_i, \tilde{q}^i \in R)$. $\Omega(t)$ denotes the time dependent angular frequency.

The equations of motion is derived from Hamilton's equations as follows,

$$\left.\begin{aligned} \frac{d^2}{dt^2}\tilde{q}^1(t) &= -\Omega^2(t)\,\tilde{q}^1(t) + \frac{2}{(\tilde{q}^1(t) - \tilde{q}^2(t))^3}, \\ \frac{d^2}{dt^2}\tilde{q}^2(t) &= -\Omega^2(t)\,\tilde{q}^2(t) + \frac{2}{(\tilde{q}^2(t) - \tilde{q}^1(t))^3}. \end{aligned}\right\} \tag{2}$$

This dynamical system admits a proper quadratic first integral

$$I = \frac{1}{2}\left(\tilde{q}^1\frac{d}{dt}\tilde{q}^2 - \tilde{q}^2\frac{d}{dt}\tilde{q}^1\right)^2 + \frac{1}{2}\left(\frac{\tilde{q}^1 + \tilde{q}^2}{\tilde{q}^1 - \tilde{q}^2}\right)^2, \tag{3}$$

(see Proposition 3 in [1]).

Secondly we give ordinary Calogero system of two degrees of freedom. In definition 1, by putting $\Omega(t) = 0$, we can get this Calogero system. Then we have,

Definition 2.

Calogero system of two degrees of freedom is defined by the following Hamiltonian H_c

$$H_c = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{(x^1 - x^2)^2}, \tag{4}$$

where y_i, x^i $(i = 1, 2)$ are generalized momentum and generalized coordinate respectively.

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* Department of Mechanical Engineering.

Physical meaning of this system should be referred in [2]. And equations of motion are given as follows,

$$\left.\begin{aligned} \frac{d^2}{dt^2}x^1 &= \frac{2}{(x^1-x^2)^3}, \\ \frac{d^2}{dt^2}x^2 &= \frac{2}{(x^2-x^1)^3}. \end{aligned}\right\} \quad (5)$$

Here we can say that Perelomov transformation means that how the time dependent angular frequency $\Omega(t)$ can be excluded from the equations of motion (2). In Perelomov transformation, it is assumed that $\Omega(t)$ is equal to a positive constant Ω_0 and by putting

$$\left.\begin{aligned} x^i(T) &= \frac{\sqrt{\Omega_0}\cdot\tilde{q}^i(t)}{\cos(\Omega_0 t)} \quad (i=1,2), \\ T &= \tan(\Omega_0 t), \end{aligned}\right\} \quad (6)$$

the equations of motion in $\tilde{q}^i(t)$ $(i = 1, 2)$ are transformed to

$$\frac{d^2}{dT^2}x^i(T) = 2\sum_{j\neq i}^{2}(x^i-x^j)^{-3} \quad (i=1,2). \qquad (7)$$

The aim of the exclusion of angular frequency $\Omega(t)$ from equations of motion can be carried out by the Perelomov transformation. But it is to be noted that the time scale is also changed.

As the third step, the assumption that $\Omega(t)$ is equal to a positive constant is not satisfied. But we can assume that $\Omega(t)$ is positive in any time t from the physical meaning.

We put

$$\Omega_E(t) = \int_0^t \Omega(s)ds, \qquad (8)$$

and instead of (6)

$$\left.\begin{aligned} x^k(T) &= \frac{\tilde{q}^k(t)}{C(t)\cos(\Omega_E(t))} \quad (k=1,2), \\ T &= \tan(\Omega_E(t)), \end{aligned}\right\} \quad (9)$$

where $C(t)$ must be determined afterwards. Namely we must decide $C(t)$ for this transformation (8) and (9) to be an extended Perelomov transformation.

From (9), after a long calculation we can get,

$$\left.\begin{aligned} \frac{d^2}{dt^2}\tilde{q}^k(t) &= \frac{C'(t)\,(\Omega_E'(t))^2}{\cos^3(\Omega_E(t))}\left(\frac{d^2}{dT^2}x^k(T)\right) \\ &+ \frac{2C'(t)\,\Omega_E'(t)+C(t)\,\Omega_E''(t)}{\cos(\Omega_E(t))}\left(\frac{d}{dT}x^k(T)\right) \\ &+ \{C''(t)\cos(\Omega_E(t)) - 2C'(t)\sin(\Omega_E(t)) \\ &\times (\Omega_E'(t)) - C(t)\cos(\Omega_E(t))\,\Omega_E'(t) \\ &- C(t)\sin(\Omega_E(t))\,\Omega_E''(t)\}\,x^k(T), \\ &\qquad (k=1,2), \end{aligned}\right\} \quad (10)$$

where

$$\left.\begin{aligned} C'(t) &= \frac{d}{dt}C(t), \\ \Omega_E''(t) &= \frac{d^2}{dt^2}\Omega_E(t), \end{aligned}\right\} \quad (11)$$

and so on.

On the other hand, from (9) and (10) we have,

$$\frac{d^2}{dt^2}\tilde{q}^k(t) + \Omega(t)^2\,\tilde{q}^k(t)$$

$$\left.\begin{aligned} &= \frac{C(t)\,(\Omega(t))^2}{\cos^3(\Omega_E(t))}\ \frac{d^2}{dT^2}x^k(T) \\ &+ \frac{2C'(t)\,\Omega_E'(t)+C(t)\,\Omega_E''(t)}{\cos(\Omega_E(t))}\ \frac{d}{dT}x^k(T) \end{aligned}\right\} \quad (12)$$

$$+\{C''(t)\cos(\Omega_E(t)) - 2C'(t)\sin(\Omega_E(t)) \times (\Omega_E'(t)) - C(t)\sin(\Omega_E(t))\,\Omega_E''(t)\}\,x^k(T), \quad (k=1,2).$$

Here let $x^k(T)$, $(k = 1, 2)$ be satisfied with the same type equations of motion (5). Namely, we can get

$$\left.\begin{aligned} \frac{d^2}{dT^2}x^k(T) &= 2\sum_{k\neq j}^{2}(\tilde{q}^k(t)-\tilde{q}^j(t))^{-3} \\ &\times (C(t)\cos(\Omega_E(t)))^3. \end{aligned}\right\} \quad (13)$$

Then by putting (13) into (12), we get

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\tilde{q}^k + \Omega^2(t)\,\tilde{q}^k \\ &= 2C(t)^4\Omega(t)^2\sum_{j\neq k}^{2}(\tilde{q}^k-\tilde{q}^j)^{-3} \\ &+ \{2C'(t)\,\Omega_E'(t) + C(t)\,\Omega_E''(t)\}\frac{d}{dT}x^k(T) \\ &+ \{C''(t)\cos(\Omega_E(t)) - 2C'(t)\sin(\Omega_E(t)) \\ &\times \Omega_E'(t) - C(t)\sin(\Omega_E(t))\,\Omega_E''(t)\}x^k(T), \\ &\qquad (k=1,2). \end{aligned}\right\} \quad (14)$$

For this transformation to be an extended Perelomov transformation, we get following three conditions (15), (16) and (17),

$$C(t) = 1/\sqrt{\Omega(t)}\,, \quad (15)$$

$$2C'(t)\,\Omega(t) + C(t)\,\Omega'(t) = 0\,, \quad (16)$$

$$\begin{aligned} &C''(t)\cos(\Omega_E(t)) - 2C'(t)\sin(\Omega_E(t))\,\Omega(t) \\ &\quad - C(t)\sin(\Omega_E(t))\,\Omega'(t) = 0\,. \end{aligned} \quad (17)$$

From (15), we get

$$C'(t) = -\frac{\Omega'(t)}{2\Omega(t)^{3/2}}\,, \quad (18)$$

then (16) is automatically derived from (18). Putting (18) into (17) leads

$$\{\frac{d^2}{dt^2}C(t)\}\cos(\Omega_E(t)) = 0\,. \quad (19)$$

Using constants c_0 and c_1, in only case of

$$C(t) = c_0t + c_1\,, \quad (20)$$

$$\Omega(t) = \frac{1}{(c_0t+c_1)^2}\,, \quad (21)$$

the transformation defined by (8) and (9) can be an extended Perelomov transformation.

Consequently, it was shown that when the time dependent angular frequency $\Omega(t)$ is given as in (21), Calogero system having harmonic type potential function can be transformed to the ordinary one by the extended Perelomov transformation defined in (8) and (9). It is to be noted that if we take $c_0 = 0$ then the extended Perelomov transformation corresponds with the original Perelomov transformation.

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