On Stochastic Parabolic Equations with Subdifferentials and Stochastic Variational Inequalities

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Masaaki ISHIKAWA *

Abstract

This paper is concerned with a mathematical formulation of stochastic parabolic equations with subdifferentials. The concept of the subdifferential strongly relates to a variational inequality which appears in modeling various kinds of nonlinear phenomena including obstacle and dam problems, so called, free boundary problems. In this paper, we will propose how to formulate the stochastic distributed parameter system with the subdifferential by the stochastic variational inequality using a random measure.

Key Words: Stochastic Parabolic Equation, Subdifferential, Stochastic Variational Inequality, Free Boundary Problem, Compactness Method

1. Introduction

Since many problems in various fields of physics and engineering have inherently nonlinear aspects, nonlinear ordinary and partial differential equations are often used in mathematical descriptions of physical phenomena. Particularly, as typical examples of problems described by nonlinear partial differential equations, free boundary problems are considered. In theoretical analysis of free boundary problems, a variational inequality have important role. Unfortunately although all free boundary problems are not necessarily transformed into the variational inequality, effectuality of the variational inequality admits of no doubt in a mathematical formulation of a large variety of physical problems. Up to date, the deterministic and stochastic variational inequalities are studied in many articles 1) ~ 6). In 4) "weak" and "almost weak" solutions for stochastic variational inequality are defined. On the other hand, in 6) the existence and uniqueness of the strong solution for the nonnegative valued stochastic variational inequality have been proved. Our approach is similar to 6), however, the considered stochastic variational inequality is somewhat general. The problem considered in 6) is an obstacle problem with the obstacle zero. The problem in this paper includes the obstacle problems. For readers' better understanding, principal symbols used here are listed below;

t;	time variable belongs to $T =]0, t_f[$
x;	spatial variable belongs to $G =]0, 1[$
$L^p(X)$;	space of the p-th power integrable functions on X
$C(X;Y)$;	space of continuous functions on X taking values in Y
$M(X)$;	space of signed measures on X which is identified with the dual of $C(X)$
$H^n(X)$;	Sobolev space of order n on X
$H_0^1(X)$;	space of functions f such that $f \in H^1(X)$, $f = 0$ on the boundary of X
$H^{-1}(X)$;	dual of $H_0^1(X)$
Ω;	sample space
$<, >$;	pairing between $H^{-1}(G)$ and $H_0^1(G)$
$(\,,\,), \lvert\ \rvert$;	inner product and norm in $L^2(G)$
$\lVert\ \rVert_X$;	norm in the space X, where X is omitted if $X = H_0^1(G)$
Δ;	Laplacian
E;	mathematical expectation

2. Mathematical Model of System Dynamics

Consider the following equation with a subdifferential

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* Department of Mechanical Engineering

$$u(t,x) - \int_0^t a\frac{\partial^2 u}{\partial x^2}\,ds + \int_0^t u\,dw + \int_0^t \partial\beta(u)ds$$
$$\ni u_0(x) + \int_0^t f ds,\ \ t\in T,\ x\in G \quad (2.1)$$

with the boundary conditions

$$u(t,0) = u(t,1) = 0 \quad (2.2)$$

where a is a positive constant, $w(t)$ is a standard Brownian motion process and $\partial\beta(u)$ denotes the subdifferential at u of a lowersemicontinuous convex function $\beta(u)$ which may takes the value $+\infty$, but not identically equal to $+\infty$. For example, in the obstacle problem[1),2)], the function β is given by

$$\beta(\lambda) = \begin{cases} 0 & , \ \lambda \leq \psi_0 \\ +\infty, & \lambda > \psi_0 \end{cases} \quad (2.3)$$

where ψ_o denotes the obstacle. From (2.3), it is easily found that β is only defined when $\lambda \leqq \psi_0$.

Let's consider the precise formulation of (2.1) with (2.2). Let V and H be two Hilbert spaces such that

$$V = H_0^1(G) \subset H = L^2(G) \quad (2.4)$$

Identifying H with its dual, we have

$$V \subset H \subset V' \ (= H^{-1}(G)) \quad (2.5)$$

Then, the subdifferential $\partial\beta$ of β precisely defined by

$$\partial\beta(u) = [\,\chi \in V' \mid \beta(v) - \beta(u) \geq <\chi, v-u>,\ {}^{\forall}v \in V\,] \quad (2.6)$$

In this paper, we assume that the function β has the following properties.

Properties

(P-1) the function $v \to \beta(v)$ is lowersemicontinuous convex function from $V \to]-\infty, \infty]$, but $\beta(v) \not\equiv \infty$.

(P-2) $0 \in \partial\beta(0)$

(P-3) $\partial\beta$ takes either a nonnegative or a nonpositive value

(P-4) there exists $h \in V'$, $r \in R^1$ such that $\beta(v) \geq <h, v> + r,\ {}^{\forall}v \in V$

(P-5) there exists $v_0 \in V$ such that $\beta(v_0) \leq C$ (Const.)

From (2.6), (2.1) can be rewritten by the following stochastic variational inequality

$$(u(t),\varphi) + \int_0^t <Au,\ \varphi> ds + \int_0^t (u,\varphi)dw$$
$$+ \int_0^t <\chi(u), \varphi> ds = (u_0,\varphi) + \int_0^t (f,\varphi)ds$$
$$\text{for } {}^{\forall}\varphi \in V \quad (2.7)$$

and for ${}^{\forall}v \in V$

$$\beta(v) - \beta(u) \geq <\chi(u),\ v-u> \quad (2.8)$$

where for ${}^{\forall}\varphi,\ \psi \in V$

$$<A\varphi,\ \psi> = a\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\psi}{\partial x}\right).$$

It is difficult to prove the existence of the solution to (2.7) and (2.8) because $\chi(u) \in L^2(\Omega \times T;\ V')$ can not be proved. Then, in order to formulate (2.7) by a weak form, we will show $\chi(u)dsdx \in L^2(\Omega; M(\bar{T} \times \bar{G}))$. We can now define precisely the solution of the stochastic variational inequality.

Definition: If the pair (u, η) satisfies the followings, (u, η) is the solution of (2.7) and (2.8);

$$\text{(i)} \begin{cases} u \in L^2(\Omega; L^2(T;V) \cap C(\bar{T}\times\bar{G})) \\ \eta \in L^2(\Omega; M(\bar{T}\times\bar{G})) \end{cases} \quad (2.9)$$

$$\text{(ii)}\quad (u(t),\varphi) + \int_0^t <Au,\varphi> ds + \int_0^t (u,\varphi)dw$$
$$+ \int_0^t\int_G \varphi\eta(ds,dx) = (u_0,\varphi) + \int_0^t (f,\varphi)ds$$
$$\text{for } {}^{\forall}\varphi \in V \quad (2.10)$$

$$\text{(iii)}\quad \int_0^{t_f} [\beta(v) - \beta(u)]\,dt \geq \int_0^{t_f}\int_G (v-u)\eta(dt,dx)$$
$$\text{for } {}^{\forall}v \in C(\bar{T}\times\bar{G}) \quad (2.11)$$

Remark: Since the region G is one-dimension, it follows from Sobolev's imbedding theorem[6)] that

$$V \subset C(\bar{G}). \tag{2.12}$$

From (2.12), (2.10) and (2.11) makes a sense.

3. Existence and Uniqueness Theorem

First, we will begin with the approximation of (2.10) and (2.11). Define for $^\forall\epsilon > 0$,

$$\beta_\epsilon(u) = \inf_{v} \left[\frac{1}{2\epsilon} \|u - v\|^2 + \beta(v),\ v \in V \right] \tag{3.1}$$

Then, from 7), we have

$$\beta_\epsilon(u) = \frac{1}{2\epsilon} \|u - J_\epsilon u\|^2 + \beta(J_\epsilon u), \tag{3.2}$$

$$\partial\beta_\epsilon(u) = \frac{1}{\epsilon} F(u - J_\epsilon u) \tag{3.3}$$

and

$$\beta(J_\epsilon u) \leq \beta_\epsilon(u) \leq \beta(u), \quad ^\forall u \in V \tag{3.4}$$

where F is a duality mapping of V defined by

$$Fu = [u^* \in V' \mid \langle u^*, u\rangle = \|u\|^2 = \|u^*\|^2_{V'}] \tag{3.5}$$

and

$$J_\epsilon u = (F + \epsilon\partial\beta)^{-1} Fu \tag{3.6}$$

With the function $\partial\beta_\epsilon$, the stochastic variational inequality (2.10) and (2.11) are approximated by

$$(u_\epsilon(t), \varphi) + \int_0^t \langle Au_\epsilon, \varphi\rangle ds + \int_0^t (u_\epsilon, \varphi) dw + \int_0^t \langle \partial\beta_\epsilon(u_\epsilon), \varphi\rangle ds = (u_0, \varphi) + \int_0^t (f, \varphi) ds \quad \text{for } ^\forall\varphi \in V \tag{3.7}$$

Lemma 3.1; With the following conditions

(C-1) $u_0 \in L^2(\Omega; H)$,

(C-2) $f \in L^2(\Omega; V')$,

there exists a unique solution u_ϵ to (3.7) such that

$$u_\epsilon \in L^2(\Omega \times T; V) \tag{3.8}$$

and

$$\frac{1}{\epsilon} E\left[\int_0^{t_f} \|u_\epsilon - J_\epsilon u_\epsilon\|^2 ds\right] \leq \text{Const.} \tag{3.9}$$

In order to prove the existence of the solution to the stochastic variational inequality (2.10) and (2.11), first we must prove the regularity property of the solution to the approximate equation (3.7). We assume here the differentiability of $\partial\beta_\epsilon$, because this assumption is justified by using the difference quotient in the case where $\partial\beta_\epsilon$ is not differentiable.

Lemma 3.2; With the conditions

(C-3) $u_0 \in L^2(\Omega; V)$,

(C-4) $f \in L^2(T; H)$,

there exists a solution to (3.7) such that

$$u_\epsilon \in L^2(\Omega \times T; H^2(G)). \tag{3.10}$$

Lemma 3.3; With (C-4) and

(C-5) $u_0 \in L^4(\Omega; V)$,

the following estimates holds

$$E\left[\sup_t \left|\frac{\partial u_\epsilon}{\partial x}\right|^4 + \left[\int_0^{t_f} |\Delta u_\epsilon|^2 dt\right]^2\right] \leq \text{Const.} \tag{3.11}$$

and

$$E\left[\left[\int_0^{t_f} |\partial\beta_\epsilon| dt\right]^2\right] \leq \text{Const.}. \tag{3.12}$$

Lemma 3.4; With the same conditions as in Lemma 3.2, $\{u_\epsilon\}$ is Cauchy in $L^2(\Omega; C(\bar{T} \times \bar{G}))$.

Theorem 3.1 With the conditions (C-1) to (C-5) there exists a unique solution u to the stochastic variational inequality (2.10) and (2.11).

Proof: Define

$$\eta_\epsilon(dt, dx) = \partial\beta_\epsilon(u_\epsilon) dt dx. \tag{3.13}$$

From Lemmas 3.1 and 3.3, because u_ϵ and η_ϵ are bounded in $L^2(\Omega \times T; V)$ and $L^2(\Omega; M(\bar{T} \times \bar{G}))$, we can extract subsequences $u_{\epsilon'}$ and $\eta_{\epsilon'}$ such that

$$u_{\epsilon'} \to u \quad \text{weakly in } L^2(\Omega \times T; V) \tag{3.14}$$

$$\eta_{\epsilon'} \to \eta \quad \text{weakly star in} \quad L^2(\Omega; M(\bar{T} \times \bar{G})) \tag{3.15}$$

From (3.14) and (3.15), taking a limit in (3.7), we have for $^\forall \varphi \in V$

$$(u(t), \varphi) + \int_0^t <Au, \varphi> ds + \int_0^t (u, \varphi) dw + \int_0^t \int_G \varphi d\eta = (u_0, \varphi) + \int_0^t (f, \varphi) ds \tag{3.16}$$

From the definition of $\partial\beta_\epsilon$, we have

$$\int_0^{t_f} [\beta_\epsilon(v) - \beta_\epsilon(u_\epsilon)]\, dt \geq \int_0^{t_f} < \partial\beta_\epsilon(u_\epsilon),\ v - u_\epsilon > dt \quad \text{for } ^\forall v \in V \tag{3.17}$$

Noting that

$$\beta(J_\epsilon u_\epsilon) \leq \beta_\epsilon(u_\epsilon) \leq \beta(u_\epsilon)$$

and from Lemma 3.4

$$u_\epsilon \to u \quad \text{strongly in} \quad L^2(\Omega; C(\bar{T} \times \bar{G}))$$

and β is lower-semicontinuous, (3.17) implies

$$\int_0^{t_f} [\beta(v) - \beta(u)]\, dt \geq \int_0^{t_f} \int_G v d\eta, \quad \text{for } ^\forall v \in C(\bar{T} \times \bar{G}).$$

The uniqueness property of the solution follows from the monotone property of $\partial\beta$.

The proof has thus been completed.

Conclusion

By introducing the random measure, it has been shown that the stochastic parabolic system with the subdifferential was formulated by the stochastic variational inequality. Furthermore, the existence and uniqueness properties of the solution to the stochastic variational inequality was proved with the help of regularity and compactness theorems. The main part of the proof relies on the Sobolev imbedding theorem of one dimensional spatial variable, so the method proposed in this paper requires the space dimension one. By guaranteeing the regularity properties of the higher order, this restriction will be removed.

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Appendices

Appendix A (Proof of Lemma 3.1): Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis of H which is made up of elements of V. Let v_0 be the function satisfies (P-4). We choose e_1 and e_2 in such a way that u_0 and v_0 belong to the the space $[e_1, e_2]$ spanned by e_1 and e_2. Then, consider the following finite dimensional stochastic equation

$$(u_\epsilon^n(t), e_i) + \int_0^t <Au_\epsilon^n, e_i> ds + \int_0^t (u_\epsilon^n, e_i) dw + \int_0^t < \partial\beta_\epsilon(u_\epsilon^n), e_i > ds = (u_0, e_i) + \int_0^t (f, e_i) ds, \quad \text{for } 1 \leq i \leq n \tag{A.1}$$

Applying the Ito lemma to $|u_\epsilon^n - v_0|^2$, we have

$$|u_\epsilon^n(t) - v_0|^2 + 2\int_0^t <Au_\epsilon^n, u_\epsilon^n - v_0> ds$$
$$+ 2\int_0^t <\partial\beta_\epsilon(u_\epsilon^n), u_\epsilon^n - v_0> ds$$
$$+ 2\int_0^t (u_\epsilon^n, u_\epsilon^n - v_0) dw$$
$$= |u_0 - v_0|^2 + 2\int_0^t (f, u_\epsilon^n - v_0) ds$$
$$+ \int_0^t |u_\epsilon^n|^2 ds \qquad (A.2)$$

It should be noted that

$$<\partial\beta_\epsilon(u_\epsilon^n), u_\epsilon^n - v_0> \geq \beta_\epsilon(u_\epsilon^n) - \beta_\epsilon(v_0)$$

(from (3.2) and (3.3))

$$\geq \frac{1}{2\epsilon} \| u_\epsilon^n - J_\epsilon u_\epsilon^n \|^2 + \beta(J_\epsilon u_\epsilon^n) - \beta(v_0)$$

(from (P.4) and (P.5))

$$\geq \frac{1}{2\epsilon} \| u_\epsilon^n - J_\epsilon u_\epsilon^n \|^2 + <h, J_\epsilon u_\epsilon^n> + r - C$$
$$\geq \frac{1}{4\epsilon} \| u_\epsilon^n - J_\epsilon u_\epsilon^n \|^2 - \epsilon \| h \|_{V'}^2$$
$$- \frac{2}{\delta_1} \| h \|_{V'}^2 - \frac{\delta_1}{2} \| u_\epsilon^n \|^2 + r - C, \qquad (A.3)$$

$$2|(f, u_\epsilon^n - v_0)| \leq \frac{1}{\delta_2} \| f \|_{V'}^2 + 2\delta_2 \| u_\epsilon^n \|^2 + 2\delta_2 \| v_0 \|^2 \qquad (A.4)$$

and

$$2|<Au_\epsilon^n, v_0>| \leq \delta_3 \| u_\epsilon^n \|^2 + \frac{1}{\delta_3} \| v_0 \|^2 \qquad (A.5)$$

Using (A.3), (A.4) and (A.5) in (A.2) and taking a mathematical expectation, we have

$$E[|u_\epsilon^n(t) - v_0|^2] + (a - \delta) E[\int_0^t \| u_\epsilon^n \|^2 ds]$$
$$+ \frac{1}{2\epsilon} E[\int_0^t \| u_\epsilon^n - J_\epsilon u_\epsilon^n \|^2 ds]$$
$$\leq E[|u_0 - v_0|^2] + 2E[\int_0^t |u_\epsilon^n - v_0|^2 ds]$$
$$+ C_1. \qquad (A.6)$$

where $\delta = \delta_1 + \delta_2 + \delta_3$ and C_1 is a constant depends on f, v_0, h and t but independent of n and ϵ.

In (A.6), choosing δ as $a - \delta \geqq \alpha > 0$ (α is constant) and using Gronwall's inequality, we have

$$E[\int_0^{t_f} \| u_\epsilon^n \|^2 ds] \leq C_2 \qquad (A.7)$$

and

$$\frac{1}{\epsilon} E[\int_0^{t_f} \| u_\epsilon^n - J_\epsilon u_\epsilon^n \|^2 ds] \leq C_3 \qquad (A.8)$$

where C_2 and C_3 are respectively constants independent of n and ϵ. Therefore, we can extract a subsequence $u_\epsilon^{n'}$ of u_ϵ^n (for fixed ϵ) such that

$$u_\epsilon^{n'} \to u_\epsilon \quad \text{weakly in } L^2(\Omega \times T; V). \qquad (A.9)$$

Because $\partial\beta_\epsilon$ is monotone, (3.9) follows from (A.9).

Appendix B (Proof of Lemma 3.2): We introduce the special base $\{e_i\}_{i=1}^\infty$ such that

$$(-\Delta) e_i = \lambda_i e_i, \quad e_i \in V \qquad (B.1)$$

Substituting this special base into (A.1) and multiplying by λ_i and using the relation (B.1), we have

$$(u_\epsilon^n(t), (-\Delta)e_i) + \int_0^t (Au_\epsilon^n, (-\Delta)e_i) ds$$
$$+ \int_0^t (u_\epsilon^n, (-\Delta)e_i) dw + \int_0^t <\partial\beta_\epsilon, (-\Delta)e_i> ds$$
$$= (u_0, (-\Delta)e_i) + \int_0^t (f, (-\Delta)e_i) ds \qquad (B.2)$$

Because $u_\epsilon^n = \sum_{i=1}^n (u_\epsilon^n, e_i) e_i$ and $e_i \in H^2(G) \cap V$, the second term of the L.H.S. of (B.2) makes a sense. From (A.1) and (B.2) (where taking a special base in (A.1)), we have

$$\left|\frac{\partial u_\epsilon^n}{\partial x}\right|^2 + 2a\int_0^t |\Delta u_\epsilon^n|^2 ds + 2\int_0^t \left|\frac{\partial u_\epsilon^n}{\partial x}\right|^2 dw$$
$$+ 2\int_0^t <\partial\beta_\epsilon(u_\epsilon^n), (-\Delta)u_\epsilon^n> ds = \left|\frac{\partial u_0}{\partial x}\right|^2$$
$$+ \int_0^t \left|\frac{\partial u_\epsilon^n}{\partial x}\right|^2 ds + 2\int_0^t (f, (-\Delta)u_\epsilon^n) ds. \qquad (B.3)$$

From the monotonicity of $\partial\beta_\epsilon$, we have

$$<\partial\beta_\epsilon(\varphi(x+h)) - \partial\beta_\epsilon(\varphi(x)), \varphi(x+h) - \varphi(x)> \geq 0 \qquad (B.4)$$

Because $\partial\beta_\epsilon$ is assumed to be differentiable, from (B.4), we obtain

$$< \frac{\partial}{\partial x}(\partial\beta_\epsilon(\varphi)), \frac{\partial\varphi}{\partial x} > \geq 0. \quad \text{(B.5)}$$

From (B.1) and (B.5), we have

$$|\frac{\partial u_\epsilon^n}{\partial x}|^2 + (2a-\delta)\int_0^t |\Delta u_\epsilon^n|^2 ds + 2\int_0^t |\frac{\partial u_\epsilon^n}{\partial x}|^2 dw \leq |\frac{\partial u_0}{\partial x}|^2 + \int_0^t |\frac{\partial u_\epsilon^n}{\partial x}|^2 ds + \frac{1}{\delta}\int_0^t |f|^2 ds. \quad \text{(B.6)}$$

Taking a mathematical expectation to the both sides of (B.6) and using the Gronwall inequality we have

$$E\left[\int_0^{t_f} |\Delta u_\epsilon^n|^2 ds\right] \leq \text{Const.} \quad \text{(B.7)}$$

In (B.7), by extracting a subsequence with respect to n, we have (3.10).

Appendix C (Proof of Lemma 3.3): It follows from (B.7) that

$$E\left[\sup_{s\leq t} |\frac{\partial u_\epsilon^n(s)}{\partial x}|^2\right] + (2a-\delta)^2 E\left[\int_0^t |\Delta u_\epsilon^n|^2 ds\right] \leq 3\left[E\left[|\frac{\partial u_0}{\partial x}|^4\right] + CE\left[\int_0^t |\frac{\partial u_\epsilon^n}{\partial x}|^4 ds\right] + \frac{1}{\delta^2}\left[\int_0^t |f|^2 ds\right]^2\right] \quad \text{(C.1)}$$

Using the Gronwall inequality, we obtain (3.11). From Lemmas 3.1 and 3.2, we can extract a subsequence $u_\epsilon^{n'}$ of u_ϵ^n such that

$$u_\epsilon^{n'} \to u_\epsilon \quad \text{weakly in } L^4(\Omega; L^2(T; H^2(G))) \quad \text{(C.2)}$$

$$u_\epsilon^{n'} \to u_\epsilon \quad \text{weakly star in } L^4(\Omega; L^\infty(T; V)) \quad \text{(C.3)}$$

and since $\partial\beta_\epsilon$ is monotone from V to V',

$$\partial\beta_\epsilon(u_\epsilon^{n'}) \to \partial\beta_\epsilon(u_\epsilon) \quad \text{weakly in } L^2(\Omega\times T; V') \quad \text{(C.4)}$$

Furthermore, from Lemma 3.2,

$$\partial\beta_\epsilon(u_\epsilon^{n'}) \to \partial\beta_\epsilon(u_\epsilon) \quad \text{weakly in } L^2(\Omega\times T; H) \quad \text{(C.5)}$$

Therefore, we have for any $\phi \in H$,

$$(u_\epsilon(t), \varphi) + \int_0^t (Au_\epsilon, \varphi)ds + \int_0^t (u_\epsilon, \varphi)dw = (u_0, \varphi) + \int_0^t (f, \varphi)ds - \int_0^t (\partial\beta_\epsilon(u_\epsilon), \varphi)ds \quad \text{(C.6)}$$

The Ito lemma implies that

$$|u_\epsilon(t)+k|^2 + 2\int_0^t (Au_\epsilon, u_\epsilon+k)ds + 2\int_0^t (u_\epsilon, u_\epsilon+k)dw + 2\int_0^t (\partial\beta_\epsilon, u_\epsilon+k)ds = |u_0+k|^2 + 2\int_0^t (f, u_\epsilon+k)ds + \int_0^t |u_\epsilon|^2 ds \quad \text{(C.7)}$$

where k is chosen as 1 (or -1) if $\partial\beta_\epsilon$ is nonnegative (or nonpositive). From (P-2) and the monotonicity of $\partial\beta_\epsilon$, we have

$$(\partial\beta_\epsilon(u_\epsilon), u_\epsilon)) \geq 0. \quad \text{(C.8)}$$

From (C.7) and (C.8), we conclude that

$$E\left[\left[\int_0^{t_f} |\partial\beta_\epsilon| ds\right]^2\right] \leq CE\left[\left[\int_0^{t_f} |Au_\epsilon|^2 ds\right]^2 + \int_0^{t_f} |u_\epsilon+k|^4 ds + \int_0^{t_f} |u_\epsilon|^4 ds + |u_0+k|^4\right] + C\int_0^{t_f} |f|^4 ds \leq \text{Const.} \quad \text{(C.9)}$$