On Construction of Completely Integrable Many-Body Systems in Two Dimensions

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ABSTRACT

Of dynamical systems, Calogero system and Toda lattice are known as examples of completely integrable many-body systems in one dimension. In this short note, it will be shown that, by making use of complexification, completely integrable many-body systems in two dimensions are gotten as extension of those in one dimension.

Key Words: Dynamical Systems, Many-Body System, Completely Integrable System.

1. INTRODUCTION.

When we investigate the structure of some dynamical system, we try to solve the equations of the motion, namely it is aimed to get the solutions of them explicitly. From this point of view, it is very important whether the dynamical system is exactly solvable, so-called completely integrable, or not. So much researches concerning complete integrability of dynamical systems have been made, see for example [1] – [6]. As we consider dynamical system with N (finite positive integer) degrees of freedom, it is said that the system is completely integrable in Liouville's sense when there exist N first integrals which are functionally independent and are involutive.

In this short note, the configuration space is restricted to a line (one dimension) or a plane (two dimensions). Calogero system is shown to be completely integrable classical system in one dimension as well as quantum case [7]. Concerning Calogero system, the author has shown that its proper quadratic first integral can be derived from those of Ermakov system [8]. In [9], F. Calogero has shown that some completely integrable system in one dimension can be extended as ones in two dimensions by making use of complexification. We will show that the other systems in one dimension can be extended as completely integrable many-body system in two dimensions.

In section 2, completely integrable systems in one dimension, which contain Calogero system and Toda lattice as some of them, are reviewed for later use as preliminaries. The results are stated in section 3.

2. PRELIMINARIES.

In this section, following M.A. Olshanetsky and A.M. Perelomov [1], completely integrable many-body system in one dimension are listed for later use.

Consider the classical nonrelativistic problem of N particles of unit mass on the line and Hamiltonian of our system is given with potential function $U(q^1, q^2, \cdots, q^N)$ as follows,

$$H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + U(q^1, \cdots, q^N), \tag{1}$$

where p_j and q^j $(j = 1, \cdots N)$ are momentum and coordinate of the j-th particle respectively. Then the potential function U is assumed to be following type,

$$U(q^1, \cdots, q^N) = \frac{g^2}{2} \sum_{j>k}^{N} v(q^j - q^k), \qquad g;\ \text{const}, \quad g \in R \tag{2}$$

where $v(q^j - q^k)$ means the interactive pairwise potential between j-th particle and k-th one.

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Next we consider the necessary and sufficient condition for our system to be completely integrable. Following [1], we can get the next six typical potentials for our system to be completely integrable,

$$v(\beta) = \begin{cases} \beta^{-2} & \text{I}, \\ a^2 \sinh^{-2}(a\beta) & \text{II}, \\ a^2 \sin^2(a\beta) & \text{III}, \\ a^2 \widetilde{P}(a\beta) & \text{IV}, \\ \beta^{-2} + \dfrac{\omega^2}{2g^2}\beta^2 & \text{V}, \\ \exp(2\beta) & \text{VI}, \end{cases} \tag{3}$$

where a, ω are constants and $\widetilde{P}$ stands for Weierstrass function. It is to be remarked that for type VI we call it nonperiodic Toda lattice and potential function is

$$U(q^1, \cdots, q^N) = \sum_{j=1}^{N-1} g_j^2\, v(q^j - q^{j+1}),\ g_j \in R, \tag{4}$$

where

$$v(\beta) = \exp(2\beta). \tag{5}$$

3. MAIN RESULTS.

In this last section, we will derive two-dimensional completely integrable many body system from one-dimensional them reviewed in section 2.

At first, let us show the extension of type V in (3) developed in [1]. Consider the classical nonrelativistic dynamical system of N particles of unit mass in the two-dimensional xy-plane and interacting pairwise via the force

$$F(r) = f(r) - \omega^2 r, \tag{6}$$

i.e.

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} - \omega^2 \begin{pmatrix} x \\ y \end{pmatrix}, \tag{7}$$

with

$$\left.\begin{aligned} f_x &= g^2 r^{-6} x(x^2 - 3y^2) = g^2 r^{-3}\cos(3\phi), \\ f_y &= g^2 r^{-6} y(y^2 - 3x^2) = -g^2 r^{-3}\sin(3\phi) \end{aligned}\right\} \tag{8}$$

Here f_x and f_y are x and y components of the force and x and y (respectively r and ϕ) are the Cartesian (respectively polar) coordinates of the interparticle vector. Note that the modulus of $f(r)$ depends only on r, but that this force is not central. The force $f(r)$ is moreover nonconservative, there are nevertheless a lot of conserved quantities.

The equations of motion of this many-body system leads

$$\ddot{r}_j = \sum_{k \neq j}^{N} [f(r_{jk}) - \omega^2 r_{jk}], \tag{9}$$

where r_j indicates the positional vector of the j-th particle and

$$r_{jk} = r_j - r_k. \tag{10}$$

They can be solved noticing that after the introduction of new coordinates

$$z_j = x_j + iy_j, \quad i = \sqrt{-1}, \tag{11}$$

we get

$$\ddot{z}_j = \sum_{k \neq j}^{N} (g^2 z_{jk}^{-3} - \omega^2 z_{jk}), \tag{12}$$

i.e. the equations of motion for this system coincide with that of one-dimensional dynamical system of type V in (3).

Secondly we concentrate on the type II and type III, whose system are generalization of type I. Because if the parameter a intends to zero in type II and III, they correspond with type I. The two-dimensional extension of the above two systems have following forces for (6),

$$F_x = \frac{2g^2 a^3 \sinh(2ax)\cos(2ay)}{\{\cosh(2ax) - \cos(2ay)\}^2} \quad \text{for type II}, \tag{13}$$

$$F_y = \frac{2g^2 a^3 \cosh(2ax)\sin(2ay)}{\{\cosh(2ax) - \cos(2ay)\}^2}$$

for type II , (14)

$$F_x = \frac{2g^2 a^3 \sin(2ax)\cosh(2ay)}{\{\cosh(2ay) - \cos(2ax)\}^2}$$

for type III , (15)

$$F_y = \frac{2g^2 a^3 \cos(2ax)\sinh(2ay)}{\{\cosh(2ay) - \cos(2ax)\}^2}$$

for type III . (16)

Then the same method of complexification for type V leads the equations of motion as follows,

$$\ddot{z}_j = g^2 a^3 \sum_{k\neq j}^{N} \sinh^{-3}(a(z_j - z_k))\cosh(a(z_j - z_k)) \quad \text{for type II} \qquad (17)$$

$$\ddot{z}_j = g^2 a^3 \sum_{k\neq j}^{N} \sin^{-3}(a(z_j - z_k))\cos(a(z_j - z_k)) \quad \text{for type III} \qquad (18)$$

These equations are easily seen to be same as ones in one-dimensional type II and III systems. So these two-dimensional systems defined above are completely integrable many-body system. But physical meaning of these systems is not clear. This point should be discussed in the future.

REFERENCES

[1] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. **71**, 313 (1981).
[2] M.A. Olshanetsky and A.M. Perelomov, Invent. Math. **37**, 93 (1976).
[3] A.M. Perelomov, Lett. Math. Phys. **1**, 531 (1977).
[4] F. Calogero, O. Ragnisco and C. Marchioro, Lett. Nuovo Cimento, **13**, 383 (1975).
[5] F. Calogero, J. Math. Phys. **12**, 419 (1971).
[6] J. Moser, Adv. Math. **16**, 197 (1975).
[7] C. Marchioro, J. Math. Phys. **11**, 2193 (1970).
[8] N. Katayama, Bul. Osaka Pref. Col. Tech. **23**, 9 (1989).
[9] F. Calogero, Lett. Nuovo Cimento, **16**, 35 (1976).