On First Integrals of Ermakov System

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ABSTRACT

From the viewpoint of the symmetry problem of dynamical systems, it is well-known that first integrals play an important role. It is shown that generalized Ermakov system is related to Calogero system through two transformations, one is canonical transformation and the other is Perelomov transformation. Then a proper quadratic first integral of Calogero system is gotten by making use of the known first integral, so-called Ermakov-Lewis invariant, of Ermakov system.

Key Words: Dynamical Systems, First Integrals, Ermakov System, Calogero System, Completely Integrable System, Canonical Transformation.

1. INTRODUCTION.

From the viewpoint of symmetry problem of dynamical systems, it well-known that first integrals play an important role [1]. Therefore first integrals have been investigated systematically within a framework of simple dynamical systems, see for example [2], [3]. In this paper, we treat Ermakov system which means time-dependent harmonic oscillator. Our system have been analyzed by several methods [4],[5]. And first integrals of Ermakov system is shown with auxiliary equation and it is called Ermakov-Lewis invariant. At first, following I.A.Pedrosa [6], generalized Ermakov system is shown to be derived from original Ermakov system by canonical transformation. Then by considering its auxiliary equation as one of equations of motion, we can get equations of motion with two degrees of freedom. Secondly the above dynamical system is transformed to the system with harmonic type potential and Calogero type potential by the other canonical transformation. Finally we can exclude harmonic type potential by Perelomov transformation and it is shown that first integral corresponding to the Ermakov-Lewis invariant is a proper quadratic first integral of Calogero system. By 'proper' quadratic first integrals, we mean first integrals in quadratic form with respect to generalized momentum, which cannot be expressed with linear first integrals except for total energy. For example, in case of Kepler motion, Laplace-Runge-Lenz vector is a proper one and in case of harmonic oscillator, each component of Fradkin's tensor is a proper one [1]. The existence of proper quadratic first integrals has been shown to be closely connected to the periodicity of the behavior of the dynamical system with central potentials, whose configuration space is a constant curvature space, by the author [7].

The contents of the paper are as follows. In section 2, generalized Ermakov system and Ermakov one are reviewed. In section 3, it is shown that Calogero system with harmonic type potential is gotten from generalized Ermakov system. In section 4, the results are stated and further discussions is given.

2. PRELIMINARIES.

In this section, following reference [6], it is aimed to give the definition of generalized Ermakov system and to show the relation between above system and original one.

Ermakov system is given as follows. For time-dependent Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2(t)}{2} q^2, \tag{1}$$

where p and q $(p, q \in R)$ are canonically

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conjugate, namely, q is a generalized coordinate and p is a generalized momentum, $\omega(t)$ is time-dependent harmonic oscillator frequency and m is a constant. Then $q(t)$ satisfies the following equation.

$$\ddot{q}_{(t)} + \omega^2(t)\, q(t) = 0, \tag{2}$$

which is obtained from (1) via Hamilton's canonical equations, where overdots stand for differentiation with respect to time t. And this system admits so-called Ermakov-Lewis invariant, (abbr. EL invariant)

$$I = \frac{1}{2}\left\{(q\dot{\alpha} - \alpha\dot{q})^2 + (\frac{q}{\alpha})^2\right\}, \tag{3}$$

where $\alpha(t)$ is any solution of the next auxiliary equation.

$$\ddot{\alpha}(t) + \omega^2(t)\alpha(t) = \frac{1}{\alpha^3(t)}. \tag{4}$$

Here we can call the pair of equations (2) and (4) Ermakov system.

As a second step of this section, we can get the generalized Ermakov system by defining modified Hamiltonian corresponding to (1).

Definition 1.

Generalized Ermakov system consists of the pair of the equations

$$\ddot{q}(t) + \gamma(t)\dot{q}(t) + \omega^2(t)q(t) = 0, \tag{5}$$

$$\ddot{\alpha}(t) + \gamma(t)\dot{\alpha}(t) + \omega^2(t)\alpha(t) = \frac{f^2(t)}{\alpha^3(t)}, \tag{6}$$

where $\gamma(t) = -\dot{f}/f$ and $f(t)$ is an arbitrary real function of time t.

Proposition 2.

Generalized Ermakov system is obtained from modified Hamiltonian

$$H = \frac{p^2}{2m} f^2(t) + f^{-1}(t)\left(\frac{m\omega^2(t)}{2}\right)q^2, \tag{7}$$

via Hamilton's canonical equations and it admits first integral

$$I_m = \frac{1}{2}\,[f^{-2}(t)\,(q\dot{\alpha} - \alpha\dot{q})^2 + (\frac{q}{\alpha})^2\,], \tag{8}$$

which corresponds to EL invariant (3).

It is to be remarked that it is easily shown that if $f(t)$ is put to equal to 1 then generalized Ermakov system coincides with original system.

Proof of proposition 2.

From (7), Hamilton's canonical equations are given as follows,

$$\dot{q}(t) = \frac{\partial H}{\partial p} = \frac{p(t)}{m} f(t), \tag{9}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q} = -f^{-1}(t)\, m\omega^2(t)q(t). \tag{10}$$

By combining (9) and (10) we can get equation (5) easily. Next we consider time-dependent canonical transformation with the generating function

$$F(q,P,t) = qPf^{-\frac{1}{2}}(t) - [m\gamma(t)]\frac{q^2 \cdot f^{-1}(t)}{4}. \tag{11}$$

Then from transformation equations $Q = \partial F/\partial P$ and $p = \partial F/\partial q$, the new canonical variables are given as follows,

$$Q = q \cdot f^{-\frac{1}{2}}(t), \tag{12}$$

$$P = pf^{\frac{1}{2}}(t) + \frac{m\gamma(t)}{2} q \cdot f^{-\frac{1}{2}}(t). \tag{13}$$

And the corresponding Hamiltonian $H_1 = H + \partial F/\partial t$ is expressed simply form with new canonical variables P, Q by straightforward calculation.

$$H_1 = \frac{1}{2m}P^2 + \frac{m}{2}\,\Omega^2(t)Q^2, \tag{14}$$

where

$$\Omega^2(t) = \omega^2(t) - \frac{\gamma^2(t)}{4} - \frac{\dot{\gamma}(t)}{2}. \tag{15}$$

Above derived system is quite same as original Ermakov system. Concerning new canonical variable Q, the pair of equations corresponding to (2) and (4) is gotten as follows,

$$\ddot{Q}(t) + \Omega^2(t) Q(t) = 0\,, \tag{16}$$

$$\ddot{\rho}(t) + \Omega^2(t)\rho(t) = \frac{1}{\rho(t)^3}\,, \tag{17}$$

where (17) is auxiliary equation. By defining $\alpha(t)$ as

$$\alpha(t) = f^{\frac{1}{2}}(t)\,\rho(t)\,, \tag{18}$$

then it is easily seen that (17) is transformed to (6). And EL invariant is expressed with $q(t)$, $\alpha(t)$ as

$$\begin{aligned} I_m &= \frac{1}{2}\,[(Q\dot{\rho} - \rho\dot{Q})^2 + (\frac{Q}{\rho})^2\,] \\ &= \frac{1}{2}\,[\,f^{-2}(t)(q\dot{\alpha} - \alpha\dot{q})^2 + (\frac{q}{\alpha})^2\,]\,. \end{aligned} \tag{19}$$

This completes the proof.

It is to be remarked that further generalization of Ermakov system is discussed in [6].

3. ERMAKOV SYSTEM AND CALOGERO SYSTEM.

In this section it is aimed to show to be transformed from generalized Ermakov system to Calogero system. From discussions in section 2, generalized Ermakov system can be transformed to the pair of equations (16) and (17). Here we regard (17) as one of equations of motion, not auxiliary equation. Namely from the viewpoint of dynamical system, $\rho(t)$ is seen to be a generalized coordinate. So we consider (16) and (17) as the equations of motion of the system with two degrees of freedom.

On the other hand, (16) and (17) are given as the equations of motion of the following dynamical system whose Hamiltonian H and symplectic 2-form $\underline{\omega}$ are given as follows,

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\Omega^2}{2}(q^1)^2 + \frac{m\Omega^2}{2}(q^2)^2 + \frac{m}{2(q^2)^2}\,, \tag{20}$$

$$\underline{\omega} = dp_1 \wedge dq^1 + dp_2 \wedge dq^2\,, \tag{21}$$

where

$$\left.\begin{aligned} q^1 &= Q, \qquad q^2 = \rho, \\ p_1 &= m\dot{Q}, \qquad p_2 = m\dot{\rho}\,. \end{aligned}\right\} \tag{22}$$

Configuration space M of our above dynamical system is expressed $M = R \times \dot{R}$, where $\dot{R}$ stands for the domain of R excluded the origin. Our system's equations of motion is expressed

$$-dH = X_H \,\lrcorner\, \underline{\omega}\,, \tag{23}$$

where X_H is Hamilton's vector fields

$$X_H = -m\Omega^2 q^1 \frac{\partial}{\partial p_1} - m\Omega^2 q^2 \frac{\partial}{\partial p_2} - \frac{m}{(q^2)^3}\frac{\partial}{\partial p_2} + \frac{p_1}{m}\frac{\partial}{\partial q^1} + \frac{p_2}{m}\frac{\partial}{\partial q^2}\,, \tag{24}$$

and $\lrcorner$ indicates the interior product. It is easily shown by straightforward calculation that the pair of equations (16) and (17) coincides with (23).

Next we show that the above derived system is shown to be transformed to Calogero system [8]. Therefore we define the next canonical transformation

$$\left.\begin{aligned} \tilde{p}_1 &= \frac{p_1 + p_2}{\sqrt{2}}\,, \\ \tilde{p}_2 &= \frac{p_1 - p_2}{\sqrt{2}}\,, \\ \tilde{q}^1 &= \frac{q^1 + q^2}{\sqrt{2}}\,, \\ \tilde{q}^2 &= \frac{q^1 - q^2}{\sqrt{2}}\,, \end{aligned}\right\} \tag{25}$$

we can easily show that $\tilde{\underline{\omega}} = d\tilde{p}_1 \wedge d\tilde{q}^1 + d\tilde{p}_2 \wedge d\tilde{q}^2 = dp_1 \wedge dq^1 + dp_2 \wedge dq^2 = \underline{\omega}$ is satisfied. Therefore the type of equations of motion is invariant. And by using of new variable $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{q}^1$, $\tilde{q}^2$, Hamiltonian is transformed as

$$H = \frac{\tilde{p}_1^2}{2m} + \frac{\tilde{p}_2^2}{2m} + \frac{m}{2}\Omega^2(t)\left((\tilde{q}^1)^2 + (\tilde{q}^2)^2\right) + \frac{m}{(\tilde{q}^1 - \tilde{q}^2)^2}. \tag{26}$$

This Hamiltonian is composed of harmonic type potential and inverse square potential. And EL invariant is given from (19) as follows,

$$I = \frac{1}{2}(\tilde{q}^1\dot{\tilde{q}}^2 - \tilde{q}^2\dot{\tilde{q}}^1)^2 + \frac{1}{2}\left(\frac{\tilde{q}^1 + \tilde{q}^2}{\tilde{q}^1 - \tilde{q}^2}\right)^2. \tag{27}$$

Therefore we can get the next Proposition 3.

Proposition 3.

Generalized Ermakov system can be transformed to the dynamical system whose Hamiltonian is (26) and symplectic 2-form $\underline{\tilde{\omega}} = d\tilde{p}_1 \wedge d\tilde{q}^1 + d\tilde{p}_2 \wedge d\tilde{q}^2$. And EL invariant is expressed as (27).

4. A PROPER QUADRATIC FIRST INTEGRAL OF CALOGERO SYSTEM.

In this section, we give the definition of Calogero system and show that the first integral corresponding to EL invariant is a proper quadratic first integral of Calogero system. At first we give Calogero system of two degrees of freedom.

Definition 4.

Calogero system of two degrees of freedom is defined as follows, whose Hamiltonian is

$$H = \frac{(y_1)^2}{2} + \frac{(y_2)^2}{2} + \frac{1}{(x^1 - x^2)^2}, \tag{28}$$

and symplectic 2-form $\underline{\omega_0}$ is

$$\underline{\omega_0} = dy_1 \wedge dx^1 + dy_2 \wedge dx^2, \tag{29}$$

where y_i $(i = 1, 2)$ is a generalized momentum and x^i $(i = 1, 2)$ is a generalized coordinate. Physical meaning is explained in [8]. Equations of motion are given from (28) and (29) as follows.

$$\begin{aligned} \ddot{x}^1 &= 2(x^1 - x^2)^{-3}, \\ \ddot{x}^2 &= 2(x^2 - x^1)^{-3}. \end{aligned} \tag{30}$$

Secondly we show that the derived system in section 3 is transformed to Calogero system defined above.

Proposition 5.

It is assumed that $\Omega(t)$ is equal to a positive constant Ω_0 and we put

$$\left.\begin{aligned} x^k(T) &= \frac{\tilde{q}^k}{C\cdot\cos(\Omega_0 t)} \quad (k = 1, 2), \\ T &= \tan(\Omega_0 t), \\ C &= \frac{1}{\sqrt{\Omega_0}} \end{aligned}\right\} \tag{31}$$

Then equations of motion in $\tilde{q}^k(t)$ $(k = 1, 2)$ are transformed to

$$\left.\begin{aligned} \left(\frac{d}{dT}\right)^2 x^1 &= 2(x^1 - x^2)^{-3}, \\ \left(\frac{d}{dT}\right)^2 x^2 &= 2(x^2 - x^1)^{-3}. \end{aligned}\right\} \tag{32}$$

And EL invariant (27) is expressed with respect to $x^k(T)$ as

$$I = \frac{1}{2}(x^1 x^{2\prime} - x^2 x^{1\prime})^2 + \frac{1}{2}\left(\frac{x^1 + x^2}{x^1 - x^2}\right)^2, \tag{33}$$

where $x^{k'}(T)$ stands for differential of $x^k(T)$ with respect to capital T.

Proof of Proposition 5

The transformation given in (31) is called Perelomov transformation and (32) and (33) are gotten by straightforward calculation.

By the above Proposition 5, we can get a proper quadratic first integral of Calogero system. In Calogero system, it is well-known that total linear momentum F_1, and total energy F_2, given as

$$F_1 = x^{1\prime} + x^{2\prime}, \tag{34}$$

$$F_2 = \frac{1}{2}((x^{1\prime})^2 + (x^{2\prime})^2) + \frac{1}{(x^1 - x^2)^2}, \tag{35}$$

are first integrals, which are involutive, namely their Poisson bracket $\{F_1, F_2\}$ banishes. This fact is important when we consider whether the dynamical system is completely integrable or not in a Liouville sense.

Next in general we consider linear first integrals of Calogero system.

Proposition 6.

Linear first integral of Calogero system is given only the next type

$$L = (x^{1\prime} + x^{2\prime})c_0 + c_1, \tag{36}$$

where c_0, c_1 are constants.

Proof of proposition 6

Linear first integral L is assumed to be

$$L = \xi_1(x)x^{1\prime} + \xi_2(x)x^{2\prime} + \eta(x), \tag{37}$$

where ξ_1, ξ_2, η are functions of only x^1, x^2. The deferential of L along the orbit of Calogero system,

$$\begin{aligned}\frac{\delta}{\delta T}L = &\frac{\partial \xi_1}{\partial x^1}(x^{1\prime})^2 + \frac{\partial \xi_2}{\partial x^2}(x^{2\prime})^2 \\ &+ \left(\frac{\partial \xi_1}{\partial x^2} + \frac{\partial \xi_2}{\partial x^1}\right)x^{1\prime}x^{2\prime} \\ &+ 2(x^1 - x^2)^{-3}(\xi_1 - \xi_2) \\ &+ \frac{\partial \eta}{\partial x^1}x^{1\prime} + \frac{\partial \eta}{\partial x^2}x^{2\prime}.\end{aligned} \tag{38}$$

Therefore, it is necessary and sufficient for L to be first integral that

$$\left.\begin{aligned}&\frac{\partial \xi_1}{\partial x^1} = 0, \qquad \frac{\partial \xi_2}{\partial x^2} = 0, \\ &\frac{\partial \xi_1}{\partial x^2} + \frac{\partial \xi_2}{\partial x^1} = 0, \quad \xi_1 - \xi_2 = 0, \\ &\frac{\partial \eta}{\partial x^1} = 0, \qquad \frac{\partial \eta}{\partial x^2} = 0,\end{aligned}\right\} \tag{39}$$

are satisfied. From these partial differential equations we can get easily

$$\xi_1 = \xi_2 = c_0, \quad \eta = c_1. \tag{40}$$

Namely the quadratic first integral given in (33) is a proper one and the results are stated in the next theorem.

Theorem 7.

Calogero system with two degrees of freedom given in definition 4 admits a proper quadratic first integral

$$I = \frac{1}{2}(x^1x^{2\prime} - x^2x^{1\prime})^2 + \frac{1}{2}\left(\frac{x^1 + x^2}{x^1 - x^2}\right)^2. \tag{41}$$

It is to be remarked that it is easily shown that F_1, F_2 and I are functionally independent.

Further discussions

We can get a proper quadratic first integral of Calogero system by making use of EL invariant of Ermakov system. In Proposition 5, it is assumed that $\Omega(t)$ is a constant. When this assumption is violated, what dynamical system is derived from Ermakov system by extended Perelomov transformation will be discussed in a forthcoming paper.

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