How many miles to βX? II - Approximations to βX versus cofinal types of sets of metrics

How many miles to βX? II —Approximations to βX versus cofinal types of sets of metrics

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Abstract

Kada, Tomoyasu and Yoshinobu proved that the Stone–Čech compactification of a locally compact separable metrizable space is approximated by the collection of $\mathfrak{d}$-many Smirnov compactifications, where $\mathfrak{d}$ is the dominating number. By refining the proof of this result, we will show that the collection of compatible metrics on a locally compact separable metrizable space has the same cofinal type, in the sense of Tukey relation, as the set of functions from ω to ω with respect to eventually dominating order.

1 Tukey relations between directed sets

We use standard terminology and refer the readers to [1] for undefined set-theoretic notions. For $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer not exceeding a, and $\lceil a \rceil$ denotes the smallest integer not below a. For $f, g \in \omega^\omega$, we say $f \leq^* g$ if for all but finitely many $n < \omega$ we have $f(n) \leq g(n)$. A subset of ω^ω is called a *dominating family* if it is cofinal in ω^ω with respect to $\leq^*$. The *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family. We let $\omega^{\uparrow\omega}$ denote the set of strictly increasing functions in ω^ω.

Let $(D, \leq)$ and $(E, \leq)$ directed partially ordered sets. A mapping φ from D to E is called a *Tukey mapping* if the image of an unbounded subset of D by φ is an unbounded subset of E, or equivalently, if the inverse image of a bounded subset of E is a bounded subset of D. We write $(D, \leq) \leq_T (E, \leq)$

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(and often say D is *Tukey below* E, or E is *cofinally finer than* D) if there is a Tukey mapping from D to E. We will write $D \leq_T E$ if referred order relations on D and E are clear from the context.

A mapping ψ from E to D is called a *convergent mapping* if the image of a cofinal subset of E by ψ is a cofinal subset of D. It is easily checked that $D \leq_T E$ if and only if there is a convergent mapping from E to D.

We write $D \equiv_T E$ (and often say D is *Tukey equivalent to* E, D is *cofinally similar to* E, or D and E have *the same cofinal type*) if both $D \leq_T E$ and $E \leq_T D$ hold. In particular, if there is a mapping from D to E which is both Tukey and convergent, then $D \equiv_T E$ holds.

It is easy to see that $(\omega^\omega, \leq^*) \equiv_T (\omega^{\uparrow\omega}, \leq^*)$ holds.

For a directed partially ordered set $(D, \leq)$, $\mathsf{add}((D, \leq))$ or $\mathsf{add}(D)$ denotes the smallest size of an unbounded subset of D, and $\mathsf{cof}((D, \leq))$ or $\mathsf{cof}(D)$ denotes the smallest size of a cofinal subset of D. It is easy to see that $D \leq_T E$ implies $\mathsf{add}(D) \geq \mathsf{add}(E)$ and $\mathsf{cof}(D) \leq \mathsf{cof}(E)$. Using this notation, the dominating number $\mathfrak{d}$ is described as $\mathfrak{d} = \mathsf{cof}((\omega^\omega, \leq^*)) = \mathsf{cof}((\omega^{\uparrow\omega}, \leq^*))$.

2 Compactifications of metrizable spaces

A *compactification* of a completely regular Hausdorff space X is a compact Hausdorff space which contains X as a dense subspace. For compactifications αX and γX of X, we write $\alpha X \leq \gamma X$ if there is a continuous surjection $f : \gamma X \to \alpha X$ such that $f \restriction X$ is the identity map on X. If such an f can be chosen to be a homeomorphism, we write $\alpha X \simeq \gamma X$. Let $\mathsf{Cpt}(X)$ denote the class of compactifications of X. When we identify $\simeq$-equivalent compactifications, we may regard $\mathsf{Cpt}(X)$ as a set, and the order structure $(\mathsf{Cpt}(X), \leq)$ is a complete upper semilattice whose largest element is the Stone–Čech compactification βX.

The *Smirnov compactification* of a metric space (X, d), denoted by $u_d X$, is the unique compactification characterized by the following property: A bounded continuous function f from X to $\mathbb{R}$ is continuously extended over $u_d X$ if and only if f is uniformly continuous with respect to the metric d.

The following theorem tells us that the Stone–Čech compactification of a metrizable space is approximated by the collection of all Smirnov compactifications. Let $\mathrm{M}(X)$ denote the set of all metrics on X which are compatible with the topology on X.

Theorem 2.1. [5, Theorem 2.11] *For a noncompact metrizable space X, we have $\beta X \simeq \sup\{u_d X : d \in \mathrm{M}(X)\}$ (the supremum is taken in the upper semilattice $(\mathsf{Cpt}(X), \leq)$).*

Now we define the following cardinal function.

Definition 2.2. [3, Definition 2.2] For a noncompact metrizable space X, let $\mathfrak{sa}(X) = \min\{|D| : D \subseteq \mathrm{M}(X) \text{ and } \beta X \simeq \sup\{u_d X : d \in D\}\}$.

For a topological space X, $X^{(1)}$ denotes the first Cantor–Bendixson derivative of X, that is, the subspace of X which consists of all nonisolated points of X. Note that $\mathfrak{sa}(X) = 1$ holds if and only if there is a metric $d \in \mathrm{M}(X)$ which makes (X, d) an Atsuji space (also called a UC-space), which is known to be equivalent to the compactness of $X^{(1)}$ [5, Corollary 3.5].

Kada, Tomoyasu and Yoshinobu [4] proved the following theorem.

Theorem 2.3. [4, Theorem 2.10] *For a locally compact separable metrizable space X such that $X^{(1)}$ is not compact, $\mathfrak{sa}(X) = \mathfrak{d}$ holds.*

For a compactification αX of X and a pair A, B of closed subsets of X, we write $A \parallel B \ (\alpha X)$ if $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$, and otherwise $A \not\parallel B \ (\alpha X)$. It is known that, for a normal space X, $\alpha X \simeq \beta X$ holds if and only if $A \parallel B \ (\alpha X)$ for any pair A, B of disjoint closed subsets of X [2, Theorem 6.5]. For Smirnov compactification $u_d X$ of (X, d), it is known that $A \parallel B \ (u_d X)$ if and only if $d(A, B) > 0$ [5, Theorem 2.5].

For $d_1, d_2 \in \mathrm{M}(X)$, we write $d_1 \preceq d_2$ if the identity function on X is uniformly continuous as a function from (X, d_2) to (X, d_1). The following equivalent conditions for $d_1 \preceq d_2$ are known.

Proposition 2.4. *For a metrizable space X and $d_1, d_2 \in \mathrm{M}(X)$, the following conditions are equivalent.*

1. $d_1 \preceq d_2$.
2. $u_{d_1} X \leq u_{d_2} X$.
3. *For closed subsets A, B of X, if $A \parallel B \ (u_{d_1} X)$ then $A \parallel B \ (u_{d_2} X)$.*
4. *For closed subsets A, B of X, if $d_1(A, B) > 0$ then $d_2(A, B) > 0$.*

For $d_1, d_2 \in \mathrm{M}(X)$, we write $d_1 \sim d_2$ if d_1 and d_2 are *uniformly equivalent*, that is, if both $d_1 \preceq d_2$ and $d_2 \preceq d_1$ hold. We will identify uniformly equivalent metrics on X and simply write $\mathrm{M}(X)$ to denote the quotient set $\mathrm{M}(X)/\sim$. Then $(\mathrm{M}(X), \preceq)$ is a directed ordered set.

Woods showed (in the proof of [5, Theorem 2.11]) that for any pair A, B of disjoint nonempty closed subsets of a metric space X there is a metric $d \in \mathrm{M}(X)$ such that $d(A, B) > 0$. Hence, if $D \subset \mathrm{M}(X)$ is cofinal with respect to $\preceq$, then $\sup\{u_d X : d \in D\} \simeq \beta X$. As a consequence, we have $\mathfrak{sa}(X) \leq \mathsf{cof}((\mathrm{M}(X), \preceq))$.

In the next section, we will prove the Tukey equivalence $(\mathrm{M}(X), \preceq) \equiv_T (\omega^\omega, \leq^*)$ for a locally compact separable metrizable space X such that $X^{(1)}$ is not compact. It will be proved by refining the proof of Theorem 2.3 ([4, Theorem 2.10]) to fit in a context of Tukey relation.

3 The main theorem

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Let X be a locally compact separable metrizable space such that $X^{(1)}$ is not compact. Then $(\mathrm{M}(X), \preceq) \equiv_T (\omega^\omega, \leq^*)$ holds.*

Throughout this section, we assume that X is a locally compact separable metrizable space and $X^{(1)}$ is not compact. Since X is embedded into the Hilbert cube $\mathbb{H} = [0,1]^\omega$ as a subspace, we fix such an embedding and regard X as a subspace of $\mathbb{H}$.

We will define a mapping from $\omega^{\uparrow\omega}$ to $\mathrm{M}(X)$ which is both Tukey and convergent, that is, the image of an unbounded set is unbounded and the image of a cofinal set is cofinal.

The following lemma, due to Kada, Tomoyasu and Yoshinobu [4, Lemma 2.8], is quite useful. Here we state this lemma in a modified and slightly strengthened form. Though it is not so difficult to modify the original proof to get the modified statement, we will present a complete proof for the reader's convenience. For a function φ from X to $\mathbb{R}$, we write $\varphi(x) \to \infty$ *as* $x \to \infty$ if, for any $M \in \mathbb{R}$ there is a compact subset K of X such that $\varphi(x) > M$ holds for all $x \in X \smallsetminus K$.

Lemma 3.2. *Suppose that X is a locally compact separable metrizable space, $d \in \mathrm{M}(X)$, $\mathrm{diam}_d(X)$ is finite, and γ is a continuous function from X to $[0, \infty)$ such that $\gamma(x) \to \infty$ as $x \to \infty$. For $n \in \omega$, Let $K_n = \{x \in X : \gamma(x) \leq \max\{n, \mathrm{diam}_d(X)\}\}$. Then we can define a mapping from $\omega^{\uparrow\omega}$ to $\mathrm{M}(X)$, which maps g to d_g, with the following properties.*

1. *If $x, y \in X \smallsetminus K_n$, then $d_g(x, y) \geq g(n) \cdot d(x, y)$.*
2. *For $x, y \in X$, $d_g(x, y) \geq |\gamma(x) - \gamma(y)|$.*
3. *For $g_1, g_2 \in \omega^{\uparrow\omega}$, $g_1 \leq^* g_2$ implies $d_{g_1} \preceq d_{g_2}$.*

Proof. We may assume that $g(0) \geq 1$. Define an increasing continuous function f_g from $[0, \infty)$ to $[1, \infty)$ in the following way: For $s \in [0, \infty)$, let $k = \lfloor 2s \rfloor$, $r = 2s - k$ and

$$f_g(s) = (1 - r) \cdot g(k) + r \cdot g(k + 1).$$

Note that, by the definition of f_g, if $g_1 \leq^* g_2$, then there is an $M \in [0, \infty)$ such that for all $s \in [M, \infty)$ we have $f_{g_1}(s) \leq f_{g_2}(s)$.

For $s \in [0, \infty)$, let

$$F_g(s) = \int_0^s f_g(t)dt.$$

Define functions ρ, ρ'_g from $X \times X$ to $[0, \infty)$ by the following:

$$\rho(x, y) = \max\{|\gamma(x) - \gamma(y)|, d(x, y)\},$$

$$\rho'_g(x, y) = f_g(\max\{\gamma(x), \gamma(y)\}) \cdot \rho(x, y).$$

ρ'_g is not necessarily a metric on X, because ρ'_g does not satisfy triangle inequality in general. So we define a function d_g from $X \times X$ to $[0, \infty)$ by the following:

$$\begin{aligned} d_g(x, y) = \inf\{&\rho'_g(x, z_0) + \cdots + \rho'_g(z_i, z_{i+1}) + \cdots + \rho'_g(z_{l-1}, y) : \\ &l < \omega \text{ and } z_0, \ldots, z_{l-1} \in X\}. \end{aligned}$$

Note that, since f_g is increasing,

$$\begin{aligned} \rho'_g(x, y) &= f_g(\max\{\gamma(x), \gamma(y)\}) \cdot \rho(x, y) \\ &\geq f_g(\max\{\gamma(x), \gamma(y)\}) \cdot |\gamma(x) - \gamma(y)| \\ &\geq |F_g(\gamma(x)) - F_g(\gamma(y))|. \end{aligned}$$

Hence we have $d_g(x, y) \geq |F_g(\gamma(x)) - F_g(\gamma(y))|$, because

$$\begin{aligned} &\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \\ &\geq |F_g(\gamma(x)) - F_g(\gamma(z_0))| + \cdots + |F_g(\gamma(z_{l-1})) - F_g(\gamma(y))| \\ &\geq |F_g(\gamma(x)) - F_g(\gamma(y))|. \end{aligned}$$

Claim 1. *For $n < \omega$ and $x, y \in X \setminus K_n$, $d_g(x, y) \geq f_g(n/2) \cdot d(x, y) = g(n) \cdot d(x, y)$.*

Proof. We may assume that $\gamma(x) = r \geq s = \gamma(y)$. Since $y \in X \setminus K_n$ and by the definition of K_n, we have $s \geq n$. Since f_g is increasing, it suffices to show that $\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq f_g(s/2) \cdot d(x, y)$ holds for any $l < \omega$, $z_0, \ldots, z_{l-1} \in X$.

Case 1. Assume that $\gamma(z_i) > s/2$ for all $i < l$. Since f_g is increasing, the definition of ρ'_g yields

$$\begin{aligned} \rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) &> f_g(s/2) \cdot (\rho(x, z_0) + \cdots + \rho(z_{l-1}, y)) \\ &\geq f_g(s/2) \cdot \rho(x, y) \\ &\geq f_g(s/2) \cdot d(x, y). \end{aligned}$$

Case 2. Assume that $\gamma(z_i) \leq s/2$ for some $i < l$. Fix such an i and then we have the following:

$$\rho'_g(x, z_0) + \cdots + \rho'_g(z_{i-1}, z_i) \geq d_g(x, z_i) \geq F_g(\gamma(x)) - F_g(\gamma(z_i)),$$
$$\rho'_g(z_i, z_{i+1}) + \cdots + \rho'_g(z_{l-1}, y) \geq d_g(z_i, y) \geq F_g(\gamma(y)) - F_g(\gamma(z_i)).$$

Hence it holds that

$$\begin{aligned}\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) &\geq (F_g(r) - F_g(\gamma(z_i))) + (F_g(s) - F_g(\gamma(z_i)))\\ &\geq (F_g(r) - F_g(s/2)) + (F_g(s) - F_g(s/2))\\ &\geq (r - s/2) \cdot f_g(s/2) + (s/2) \cdot f_g(s/2)\\ &= r \cdot f_g(s/2).\end{aligned}$$

On the other hand, $d(x, y) \leq r$, because $x \in X \setminus K_n$ and hence $r = \gamma(x) \geq \mathrm{diam}_d(X)$ by the definition of K_n. So we have

$$\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq f_g(s/2) \cdot d(x, y).$$

This concludes the proof of the claim. $\square$

Clearly d_g is symmetric and satisfies the triangle inequality. Since $f_g(s) \geq 1$ for all $s \in [0, \infty)$, Claim 1 implies that d_g is a metric on X. It is easy to see that d_g is compatible with the topology of (X, d).

It is easy to check that, if $g_1 \leq^* g_2$, then there is a compact subset K of X such that for any $x, y \in X \setminus K$ we have $d_{g_1}(x, y) \leq d_{g_2}(x, y)$. Therefore, $g_1 \leq^* g_2$ implies $d_{g_1} \preceq d_{g_2}$.

Finally, for any $x, y \in X$ we have $d_g(x, y) \geq \rho(x, y) \geq |\gamma(x) - \gamma(y)|$. $\square$

Now we work on a fixed locally compact separable metrizable space X such that $X^{(1)}$ is not compact. We regard X as a subspace of the Hilbert cube $\mathbb{H}$. Let μ be a fixed metric function on $\mathbb{H}$. Since $\mathbb{H}$ is compact, clearly $\mathrm{diam}_\mu(X)$ is finite.

Let E be a countable discrete closed subset of $X^{(1)}$. Such a set E exists by our assumption. We can find a continuous function γ from X to $[0, \infty)$ and a sequence $\{e_n : n < \omega\} \subseteq E$ with the following properties:

1. $\gamma(x) \to \infty$ as $x \to \infty$,

2. For each n, $\gamma(e_n) = n + 1/2$.

For each n, choose a sequence $\langle e_{n,j} : j \in \omega \rangle$ in X so that:

1. $\langle e_{n,j} : j \in \omega \rangle$ converges to e_n,

2. For all j, $n < \gamma(e_{n,j}) < n+1$.

Now we consider the mapping from $(\omega^{\uparrow\omega}, \leq^*)$ to $(\mathrm{M}(X), \preceq)$ obtained by applying Lemma 3.2 for X and μ, which maps $g \in \omega^{\uparrow\omega}$ to $\mu_g \in \mathrm{M}(X)$. We will show that it is both a Tukey and a convergent mapping, which concludes the proof of Theorem 3.1.

To show this, we define two auxiliary mappings from $\mathrm{M}(X)$ to $\omega^{\uparrow\omega}$ as follows. For $n < \omega$, let K_n be the one which appears in the statement of Lemma 3.2. For $\rho \in \mathrm{M}(X)$, define h_ρ recursively by letting $h(0) = 0$ and

$$h_\rho(n) = \min\{l : l > h_\rho(n-1) \text{ and } \forall x, y \in K_{n+2}\, (\rho(x,y) \geq 1/n \to \mu(x,y) \geq 1/l)\}$$

for $n \geq 1$. The set of l's in the definition of $h_\rho(n)$ is nonempty because of compactness, and so h_ρ is well-defined. Also, for $\rho \in \mathrm{M}(X)$, define H_ρ recursively in the following way. For each $n \geq 1$, define $j_n^\rho \in \omega$ by

$$j_n^\rho = \min\{j : \rho(e_{n,j}, e_n) \leq 1/n\}.$$

Let $H(0) = 0$ and

$$H_\rho(n) = \max\left\{H_\rho(n-1) + 1, \left\lceil 1/\mu(e_{n,j_n^\rho}, e_n) \right\rceil\right\}$$

for $n \geq 1$.

Lemma 3.3. *The mapping from $\omega^{\uparrow\omega}$ to $\mathrm{M}(X)$ which maps g to μ_g is a convergent mapping, that is, the image of a cofinal subset of $\omega^{\uparrow\omega}$ is a cofinal subset of $\mathrm{M}(X)$.*

Proof. It suffices to show that, for $\rho \in \mathrm{M}(X)$ and $g \in \omega^{\uparrow\omega}$, if $h_\rho \leq^* g$ then $\rho \preceq \mu_g$.

Suppose that $\rho \in \mathrm{M}(X)$, $g \in \omega^{\uparrow\omega}$ and $h_\rho \leq^* g$. To show $\rho \preceq \mu_g$, take any pair A, B of closed subsets of X which satisfies $\rho(A,B) > 0$, and we shall show $\mu_g(A,B) > 0$.

Take $k \in \omega$ so that $\rho(A,B) > 1/k$ and $g(n) \geq h_\rho(n)$ for all $n \geq k$. By the definition of h_ρ, for all $n \geq k$ and $x, y \in K_{n+2} \setminus K_n$, if $\rho(x,y) \geq 1/n$ then $\mu(x,y) \geq 1/h_\rho(n)$. So we have

$$\mu(A \cap (K_{n+2} \setminus K_n), B \cap (K_{n+2} \setminus K_n)) \geq 1/h_\rho(n).$$

Since $g(n) \geq h_\rho(n)$ for all $n \geq k$ and by the property (1) in Lemma 3.2, we have

$$\mu_g(A \cap (K_{n+2} \setminus K_n), B \cap (K_{n+2} \setminus K_n)) \geq 1.$$

for all $n \geq k$. Also, by the property (2) in Lemma 3.2 and the definition of K_n's, for $m, n \in \omega$ with $k \leq m < n$ we have $\mu_g(X \setminus K_n, K_m) \geq n - m$ and so

$$\mu_g(A \cap (K_{n+2} \setminus K_{n+1}), B \cap (K_{m+1} \setminus K_m)) \geq 1$$

and

$$\mu_g(A \cap (K_{m+1} \setminus K_m), B \cap (K_{n+2} \setminus K_{n+1})) \geq 1.$$

Hence $\mu_g(A, B) \geq \min\{1, \mu_g(A \cap K_{k+1}, B \cap K_{k+1})\} > 0$. □

Lemma 3.4. *The mapping from $\omega^{\uparrow\omega}$ to $\mathrm{M}(X)$ which maps g to μ_g is a Tukey mapping, that is, the image of an unbounded subset of $\omega^{\uparrow\omega}$ is an unbounded subset of $\mathrm{M}(X)$.*

Proof. It suffices to show that, for $\rho \in \mathrm{M}(X)$ and $g \in \omega^{\uparrow\omega}$, if $g \not\leq^* H_\rho$ then $\mu_g \not\preceq \rho$.

Suppose that $\rho \in \mathrm{M}(X)$, $g \in \omega^{\uparrow\omega}$ and $g \not\leq^* H_\rho$. To show $\mu_g \not\preceq \rho$, we shall find a pair A, B of closed subsets of X such that $\rho(A, B) = 0$ but $\mu_g(A, B) > 0$.

Let $U = \{n : H_\rho(n) < g(n)\}$, $A = \{e_{n,j_n^\rho} : n \in U\}$ and $B = \{e_n : n \in U\}$. Since $g \not\leq^* H_\rho$, U is an infinite subset of ω. By the choice of j_n^ρ, for each $n \in U$ we have $\rho(e_{n,j_n^\rho}, e_n) \leq 1/n$, and hence $\rho(A, B) = 0$. On the other hand, for each $n \in U$, since $g(n) > H_\rho(n) \geq 1/\mu(e_{n,j_n^\rho}, e_n)$ and by the property (1) in Lemma 3.2, we have $\mu_g(e_{n,j_n^\rho}, e_n) \geq g(n) \cdot \mu(e_{n,j_n^\rho}, e_n) \geq 1$. By the choice of $e_{n,j}$'s and the property (2) in Lemma 3.2, for any n, m, j with $n \neq m$ we have $\mu_g(e_{n,j}, e_m) > 1/2$. Hence $\mu_g(A, B) > 1/2$. □

This concludes the proof of Theorem 3.1.

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