How many miles to βX? : d miles, or just one foot

How many miles to βX? — $\mathfrak{d}$ miles, or just one foot

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Abstract

It is known that the Stone–Čech compactification βX of a metrizable space X is approximated by the collection of Smirnov compactifications of X for all compatible metrics on X. If we confine ourselves to locally compact separable metrizable spaces, the corresponding statement holds for Higson compactifications. We investigate the smallest cardinality of a set D of compatible metrics on X such that βX is approximated by Smirnov or Higson compactifications for all metrics in D. We prove that it is either the dominating number or 1 for a locally compact separable metrizable space.

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1 Introduction

A *compactification* of a completely regular Hausdorff space X is a compact Hausdorff space which contains X as a dense subspace. For compactifications αX and γX of X, we write $\alpha X \leq \gamma X$ if there is a continuous surjection $f : \gamma X \to \alpha X$ such that $f \restriction X$ is the identity map on X. If such an f can be chosen to be a homeomorphism, we write $\alpha X \simeq \gamma X$. Let $\mathcal{K}(X)$ denote the class of compactifications of X. When we identify $\simeq$-equivalent compactifications, we may regard $\mathcal{K}(X)$ as a set, and the order structure $(\mathcal{K}(X), \leq)$ is a complete upper semilattice whose largest element is the Stone–Čech compactification βX.

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Let $C^*(X)$ denote the ring of bounded continuous functions from X to $\mathbb{R}$. A subring R of $C^*(X)$ is called *regular* if R is closed in the sense of uniform norm topology, contains all constant functions, and generates the topology of X. Let $\mathcal{R}(X)$ denote the class of regular subrings of $C^*(X)$. Then it is known that $(\mathcal{K}(X), \leq)$ is isomorphic to $(\mathcal{R}(X), \subseteq)$, by mapping each $\alpha X \in \mathcal{K}(X)$ to the set of bounded continuous functions from X to $\mathbb{R}$ which are continuously extended over αX (cf. [1, Theorem 3.7], [2, Theorem 2.5]). In particular, βX corresponds to the whole $C^*(X)$. (See [2, 4] for more details.)

For a compactification αX of X and two closed subsets A, B of X, we write $A \parallel B \ (\alpha X)$ if $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$, and otherwise $A \nparallel B \ (\alpha X)$.

For a metric space (X, d), $U_d^*(X)$ denotes the set of all bounded uniformly continuous functions from (X, d) to $\mathbb{R}$. $U_d^*(X)$ is a regular subring of $C^*(X)$. The *Smirnov compactification $u_d X$ of* (X, d) is the unique compactification associated with the subring $U_d^*(X)$. For disjoint closed subsets A, B of X, $A \parallel B \ (u_d X)$ if and only if $d(A, B) > 0$ [7, Theorem 2.5].

The following theorem tells us that we can approximate the Stone–Čech compactification of a metrizable space by the collection of all Smirnov compactifications. Let $\mathrm{M}(X)$ denote the set of all metrics on X which are compatible with the topology on X.

Theorem 1.1. [7, Theorem 2.11] *For a noncompact metrizable space X, we have $\beta X \simeq \sup\{u_d X : d \in \mathrm{M}(X)\}$ (the supremum is taken in the lattice $(\mathcal{K}(X), \leq)$).*

Now we define the following cardinal function.

Definition 1.2. [5, Definition 2.2] For a noncompact metrizable space X, let $\mathfrak{sa}(X) = \min\{|D| : D \subseteq \mathrm{M}(X) \text{ and } \beta X \simeq \sup\{u_d X : d \in D\}\}$.

For a metrizable space X, a metric d on X is called *proper* if each d-bounded set has compact closure. A *proper metric space* means a metric space whose metric is proper.

For a function f and a subset A of the domain of f, $f''A$ denotes the image of A by f.

Let (X, d) be a proper metric space and (Y, ρ) a metric space. We say a function f from X to Y is *slowly oscillating* if it satisfies the following condition:

$$\forall r > 0 \, \forall \varepsilon > 0 \, \exists K \text{ a compact subset of } X \, \forall x \in X \setminus K \, (\mathrm{diam}_\rho(f'' \mathrm{B}_d(x, r)) < \varepsilon).$$

For a proper metric space (X, d), let $C_d^*(X)$ be the set of all bounded continuous slowly oscillating functions from (X, d) to $\mathbb{R}$. $C_d^*(X)$ is a regular subring of $C^*(X)$. The *Higson compactification* $\overline{X}^d$ *of* (X, d) is the unique compactification associated with the subring $C_d^*(X)$. For disjoint closed subsets A, B of X, $A \parallel B$ $(\overline{X}^d)$ if and only if for any $R > 0$ there is a compact subset K_R of X such that $d(x, A) + d(x, B) > R$ holds for all $x \in X \setminus K_R$ [3, Proposition 2.3].

The following corresponds to Theorem 1.1 for Higson compactifications. Note that a proper metric space is locally compact and separable. Let $\mathrm{PM}(X)$ be the set of all proper metrics compatible with the topology of X.

Theorem 1.3. [6, Theorem 3.2] *For a noncompact locally compact separable metrizable space X, we have $\beta X \simeq \sup\{\overline{X}^d : d \in \mathrm{PM}(X)\}$.*

So we consider the following cardinal function.

Definition 1.4. [5, Definition 6.2] For a noncompact locally compact separable metrizable space X, let $\mathfrak{ha}(X) = \min\{|D| : D \subseteq \mathrm{PM}(X)$ and $\beta X \simeq \sup\{\overline{X}^d : d \in D\}\}$.

We have $\mathfrak{sa}(X) \leq \mathfrak{ha}(X)$ for each locally compact separable metrizable space X [5, Lemma 6.3].

For $f, g \in \omega^\omega$, we say $f \leq^* g$ if for all but finitely many $n < \omega$ we have $f(n) \leq g(n)$. The *dominating number* $\mathfrak{d}$ is the smallest size of a subset of ω^ω which is cofinal in ω^ω with respect to $\leq^*$.

In Section 2 we will show that, for a locally compact separable metrizable space X, either $\mathfrak{sa}(X) = \mathfrak{ha}(X) = \mathfrak{d}$ or $\mathfrak{sa}(X) = \mathfrak{ha}(X) = 1$ holds. In Section 3 we will give an example of a nonseparable metrizable space X for which $\mathfrak{sa}(X) > \mathfrak{d}$ holds.

2 Dichotomy for locally compact separable spaces

It is easily seen that $\mathfrak{sa}(\omega) = \mathfrak{ha}(\omega) = 1$. In fact, the following two theorems give equivalent conditions respectively for $\mathfrak{sa}(X) = 1$ and $\mathfrak{ha}(X) = 1$.

For a space X, $X^{(1)}$ denotes the first Cantor–Bendixson derivative of X, that is, the subspace of X which consists of nonisolated points of X.

Theorem 2.1. [7, Corollary 3.5] *For a metrizable space X, the following conditions are equivalent.*

1. *There is a metric $d \in \mathrm{M}(X)$ for which $u_d X \simeq \beta X$ holds.*

2. *$X^{(1)}$ is compact.*

Theorem 2.2. [6, Proposition 2.6] *For a locally compact separable metrizable space X, the following conditions are equivalent.*

1. *There is a proper metric $d \in \mathrm{PM}(X)$ for which $\overline{X}^d \simeq \beta X$ holds.*
2. *$X^{(1)}$ is compact.*

In the paper [5] we proved the following proposition.

Proposition 2.3. [5, Examples 2.3 and 6.4] $\mathfrak{sa}([0, \infty)) = \mathfrak{ha}([0, \infty)) = \mathfrak{d}$.

In this section we prove that, assuming that X is locally compact and separable, $\mathfrak{sa}(X) = \mathfrak{ha}(X) = \mathfrak{d}$ unless $X^{(1)}$ is compact. In particular, since $\mathfrak{ha}(X)$ is defined only when X is locally compact and separable, $\mathfrak{ha}(X)$ is either $\mathfrak{d}$ or 1 when it is defined.

We will use the following two lemmas.

Lemma 2.4. [5, Lemma 1.1] *For a compactification αX of a space X and closed subsets A, B of X, the following conditions are equivalent*:

1. *$A \parallel B \ (\alpha X)$.*
2. *There are $g \in C^*(X)$ and $a, b \in \mathbb{R}$ such that $a > b$, $g(x) \geq a$ for all $x \in A$, $g(x) \leq b$ for all $x \in B$ and g is continuously extended over αX.*

Note that, for a normal space X, $\alpha X \simeq \beta X$ if and only if $A \parallel B \ (\alpha X)$ for any disjoint closed subsets A, B of X.

Lemma 2.5. [5, Lemma 1.2] *Suppose that $\mathcal{C}$ is a set of compactifications of a space X. For closed sets A, B of X, the following conditions are equivalent:*

1. *$A \parallel B \ (\sup \mathcal{C})$.*
2. *$A \parallel B \ (\sup \mathcal{F})$ for some nonempty finite subset $\mathcal{F}$ of $\mathcal{C}$.*

Since $\mathfrak{sa}(X) \leq \mathfrak{ha}(X)$ holds if both are defined, it suffices to show that $\mathfrak{sa}(X) \geq \mathfrak{d}$ and $\mathfrak{ha}(X) \leq \mathfrak{d}$.

First we show that $\mathfrak{sa}(X) \geq \mathfrak{d}$ unless $\mathfrak{sa}(X) = 1$. This holds for all metrizable spaces.

Lemma 2.6. *Let X be a metrizable space. If $X^{(1)}$ is not compact, then $\mathfrak{sa}(X) \geq \mathfrak{d}$.*

Proof. Since $X^{(1)}$ is not compact, there is a countable subset A of $X^{(1)}$ which has no accumulating point in X. Note that A is closed in X. Enumerate A as $\{a_n : n < \omega\}$.

Claim 1. *There are a neighborhood* U_n *of* a_n *and a sequence* $\langle b_{n,i} : i < \omega\rangle$ *in* $U_n \smallsetminus \{a_n\}$ *for* $n < \omega$ *such that,*

1. *for each* $n < \omega$, $\langle b_{n,i} : i < \omega\rangle$ *converges to* a_n,
2. *if* $n < m < \omega$ *then* $U_n \cap U_m = \emptyset$, *and*
3. *for any* $f \in \omega^\omega$, *the set* $B_f = \{b_{n,f(n)} : n < \omega\}$ *has no accumulating point.*

Proof. Fix a metric $\rho \in \mathrm{M}(X)$. For each $n < \omega$, let $\delta_n = \frac{1}{3} \cdot \rho(a_n, A \smallsetminus \{a_n\})$. By the choice of A, we have $\delta_n > 0$. Let $U_n = \mathrm{B}_\rho(a_n, \delta_n)$. Then $n \neq m$ implies $U_n \cap U_m = \emptyset$. Since a_n is not isolated in X, we can choose a sequence $\langle b_{n,i} : i < \omega\rangle$ in $U_n \smallsetminus \{a_n\}$ which converges to a_n. Fix an arbitrary $f \in \omega^\omega$. By the choice of δ_n's, if B_f accumulates to a point, then A must accumulate to the same point. Hence B_f has no accumulating point. $\square$

Fix $\kappa < \mathfrak{d}$ and a set $D \subseteq \mathrm{M}(X)$ of size κ. We show that $\beta X \not\preceq \sup\{u_d X : d \in D\}$.

For each $d \in D$, define a function $g_d \in \omega^\omega$ by letting

$$g_d(n) = \min\left\{m < \omega : \forall i \geq m \ \left(d(a_n, b_{n,i}) < \tfrac{1}{n+1}\right)\right\}$$

for $n < \omega$. For each nonempty finite subset F of D, let $g_F = \max\{g_f : f \in F\}$ (where max is the pointwise maximum). Since $|[D]^{<\omega}| = |D| = \kappa < \mathfrak{d}$, there is an $f \in \omega^\omega$ which satisfies $f \not\leq^* g_F$ for every nonempty finite subset F of D.

Let $B = B_f = \{b_{n,f(n)} : n < \omega\}$. Then B is closed and disjoint from A.

For an arbitrary nonempty finite subset F of D, the set $I_F = \{n < \omega : g_F(n) < f(n)\}$ is infinite. Let $C = \mathrm{cl}\langle \bigcup\{U_d^*(X) : d \in F\}\rangle$. Then C is the closed subring of $C^*(X)$ associated with $\sup\{u_d X : d \in F\}$. By the definition of g_F, each $n \in I_F$ satisfies $d(a_n, b_{n,f(n)}) < \frac{1}{n+1}$ for all $d \in F$. If $\psi \in \bigcup\{U_d^*(X) : d \in F\}$, then the sequence $\langle \psi(a_n) - \psi(b_{n,f(n)}) : n \in I_F\rangle$ converges to 0. So for all $\varphi \in C$, $\langle \varphi(a_n) - \varphi(b_{n,f(n)}) : n \in I_F\rangle$ converges to 0. This means that there are no $\varphi \in C$ and $a, b \in \mathbb{R}$ such that $a > b$, $\varphi(x) \geq a$ for all $x \in A$, and $\varphi(x) \leq b$ for all $x \in B$. By Lemma 2.4, this means $A \not\parallel B$ $(\sup\{u_d X : d \in F\})$. Since F is an arbitrary nonempty finite subset of D and by Lemma 2.5, we have $A \not\parallel B$ $(\sup\{u_d X : d \in D\})$, and hence $\beta X \not\preceq \sup\{u_d X : d \in D\}$. $\square$

We turn to the proof of the inequality $\mathfrak{ha}(X) \leq \mathfrak{d}$.

For notational convenience, in the following lemmas and proofs, we let $C_n = K_n = \emptyset$ for $n = -1, -2, \ldots$

Lemma 2.7. *Suppose that X is a normal space, and a sequence $\langle C_n : n < \omega \rangle$ of closed subsets of X satisfies $C_n \subseteq \operatorname{int} C_{n+1}$ for all $n < \omega$ and $X = \bigcup\{C_n : n < \omega\}$. Then, for an increasing sequence $\langle r_n : n < \omega \rangle$ of nonnegative real numbers, there is a continuous function φ from X to $[0, \infty)$ such that, for each $n < \omega$ we have $\varphi''(C_n \smallsetminus \operatorname{int} C_{n-1}) \subseteq [r_n, r_{n+1}]$.*

Proof. For each $n < \omega$, choose a continuous function φ_n from X to $[0, r_n]$ so that $\varphi_n{}'' C_{n-2} = \{0\}$ and $\varphi_n{}''(X \smallsetminus \operatorname{int} C_{n-1}) = \{r_n\}$. Note that, if $x \in C_m$, then for all $n \geq m + 2$ we have $\varphi_n(x) = 0$. So we can define a continuous function φ from X to $[0, \infty)$ as the pointwise maximum of $\{\varphi_n : n < \omega\}$, and then φ satisfies the requirement. $\square$

Suppose that X is a locally compact separable metrizable space. Since X is σ-compact, there is a sequence $\langle K_n : n < \omega \rangle$ of compact subsets of X such that, for each $n < \omega$ we have $K_n \subseteq \operatorname{int} K_{n+1}$, and $X = \bigcup\{K_n : n < \omega\}$.

Lemma 2.8. *Let (X, d) be a locally compact separable metric space, and $\langle K_n : n < \omega \rangle$ a sequence of compact subsets of X such that, for each $n < \omega$ we have $K_n \subseteq \operatorname{int} K_{n+1}$, and $X = \bigcup\{K_n : n < \omega\}$. Then, for each $g \in \omega^\omega$, there is a proper metric d_g which satisfies the following:*

1. *d_g is compatible with the topology of X.*
2. *For $n < \omega$ and $x, y \in X \smallsetminus K_{n-1}$ we have $d_g(x, y) \geq g(n) \cdot d(x, y)$.*
3. *For $n < \omega$ we have $d_g(K_{n-1}, X \smallsetminus K_n) \geq n$.*

Proof. Let $R_n = \max\{n, \operatorname{diam}_d(K_n)\}$ for each $n < \omega$, and let c be the continuous function from X to $[0, \infty)$ which is obtained by applying Lemma 2.7 to $\langle K_n : n < \omega \rangle$ and $\langle R_n : n < \omega \rangle$.

We may assume that g is increasing and $g(0) \geq 1$. Choose an increasing continuous function f from $[0, \infty)$ to $[1, \infty)$ such that $f(\frac{n}{2}) \geq g(n)$ for all $n < \omega$. For $s \in [0, \infty)$, let

$$F(s) = \int_0^s f(t)dt.$$

Define functions ρ, ρ'_g from $X \times X$ to $[0, \infty)$ by the following:

$$\rho(x, y) = \max\{|c(x) - c(y)|, d(x, y)\},$$

$$\rho'_g(x, y) = f(\max\{c(x), c(y)\}) \cdot \rho(x, y).$$

It is easy to see that ρ is a proper metric on X and compatible with the topology on X. However, ρ'_g is not necessarily a metric on X, because ρ'_g

does not satisfy triangle inequality in general. So we define a function ρ_g from $X \times X$ to $[0, \infty)$ by the following:

$$\rho_g(x,y) = \inf\{\rho'_g(x,z_0) + \cdots + \rho'_g(z_i,z_{i+1}) + \cdots + \rho'_g(z_{l-1},y) : \\ l < \omega \text{ and } z_0, \ldots, z_{l-1} \in X\}.$$

Note that, since f is increasing, $\rho'_g(x,y) \geq f(\max\{c(x), c(y)\}) \cdot |c(x) - c(y)| \geq |F(c(x)) - F(c(y))|$. Hence we have $\rho_g(x,y) \geq |F(c(x)) - F(c(y))|$, because

$$\begin{aligned} &\rho'_g(x,z_0) + \cdots + \rho'_g(z_{l-1},y) \\ &\geq |F(c(x)) - F(c(z_0))| + \cdots + |F(c(z_{l-1})) - F(c(y))| \\ &\geq |F(c(x)) - F(c(y))|. \end{aligned}$$

Claim 1. *Let x, y be points of X. If $x, y \in X \setminus K_{n-1}$, $n < \omega$, then $\rho_g(x,y) \geq f(\frac{n}{2}) \cdot d(x,y) \geq g(n) \cdot d(x,y)$.*

Proof. We may assume that $c(x) = r \geq s = c(y)$, $x \in K_m \setminus K_{m-1}$ and $y \in K_m$ for some $m \geq n$. By the definition of c, we have $s \geq n$. Since f is increasing, it suffices to show that $\rho'_g(x,z_0) + \cdots + \rho'_g(z_{l-1},y) \geq f(\frac{s}{2}) \cdot d(x,y)$ for any $l < \omega$, $z_0, \ldots, z_{l-1} \in X$.

Case 1. Assume that $c(z_i) > \frac{s}{2}$ for all $i < l$. Since f is increasing, the definition of ρ'_g yields

$$\begin{aligned} \rho'_g(x,z_0) + \cdots + \rho'_g(z_{l-1},y) &> f(\tfrac{s}{2}) \cdot (\rho(x,z_0) + \cdots + \rho(z_{l-1},y)) \\ &\geq f(\tfrac{s}{2}) \cdot \rho(x,y) \\ &\geq f(\tfrac{s}{2}) \cdot d(x,y). \end{aligned}$$

Case 2. Assume that $c(z_i) \leq \frac{s}{2}$ for some $i < l$. Fix such an i and then we have the following:

$$\begin{aligned} \rho'_g(x,z_0) + \cdots + \rho'_g(z_{i-1},z_i) \geq \rho_g(x,z_i) \geq F(c(x)) - F(c(z_i)), \\ \rho'_g(z_i,z_{i+1}) + \cdots + \rho'_g(z_{l-1},y) \geq \rho_g(z_i,y) \geq F(c(y)) - F(c(z_i)). \end{aligned}$$

Hence it holds that

$$\begin{aligned} \rho'_g(x,z_0) + \cdots + \rho'_g(z_{l-1},y) &\geq (F(r) - F(c(z_i))) + (F(s) - F(c(z_i))) \\ &\geq (F(r) - F(\tfrac{s}{2})) + (F(s) - F(\tfrac{s}{2})) \\ &\geq (r - \tfrac{s}{2}) f(\tfrac{s}{2}) + \tfrac{s}{2} f(\tfrac{s}{2}) \\ &= r f(\tfrac{s}{2}). \end{aligned}$$

On the other hand, $d(x,y) \leq r$, because $x, y \in K_m$ and $r = c(x) \geq \operatorname{diam}_d K_m$. So we have

$$\rho'_g(x,z_0) + \cdots + \rho'_g(z_{l-1},y) \geq f(\tfrac{s}{2}) \cdot d(x,y).$$

This concludes the proof of the claim. □

Clearly ρ_g is symmetric and satisfies the triangle inequality. Since $f(s) \geq 1$ for all $s \in [0, \infty)$, Claim 1 implies that ρ_g is a metric on X. Moreover, ρ_g is proper because $\rho_g \geq \rho$ and ρ is proper. It is easy to see that ρ_g is compatible with the topology of (X, d).

Finally, we define a metric d_g using ρ_g. Let δ be the continuous function from X to $[0, \infty)$ which is obtained by applying Lemma 2.7 to $\langle K_n : n < \omega \rangle$ and $\langle n^2 : n < \omega \rangle$. Note that, for $n < \omega$, $x \in K_{n-1}$ and $y \in X \setminus K_n$ we have $\delta(y) - \delta(x) \geq n$. Define d_g by letting $d_g(x, y) = \max\{|\delta(x) - \delta(y)|, \rho_g(x, y)\}$ for $x, y \in X$. Then d_g satisfies all requirements of the lemma. $\square$

Lemma 2.9. *For any locally compact separable metrizable space X, we have $\mathfrak{ha}(X) \leq \mathfrak{d}$.*

Proof. Fix a metric d on X, and choose a sequence $\langle K_n : n < \omega \rangle$ of compact sets of X that meets the requirement in Lemma 2.8. For each $g \in \omega^\omega$, let d_g be the metric on X which is obtained by applying Lemma 2.8 to (X, d), $\langle K_n : n < \omega \rangle$ and g.

Choose a subset $\mathcal{F}$ of ω^ω of size $\mathfrak{d}$ which is cofinal with respect to $\leq^*$. We will prove that $\beta X \simeq \sup\{\overline{X}^{d_g} : g \in \mathcal{F}\}$. It suffices to show that, for any two disjoint closed sets A, B of X there is a $g \in \mathcal{F}$ such that $A \parallel B$ $(\overline{X}^{d_g})$.

For $n < \omega$, let $\Delta_n = K_{n+2} \setminus \operatorname{int} K_n$. Note that $\Delta_n \subseteq X \setminus K_{n-1}$ for each $n < \omega$. Since A, B are disjoint closed sets and each Δ_n is compact, we have $d(A \cap \Delta_n, B \cap \Delta_n) > 0$ if $A \cap \Delta_n \neq \emptyset \neq B \cap \Delta_n \neq \emptyset$. Define $h_{A,B} \in \omega^\omega$ as follows: For $n < \omega$ with $A \cap \Delta_n \neq \emptyset \neq B \cap \Delta_n \neq \emptyset$, let

$$h_{A,B}(n) = \left\lceil \frac{n}{d(A \cap \Delta_n, B \cap \Delta_n)} \right\rceil,$$

(where $\lceil r \rceil$ denotes the smallest integer not smaller than r) and otherwise $h_{A,B}(n)$ is arbitrary. Find $g \in \mathcal{F}$ and $N < \omega$ such that $h_{A,B}(n) \leq g(n)$ for $n > N$.

We claim that, for every $M \geq N$ and $x \in X \setminus K_{M+1}$ we have $d_g(x, A) + d_g(x, B) \geq M$, and hence $A \parallel B$ $(\overline{X}^{d_g})$. Fix $M < \omega$ and $x \in X \setminus K_{M+1}$. Since d_g is a proper metric, we can find $a \in A$ and $b \in B$ such that $d_g(x, A) + d_g(x, B) = d_g(x, a) + d_g(x, b)$ holds. Choose $n_a, n_b < \omega$ so that $a \in K_{n_a} \setminus K_{n_a - 1}$ and $b \in K_{n_b} \setminus K_{n_b - 1}$, and let $n = \min\{n_a, n_b\}$.

Case 1. $n \leq M$. Since $x \in X \setminus K_{M+1}$ and $d_g(K_M, X \setminus K_{M+1}) \geq M$, we have $d_g(a, x) \geq M$ or $d_g(b, x) \geq M$.

Case 2. $n > M$. By the triangle inequality, it suffices to show that

$d_g(a,b) \geq M$. If $|n_a - n_b| \leq 1$, then $a, b \in \Delta_{n-1}$, and hence we have

$$\begin{aligned} d_g(a,b) &\geq g(n-1) \cdot d(a,b) \\ &\geq h_{A,B}(n-1) \cdot d(a,b) \\ &\geq h_{A,B}(n-1) \cdot d(A \cap \Delta_{n-1}, B \cap \Delta_{n-1}) \\ &\geq n-1 \geq M. \end{aligned}$$

Otherwise, we have $d_g(a,b) \geq d_g(K_n, X \setminus K_{n+1}) \geq n > M$.

This concludes the proof. □

Now we have the following theorem.

Theorem 2.10. *Let X be a locally compact separable metrizable space. If $X^{(1)}$ is not compact, then $\mathfrak{sa}(X) = \mathfrak{ha}(X) = \mathfrak{d}$, and otherwise $\mathfrak{sa}(X) = \mathfrak{ha}(X) = 1$.*

3 It may be further than $\mathfrak{d}$ miles

The cardinal $\mathfrak{ha}(X)$ is defined for locally compact separable metrizable spaces X, while $\mathfrak{sa}(X)$ is defined for any metrizable space X. By Theorem 2.1 and Lemma 2.6, either $\mathfrak{sa}(X) \geq \mathfrak{d}$ or $\mathfrak{sa}(X) = 1$ holds for any X. In this section, we show the existence of a metrizable space X for which $\mathfrak{sa}(X) > \mathfrak{d}$ holds.

For a topological space X, $e(X)$, the *extent* of X, is defined by $e(X) = \sup\{|D| : D \subseteq X \text{ and } D \text{ is closed discrete}\} + \aleph_0$.

Definition 3.1. For an infinite cardinal κ, define $\log\kappa$ by letting $\log\kappa = \min\{\theta : 2^\theta \geq \kappa\}$.

It is easy to see that, for a set C of infinite cardinals, we have $\log(\sup C) = \sup\{\log\kappa : \kappa \in C\}$.

Proposition 3.2. *Let X be a metrizable space. If $X^{(1)}$ is not compact, then $\mathfrak{sa}(X) \geq \log e(X^{(1)})$.*

Proof. It suffices to show that, for infinite cardinals κ and λ, if $X^{(1)}$ has a closed discrete subset of size κ and $\lambda = \log\kappa$, then $\mathfrak{sa}(X) \geq \lambda$.

Suppose that D is a set of compatible metrics on X and $|D| = \mu < \lambda$. We will show that $\beta X \not\leq \sup\{u_\rho X : \rho \in D\}$. Since we have $\mathfrak{sa}(X) \geq \mathfrak{d}$ by Lemma 2.6, we may assume that $\mu \geq \mathfrak{d}$.

Choose a subset H of ω^ω of size $\mathfrak{d}$ which is cofinal with respect to $\leq$.

Fix a closed discrete subset $A = \{a_\xi : \xi < \kappa\}$ of $X^{(1)}$. As in the proof of Lemma 2.6, we choose a neighborhood U_ξ of a_ξ and a sequence $\langle b_{\xi,i} : i < \omega\rangle$ in $U_\xi \setminus \{a_\xi\}$ for $\xi < \kappa$ so that,

1. for each $\xi < \kappa$, $\langle b_{\xi,i} : i < \omega \rangle$ converges to a_ξ,

2. if $\xi < \eta < \kappa$ then $U_\xi \cap U_\eta = \emptyset$, and

3. for any $\varphi \in \omega^\kappa$, the set $\{b_{\xi,\varphi(\xi)} : \xi < \kappa\}$ has no accumulating point.

For each $\rho \in D$ and $\xi < \kappa$, define $g^\xi_\rho \in \omega^\omega$ by letting

$$g^\xi_\rho(m) = \min\left\{k < \omega : \forall i \geq k \left(\rho(a_\xi, b_{\xi,i}) < \frac{1}{m+1}\right)\right\}$$

for $m < \omega$, and choose $h^\xi_\rho \in H$ so that $g^\xi_\rho \leq h^\xi_\rho$.

Since $\mathfrak{d} \leq \mu = |D| < \lambda = \log \kappa$, we have $\mathfrak{d}^\mu = 2^\mu < \kappa$, and hence there are $K \in [\kappa]^\kappa$ and $\{h^\xi : \xi \in K\}$ such that, for each $\xi \in K$, $h^\xi_\rho = h^\xi$ for all $\rho \in D$.

Fix a countable set $\{\xi_n : n < \omega\} \subseteq K$. Let $b_n = b_{\xi_n, h^{\xi_n}(n)}$ and $B = \{b_n : n < \omega\}$. By the choice of A, U_ξ's and $b_{\xi,i}$'s, $A \cap B = \emptyset$ and B is closed in X. Also, by the choice of h^ξ_ρ's, for each $\rho \in D$ and $n < \omega$ we have $\rho(a_{\xi_n}, b_n) \leq \frac{1}{n+1}$.

Now it is easy to see that $A \not\delta B$ $(\sup\{u_\rho X : \rho \in D\})$, and hence $\beta X \not\simeq \sup\{u_\rho X : \rho \in D\}$. □

Corollary 3.3. *Let $X_\kappa = \kappa \times (\omega + 1)$, where κ is equipped with the discrete topology and $\omega + 1$ is equipped with the usual order topology. If $\kappa > 2^{\mathfrak{d}}$, then $\mathfrak{sa}(X_\kappa) > \mathfrak{d}$.*

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