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混合正規分布の漸近分散および共分散の性質について

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混合正規分布の漸近分散及び共分散の性質について*

Some Properties of the Moment Estimators for the Parameters of a Mixture of Two Normal Distributions

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1. 概　　要

分散の等しい2つの正規分布 $N(\theta_1, \sigma^2)$, $N(\theta_2, \sigma^2)$ （ただし $\theta_1<\theta_2$）が未知の割合 $p:(1-p)$ で混合している mixture は moment 法により Rao〔2〕津村，管野〔4〕，等によって解かれている．ここでは Blischke〔1〕の方法によって asymptotic variance matrix を explicitに計算し（ただし n を標本の大きさとしたとき n^{-1} の order まで），さらに $\Delta=\frac{\theta_2-\theta_1}{\sigma}\to\infty$ のとき moment 法による estimator の JARE が1に近づくことを示す．〔1〕，〔3〕．

2. mixture の distribution の moment estimator とその asymptotic covariance matrix

mixture の密度関数を

$$f(x)=pf_1(x)+(1-p)f_2(x),\ 0<p<1$$

ただし，$f_i(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\theta_i)^2}{2\sigma^2}}$　　$i=1,\ 2,\ \theta_1<\theta_2$

とする．

この cumulant は

$$\kappa_1=p\theta_1+(1-p)\theta_2$$

$$\kappa_2=\sigma^2+p(1-p)(\theta_1-\theta_2)^2$$

$$\kappa_3=p(1-p)(1-2p)(\theta_1-\theta_2)^3$$

$$\kappa_4=p(1-p)(1-6p+6p^2)(\theta_1-\theta_2)^4$$

$$\kappa_5=p(1-p)(1-2p)(1-12p+12p^2)(\theta_1-\theta_2)^5$$

$$\kappa_6=p(1-p)(1-30p+150p^2-240p^3+120p^4)(\theta_1-\theta_2)^6$$

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* 1972年日本統計学会において講演

** 一般教養科

$$\kappa_7=p(1-p)(1-2p)(1-60p+420p^2-720p^3+360p^4)(\theta_1-\theta_2)^7$$

$$\kappa_8=p(1-p)(1-126p+1806p^2-8400p^3+16800p^4-15120p^5+5040p^6)(\theta_1-\theta_2)^8$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

となる．

moment 法による，$\theta_1, \theta_2, \sigma^2, p$ の推定量をそれぞれ $\hat{\theta}_1, \hat{\theta}_2, \hat{\sigma}^2, \hat{p}$ とし K－統計量を $K_1, K_2, K_3, K_4 \cdots\cdots\cdots\cdots$ であらわすと $E(K_i)=\kappa_i$ $(i=1, 2, 3, 4, 5\cdots\cdots\cdots\cdots)$，を用いて次の連立方程式を得る．

$$K_1=\hat{p}\hat{\theta}_1+(1-\hat{p})\hat{\theta}_2$$

$$K_2=\hat{\sigma}^2+\hat{p}(1-\hat{p})\hat{\delta}^2$$

$$K_3=-\hat{p}(1-\hat{p})(1-2\hat{p})\hat{\delta}^3$$

$$K_4=\hat{p}(1-\hat{p})(1-6\hat{p}+6\hat{p}^2)\hat{\delta}^4$$

ただし $\hat{\delta}=\hat{\theta}_2-\hat{\theta}_1$

$x=(\hat{\theta}_1-K_1)(\hat{\theta}_2-K_1)$ とおけば x は3次方程式

$$x^3+\frac{1}{2}K_4x+\frac{1}{2}K_3^2=0$$

の負根として一意的に求まる．

$d_1=\hat{\theta}_1-K_1$, $d_2=\hat{\theta}_2-K_1$ とおく．x が求められると d_1 は2次方程式

$$d_1^2+\frac{K_3}{x}d_1+x=0$$

の負根として求まり，したがって

$$d_2=-\frac{K_3}{x}-d_1$$

よって推定量 $\hat{\theta}_1, \hat{\theta}_2, \hat{\sigma}^2, \hat{p}$ は次式で求めることが出来る．

$$\hat{\theta}_1=K_1+d_1 \qquad \hat{\theta}_2=K_1+d_2$$

$$\hat{\sigma}^2=K_2+x \qquad \hat{p}=\frac{d_2}{d_2-d_1}$$

また上記の連立方程式を K_i $(i=1, 2, 3, 4)$ でそれぞれを偏微分して，それらの連立方程式を解くことにより次式を得る．

$$\frac{\partial\hat{\theta}_1}{\partial K_1}=1,\ \frac{\partial\hat{\theta}_1}{\partial K_2}=0,\ \frac{\partial\hat{\theta}_1}{\partial K_3}=-\frac{1}{\hat{\delta}^2\hat{p}}(1-10\hat{p}+12\hat{p}^2),\ \frac{\partial\hat{\theta}_1}{\partial K_4}=-\frac{1}{\hat{\delta}^3\hat{p}}(1-3\hat{p})$$

$$\frac{\partial\hat{\theta}_2}{\partial K_1}=1,\ \frac{\partial\hat{\theta}_2}{\partial K_2}=0,\ \frac{\partial\hat{\theta}_2}{\partial K_3}=-\frac{1}{\hat{\delta}^2(1-\hat{p})}(3-14\hat{p}+12\hat{p}^2),\ \frac{\partial\hat{\theta}_2}{\partial K_4}=-\frac{1}{\hat{\delta}^3(1-\hat{p})}(2-3\hat{p})$$

$$\frac{\partial\hat{p}}{\partial K_1}=0,\ \frac{\partial\hat{p}}{\partial K_2}=0,\ \frac{\partial\hat{p}}{\partial K_3}=-\frac{4}{\hat{\delta}^3}(1-6\hat{p}+6\hat{p}^2),\ \frac{\partial\hat{p}}{\partial K_4}=-\frac{3}{\hat{\delta}^4}(1-2\hat{p})$$

$$\frac{\partial\hat{\sigma}^2}{\partial K_1}=0,\ \frac{\partial\hat{\sigma}^2}{\partial K_2}=1,\ \frac{\partial\hat{\sigma}^2}{\partial K_3}=\frac{2}{\hat{\delta}}(1-2\hat{p}),\ \frac{\partial\hat{\sigma}^2}{\partial K_4}=\frac{1}{\hat{\delta}^2}$$

$(\hat{\theta}_1, \hat{\theta}_2, \hat{p}_1\ \hat{\sigma}^2)=(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4)$ と記すと asymptotic variance matrix

$\Sigma=[\sigma_{ij}]_{j=1,2,3,4}^{i=1,2,3,4}$ $\sigma_{ij}=\mathrm{Cov}(\hat{\theta}_i, \hat{\theta}_j)$ を求めるために Cramer の公式 (n^{-1} の order まで)

$$\sigma_{ij}=\sum_{h=1}^{4}V(K_h)\left[\frac{\partial\hat{\theta}_i}{\partial K_h}\cdot\frac{\partial\hat{\theta}_j}{\partial K_h}\right]_{\hat{\theta}=\theta}+\sum_{h\neq h}\sum\left[\frac{\partial\hat{\theta}_i}{\partial K_h}\cdot\frac{\partial\hat{\theta}_j}{\partial K_{h'}}\right]_{\hat{\theta}=\theta}\mathrm{Cov}(K_i, K_j)$$ を用いる．また

$$V(K_1)=\frac{1}{n}\kappa_2$$

$$V(K_2)=\frac{1}{n}(\kappa_4+2\kappa_2^2)$$

$$V(K_3)=\frac{1}{n}(\kappa_6+9\kappa_4\kappa_2+9\kappa_3^2+6\kappa_2^3)$$

$$V(K_4)=\frac{1}{n}(\kappa_8+16\kappa_6\kappa_2+48\kappa_5\kappa_3+34\kappa_4^2+72\kappa_4\kappa_2^2+144\kappa_3^2\kappa_2+24\kappa_2^4)$$

$$\mathrm{Cov}\ (K_1, K_2)=\frac{1}{n}\kappa_3$$

$$\mathrm{Cov}\ (K_1, K_3)=\frac{1}{n}\kappa_4$$

$$\mathrm{Cov}\ (K_1, K_4)=\frac{1}{n}\kappa_5$$

$$\mathrm{Cov}\ (K_2, K_3)=\frac{1}{n}(\kappa_5+6\kappa_2\kappa_3)$$

$$\mathrm{Cov}\ (K_2, K_4)=\frac{1}{n}(\kappa_6+8\kappa_4\kappa_2+6\kappa_3^2)$$

$$\mathrm{Cov}\ (K_3, K_4)=\frac{1}{n}(\kappa_7+12\kappa_5\kappa_2+30\kappa_4\kappa_3+36\kappa_3\kappa_2^2)$$

ただし n^{-1} の order の近似式で n は標本の大きさである．
上記の式を用いて次の結果を得る．

$$\sigma_{11}=V(\hat{\theta}_1)=\frac{1}{n}\left\{\frac{\sigma^2}{p}+\frac{18(1-p)\sigma^2}{p\Delta^2}+\frac{6(1-4p+12p^2)\sigma^2}{p^2\Delta^4}+\frac{24(1-3p)^2\sigma^2}{p^2\Delta^6}\right\}$$

$$\sigma_{22}=V(\hat{\theta}_2)=\frac{1}{n}\left\{\frac{\sigma^2}{1-p}+\frac{18p\sigma^2}{(1-p)\Delta^2}+\frac{6(9-20p+12p^2)\sigma^2}{(1-p)^2\Delta^4}+\frac{24(2-3p)^2\sigma^2}{(1-p)^2\Delta^6}\right\}$$

$$\sigma_{33}=V(\hat{p})=\frac{1}{n}\left\{p(1-p)+72p(1-p)\frac{1}{\Delta^4}+96(1-3p+3p^2)\frac{1}{\Delta^6}+216(1-2p)^2\frac{1}{\Delta^8}\right\}$$

$$\sigma_{44}=V(\hat{\sigma}^2)=\frac{1}{n}\left\{2\sigma^4+\frac{24\sigma^6}{\Delta^2}+\frac{24\sigma^8}{\Delta^4}\right\}$$

$$\sigma_{12}=\mathrm{Cov}\ (\hat{\theta}_1,\hat{\theta}_2)=\frac{1}{n}\left\{\frac{18\sigma^2}{\Delta^2}+\frac{18(1-2p)^2\sigma^2}{p(1-p)\Delta^4}+\frac{24(2-9p+9p^2)\sigma^2}{p(1-p)\Delta^6}\right\}$$

$$\sigma_{13}=\mathrm{Cov}\ (\hat{\theta}_1, \hat{p})=\frac{1}{n}\left\{\frac{36(1-p)\sigma}{\Delta^3}+\frac{24(1-4p+6p^2)\sigma}{p\Delta^5}+\frac{72(1-3p)(1-2p)\sigma}{p\Delta^7}\right\}$$

$$\sigma_{23}=\mathrm{Cov}\,(\hat{\theta}_2,\hat{p})=\frac{1}{n}\left\{\frac{36p\sigma}{\Delta^3}+\frac{24(3-8p+6p^2)\sigma}{(1-p)\Delta^5}+\frac{72(2-3p)(1-2p)\sigma}{(1-p)\Delta^7}\right\}$$

$$\sigma_{14}=\mathrm{Cov}\,(\hat{\theta}_1,\hat{\sigma}^2)=\frac{1}{n}\left\{-\frac{12(1-4p)\sigma^3}{p\Delta^3}-\frac{24(1-3p)\sigma^3}{p\Delta^5}\right\}$$

$$\sigma_{24}=\mathrm{Cov}\,(\hat{\theta}_2,\hat{\sigma}^2)=\frac{1}{n}\left\{\frac{12(3-4p)\sigma^3}{(1-p)\Delta^3}+\frac{24(2-3p)\sigma^3}{(1-p)\Delta^5}\right\}$$

$$\sigma_{34}=\mathrm{Cov}\,(\hat{p},\hat{\sigma}^2)=\frac{1}{n}\left\{-24(1-2p)\left(\frac{2}{\Delta^4}+\frac{3}{\Delta^6}\right)\sigma^2\right\}$$

ただし $\Delta=(\theta_2-\theta_1)/\sigma$ である.

よって

$$n\Sigma=\begin{pmatrix}\frac{\sigma^2}{p} & 0 & 0 & 0\\ 0 & \frac{\sigma^2}{1-p} & 0 & 0\\ 0 & 0 & p(1-p) & 0\\ 0 & 0 & 0 & 2\sigma^4\end{pmatrix}+\begin{pmatrix}0(\Delta^{-2}) & 0(\Delta^{-2}) & 0(\Delta^{-3}) & 0(\Delta^{-3})\\ 0(\Delta^{-2}) & 0(\Delta^{-2}) & 0(\Delta^{-3}) & 0(\Delta^{-3})\\ 0(\Delta^{-3}) & 0(\Delta^{-3}) & 0(\Delta^{-4}) & 0(\Delta^{-4})\\ 0(\Delta^{-3}) & 0(\Delta^{-3}) & 0(\Delta^{-4}) & 0(\Delta^{-2})\end{pmatrix}$$

3. information matrix と asymptotic relative efficiency

$(\theta_1,\ \theta_2,\ \theta_3,\ \theta_4)=(\theta_1\ \theta_2\ p,\ \sigma^2)$ の information matrix は

$$G=\left[E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_i}\right)\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_j}\right)\right]_{\substack{i=1,\,2,\,3,\,4\\ j=1,\,2,\,3,\,4}}$$

である. この maxtrix の element を求めると次のようになる.

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_1}\right)^2=\frac{p^2}{\sigma^2}\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{f_1^2(x)}{f(x)}dx$$

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_2}\right)^2=\frac{(1-p)^2}{\sigma^2}\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_2^2(x)}{f(x)}dx$$

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial p}\right)^2=\int_{-\infty}^{\infty}\frac{\{f_1(x)-f_2(x)\}^2}{f(x)}dx$$

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial\sigma^2}\right)^2=\frac{1}{4\sigma^4}\int_{-\infty}^{\infty}\frac{\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2f_2(x)-f(x)\right\}^2}{f(x)}dx$$

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_1}\right)\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_2}\right)=\frac{p(1-p)}{\sigma^2}\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx$$

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_1}\right)\left(\frac{\partial\log f(\mathrm{x})}{\partial p}\right)=\frac{p}{\sigma}\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)\{f_1(x)-f_2(x)\}}{f(x)}dx$$

$$E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_1}\right)\left(\frac{\partial\log f(\mathrm{x})}{\partial\sigma^2}\right)$$

$$=\frac{p}{2\sigma^2}\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2f_2(x)-f(x)\right\}}{f(x)}dx$$

$$E\left(\frac{\partial \log f(x)}{\partial \theta_2}\right)\left(\frac{\partial \log f(x)}{\partial p}\right)=\frac{1-p}{\sigma}\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_2(x)\{f_1(x)-f_2(x)\}}{f(x)}dx$$

$$E\left(\frac{\partial \log f(x)}{\partial \theta_2}\right)\left(\frac{\partial \log f(x)}{\partial \sigma^2}\right)$$

$$=\frac{1-p}{2\sigma^2}\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_2(x)\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)-f(x)\right\}}{f(x)}dx$$

$$E\left(\frac{\partial \log f(x)}{\partial p}\right)\left(\frac{\partial \log f(x)}{\partial \sigma^2}\right)$$

$$=\frac{1}{2\sigma^2}\int_{-\infty}^{\infty}\frac{\left\{f_1(x)-f_2(x)\right\}\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1+p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)-f(x)\right\}}{f(x)}dx$$

このことと後で述べる証明を用いて

$$G=\begin{pmatrix}\frac{p}{\sigma^2} & 0 & 0 & 0\\ 0 & \frac{1-p}{\sigma^2} & 0 & 0\\ 0 & 0 & \frac{1}{p(1-p)} & 0\\ 0 & 0 & 0 & \frac{1}{2\sigma^4}\end{pmatrix}+\begin{pmatrix}0(\Delta^2 e^{-\frac{\Delta^2}{2}}) & 0(\Delta^2 e^{-\frac{\Delta^2}{2}}) & 0(\Delta e^{-\frac{\Delta^2}{2}}) & 0(\Delta^3 e^{-\frac{\Delta^2}{2}})\\ 0(\Delta^2 e^{-\frac{\Delta^2}{2}}) & 0(\Delta^2 e^{-\frac{\Delta^2}{2}}) & 0(\Delta e^{-\frac{\Delta^2}{2}}) & 0(\Delta^3 e^{-\frac{\Delta^2}{2}})\\ 0(\Delta e^{-\frac{\Delta^2}{2}}) & 0(\Delta e^{-\frac{\Delta^2}{2}}) & 0(e^{-\frac{\Delta^2}{2}}) & 0(\Delta^2 e^{-\frac{\Delta^2}{2}})\\ 0(\Delta^3 e^{-\frac{\Delta^2}{2}}) & 0(\Delta^3 e^{-\frac{\Delta^2}{2}}) & 0(\Delta^2 e^{-\frac{\Delta^2}{2}}) & 0(\Delta^4 e^{-\frac{\Delta^2}{2}})\end{pmatrix}$$

よって $\Delta=\frac{\theta_2-\theta_1}{\sigma}\to\infty$ のとき

$$Det.\,[n\Sigma\cdot G]^{-1}\to 1$$

となる.

4. 証明の補足

$$\mathrm{JARE}(\hat{\theta}_1,\ \hat{\theta}_2,\ \hat{p},\ \hat{\sigma}^2)=\mathrm{Det.}\,[n\Sigma\cdot \mathrm{G}]^{-1}$$

とする. moment 法による推定においては一般に Det. $[n\Sigma\cdot \mathrm{G}]^{-1}\leqq 1$ であるが $\Delta=\frac{\theta_2-\theta_1}{\sigma}\to\infty$ のとき Det. $[n\Sigma\cdot \mathrm{G}]^{-1}\to 1$ になることを示す. このため次の lemma をあげておく.

Lemma. (Markov の不等式)

$c>o$ ならば

(1) $\int_c^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\leqq e^{-\frac{c^2}{2}}$

(2) $\int_c^{\infty}z\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=\frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}$

(3) $\int_c^\infty z^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \leqq \frac{1}{\sqrt{2\pi}}(c+1) e^{-\frac{c^2}{2}}$

(4) $\int_c^\infty z^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}}(c^2+2) e^{-\frac{c^2}{2}}$

(5) $\int_c^\infty z^4 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \leqq \frac{1}{\sqrt{2\pi}}(c^3+3c+3) e^{-\frac{c^2}{2}}$

$c < o$ ならば

(1)′ $\int_{-\infty}^c \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \leqq e^{-\frac{c^2}{2}}$

(2)′ $\int_{-\infty}^c z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}$

(3)′ $\int_{-\infty}^c z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \leqq \frac{1}{\sqrt{2\pi}}(-c+1) e^{-\frac{c^2}{2}}$

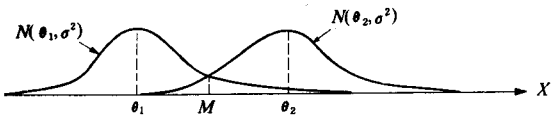
(4)′ $\int_{-\infty}^c z^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}}(c^2+2) e^{-\frac{c^2}{2}}$

(5)′ $\int_{-\infty}^c z^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \leqq \frac{1}{\sqrt{2\pi}}(-c^3-3c+3) e^{-\frac{c^2}{2}}$

この lemma を用いて次式を証明する.

ただし, $\theta_1<\theta_2$ でMは $f_1(x)=\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta_1)^2}{2\sigma^2}}$ 及び $f_2(x)=\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta_2)^2}{2\sigma^2}}$ の交点のX座標であり

$c_1=\frac{M-\theta_1}{\sigma}$,　$c_2=\frac{M-\theta_2}{\sigma}$,　$\Delta=\frac{\theta_2-\theta_1}{\sigma}$ とする.



(6) $\left| E\left(\frac{\partial \log f(x)}{\partial \theta_1}\right)^2 - \frac{p}{\sigma^2} \right|$

$= \frac{p^2}{\sigma^2} \left| \int_{-\infty}^\infty \left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x) \left\{\frac{f_1(x)}{f(x)} - \frac{1}{p}\right\} dx \right|$

$= \frac{p^2}{\sigma^2} \left| \int_{-\infty}^\infty \left(\frac{x-\theta_1}{\sigma}\right)^2 \frac{(1-p) f_1(x) f_2(x)}{p f(x)} dx \right|$

$$=\frac{p(1-p)}{\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\leqq\frac{p(1-p)}{\sigma^2}\left|\int_{-\infty}^{M}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{1}{p}f_2(x)\,dx\right|+\frac{p(1-p)}{\sigma^2}\left|\int_{M}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{1}{1-p}f_1(x)\,dx\right.$$

$$=\frac{1-p}{\sigma^2}\int_{-\infty}^{M}\left(\frac{x-\theta_1}{\sigma}+\frac{\theta_2-\theta_1}{\sigma}\right)^2f_2(x)\,dx+\frac{p}{\sigma^2}\int_{M}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2f_1(x)\,dx$$

$$\leqq\frac{1-p}{\sigma^2}\int_{-\infty}^{\frac{M-\theta_2}{\sigma}}\left\{z^2+\frac{2(\theta_2-\theta_1)}{\sigma}z+\left(\frac{\theta_2-\theta_1}{\sigma}\right)^2\right\}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$+\frac{p}{\sigma^2}\int_{\frac{M-\theta_1}{\sigma}}^{\infty}z^2\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$\leqq\frac{1-p}{\sigma^2}\left\{\frac{-c_2+1}{\sqrt{2\pi}}e^{-\frac{c_2^2}{2}}+\frac{2(\theta_2-\theta_1)}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{c_2^2}{2}}+\left(\frac{\theta_2-\theta_1}{\sigma}\right)^2e^{-\frac{c_1^2}{2}}\right\}$$

$$+\frac{p}{\sigma^2}\frac{c_1+1}{\sqrt{2\pi}}e^{-\frac{c_1^2}{2}}=0(\Delta^2e^{-\frac{\Delta^2}{2}})$$

同様にして

(7) $$\left|E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_2}\right)^2-\frac{1-p}{\sigma^2}\right|\leqq0(\Delta^2e^{-\frac{\Delta^2}{2}})$$

(8) $$\left|E\left(\frac{\partial\log f(\mathrm{x})}{\partial p}\right)^2-\frac{1}{p(1-p)}\right|$$

$$=\left|\int_{-\infty}^{\infty}\frac{\{f_1(x)-f_2(x)\}^2}{f(x)}dx-\frac{1}{p}-\frac{1}{1-p}\right|$$

$$=\left|\int_{-\infty}^{\infty}\left\{\frac{(f_1(x)-f_2(x))^2}{f(x)}-\frac{f_1(x)}{p}-\frac{f_2(x)}{1-p}\right\}dx\right|$$

$$\leqq\frac{1}{p(1-p)}\left\{\left|\int_{-\infty}^{M}\frac{f_1(x)f_2(x)}{pf_1(x)}dx\right|+\left|\int_{M}^{\infty}\frac{f_1(x)f_2(x)}{(1-p)f_2(x)}dx\right|\right\}$$

$$=\frac{1}{p^2(1-p)}\int_{-\infty}^{\frac{M-\theta_2}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz+\frac{1}{p(1-p)^2}\int_{\frac{M-\theta_1}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$\leqq\frac{1}{p^2(1-p)}e^{-\frac{c_2^2}{2}}+\frac{1}{p(1-p)^2}e^{-\frac{c_1^2}{2}}$$

$$=0(e^{-\frac{\Delta^2}{2}})$$

(9) $$\left|E\left(\frac{\partial\log f(\mathrm{x})}{\partial\sigma^2}\right)^2-\frac{1}{2\sigma^4}\right|$$

$$=\frac{1}{4\sigma^4}\left|\int_{-\infty}^{\infty}\frac{\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)-f(x)\right\}^2}{f(x)}dx-2\right|$$

$$=\frac{1}{4\sigma^4}\left|\int_{-\infty}^{\infty}\left[\frac{1}{f(x)}\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)\right\}^2-2\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)\right.\right.\right.$$

$$\left.\left.\left.+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)\right\}\right]dx+1-2\right|$$

$$=\frac{1}{4\sigma^4}\left|\int_{-\infty}^{\infty}\frac{1}{f(x)}\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)\right\}^2 dx-3\right|$$

$$=\frac{1}{4\sigma^4}\left|\int_{-\infty}^{\infty}\left[\frac{1}{f(x)}\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)f_2(x)\right\}^2\right.\right.$$

$$\left.\left.-\frac{3\{pf_1(x)+(1-p)f_2(x)\}^2}{f(x)}\right]dx\right|$$

$$=\frac{1}{4\sigma^4}\left|p^2\int_{-\infty}^{\infty}\left\{\left(\frac{x-\theta_1}{\sigma}\right)^4-3\right\}\frac{f_1^2(x)}{f(x)}dx+(1-p)^2\int_{-\infty}^{\infty}\left\{\left(\frac{x-\theta_2}{\sigma}\right)^4-3\right\}\frac{f_2^2(x)}{f(x)}dx\right.$$

$$\left.+2p(1-p)\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx-6p(1-p)\int_{-\infty}^{\infty}\frac{f_1(x)-f_2(x)}{f(x)}dx\right|$$

$$\leqq\frac{1}{4\sigma^4}\left[p^2\left|\int_{-\infty}^{\infty}\left\{\left(\frac{x-\theta_1}{\sigma}\right)^4-3\right\}\frac{f_1^2(x)}{f(x)}dx\right|+(1-p)^2\left|\int_{-\infty}^{\infty}\left\{\left(\frac{x-\theta_2}{\sigma}\right)^4-3\right\}\frac{f_2^2(x)}{f(x)}dx\right|\right.$$

$$\left.+2p(1-p)\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|+6p(1-p)\left|\int_{-\infty}^{\infty}\frac{f_1(x)f_2(x)}{f(x)}dx\right|\right]$$

$$\leqq\frac{1}{4\sigma^4}\left[p(1-p)\left|\int_{-\infty}^{\infty}\left\{\left(\frac{x-\theta_1}{\sigma}\right)^4-3\right\}\frac{f_1(x)f_2(x)}{f(x)}dx\right|+p(1-p)\left|\int_{-\infty}^{\infty}\left\{\left(\frac{x-\theta_2}{\sigma}\right)^4-3\right\}\right.\right.$$

$$\left.\frac{f_1(x)f_2(x)}{f(x)}dx\right|+2p(1-p)\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\left.+6p(1-p)\left|\int_{-\infty}^{\infty}\frac{f_1(x)f_2(x)}{f(x)}dx\right|\right]$$

$$\leqq\frac{1}{4\sigma^4}p(1-p)\left[\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^4\frac{f_1(x)f_2(x)}{f(x)}dx\right|+\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)^4\frac{f_1(x)f_2(x)}{f(x)}dx\right|\right.$$

$$\left.+2\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|+12\left|\int_{-\infty}^{\infty}\frac{f_1(x)f_2(x)}{f(x)}dx\right|\right]$$

$$=0\left(\Delta^4 e^{-\frac{\Delta^2}{2}}\right)$$

(10) $$\left|E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_1}\right)\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_2}\right)\right|$$

$$=\frac{p(1-p)}{\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\leqq\frac{p(1-p)}{\sigma^2}\left[\left|\int_{-\infty}^{M}\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|\right.$$

$$+\left|\int_M^\infty\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|\Bigg]$$

$$=\frac{p(1-p)}{\sigma^2}\Bigg[\left|\int_{-\infty}^M\left(\frac{x-\theta_2}{\sigma}+\frac{\theta_2-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$+\left|\int_M^\infty\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_1}{\sigma}-\frac{\theta_2-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|\Bigg]$$

$$\leqq\frac{p(1-p)}{\sigma^2}\Bigg[\frac{1}{p}\left|\int_{-\infty}^M\left(\frac{x-\theta_2}{\sigma}\right)^2f_2(x)dx\right|+\frac{\theta_2-\theta_1}{\sigma p}\left|\int_{-\infty}^M\left(\frac{x-\theta_2}{\sigma}\right)f_2(x)dx\right|$$

$$+\frac{1}{1-p}\left|\int_M^\infty\left(\frac{x-\theta_1}{\sigma}\right)^2f_1(x)dx\right|+\frac{\theta_2-\theta_1}{\sigma(1-p)}\left|\int_M^\infty\left(\frac{x-\theta_1}{\sigma}\right)f_1(x)dx\right|\Bigg]$$

$$=0(\Delta^2e^{-\frac{\Delta^2}{2}})$$

(11) $$\left|E\left(\frac{\partial\log f(\mathrm{x})}{\partial\theta_1}\right)\left(\frac{\partial\log f(\mathrm{x})}{\partial p}\right)\right|$$

$$=\frac{p}{\sigma}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1^2(x)-f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\leqq\frac{p}{\sigma}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1^2(x)}{f(x)}dx\right|+\frac{p}{\sigma}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$=\frac{p}{\sigma}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\left\{\frac{f_1(x)}{p}-\frac{1-p}{p}\frac{f_1(x)f_2(x)}{f(x)}\right\}dx\right|+\frac{p}{\sigma}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right.$$

$$=\frac{p}{\sigma}\Bigg[\frac{1}{p}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)f_1(x)dx\right|+\frac{1-p}{p}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$+\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|\Bigg]$$

$$=\frac{1}{\sigma}\left|\int_{-\infty}^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\leqq\frac{1}{\sigma}\Bigg[\left|\int_{-\infty}^M\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|+\left|\int_M^\infty\left(\frac{x-\theta_1}{\sigma}\right)\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\leqq\frac{1}{\sigma}\Bigg[\frac{1}{p}\left|\int_{-\infty}^M\left(\frac{x-\theta_1}{\sigma}\right)f_2(x)dx\right|+\frac{\theta_2-\theta_1}{\sigma p}\left|\int_{-\infty}^M f_2(x)dx\right|$$

$$+\frac{1}{1-p}\left|\int_M^\infty\left(\frac{x-\theta_1}{\sigma}\right)f_1(x)dx\right|\Bigg]$$

$$=\frac{1}{\sigma}\Bigg[\frac{1}{p}\left|\int_{-\infty}^{\frac{M-\theta_2}{\sigma}}z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right|+\frac{1}{p}\left(\frac{\theta_2-\theta_1}{\sigma}\right)\left|\int_{-\infty}^{\frac{M-\theta_2}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right|$$

$$+\frac{1}{1-p}\left|\int_{\frac{M-\theta_1}{\sigma}}^\infty z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right|\Bigg]$$

$$\leqq \frac{1}{\sigma}\left[\frac{1}{p}\ \frac{1}{\sqrt{2\pi}}e^{-\frac{c_2^2}{2}}+\frac{1}{p}\left(\frac{\theta_2-\theta_1}{\sigma}\right)e^{-\frac{c_2^2}{2}}+\frac{1}{1-p}\ \frac{1}{\sqrt{2\pi}}e^{-\frac{c_1^2}{2}}\right]$$

$$=0(\Delta e^{-\frac{\Delta^2}{2}})$$

同様にして

(12) $$\left| E\left(\frac{\partial \log f(\mathrm{x})}{\partial \theta_2}\right)\left(\frac{\partial \log f(\mathrm{x})}{\partial p}\right)\right| \leqq 0(\Delta e^{-\frac{\Delta^2}{2}})$$

(13) $$\left| E\left(\frac{\partial \log f(\mathrm{x})}{\partial \theta_1}\right)\left(\frac{\partial \log f(\mathrm{x})}{\partial \sigma^2}\right)\right|$$

$$=\frac{p}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)-f(x)\right\}\frac{f_1(x)}{f(x)}dx\right|$$

$$\leqq \frac{p^2}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^3\frac{f_1^2(x)}{f(x)}dx\right|+\frac{p(1-p)}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$+\frac{p}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)f_1(x)dx\right|$$

$$\leqq \frac{p}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^3 f_1(x)dx\right|+\frac{p(1-p)}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^3\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$+\frac{p(1-p)}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$\leqq \frac{p(1-p)}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^3\frac{f_1(x)f_2(x)}{f(x)}dx\right|+\frac{p(1-p)}{2\sigma^2}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)^3\frac{f_1(x)f_2(x)}{f(x)}dx\right.$$

$$+\frac{p(1-p)(\theta_2-\theta_1)}{2\sigma^3}\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|$$

$$=0\ \ (\Delta^3 e^{-\frac{\Delta^2}{2}})$$

同様にして

(14) $$\left| E\left(\frac{\partial \log f(\mathrm{x})}{\partial \theta_2}\right)\left(\frac{\partial \log f(\mathrm{x})}{\partial \sigma^2}\right)\right| \leqq 0(\Delta^3 e^{-\frac{\Delta^2}{2}})$$

(15) $$\left| E\left(\frac{\partial \log f(\mathrm{x})}{\partial p}\right)\left(\frac{\partial \log f(\mathrm{x})}{\partial \sigma^2}\right)\right|$$

$$=\frac{1}{2\sigma^2}\left|\int_{-\infty}^{\infty}\frac{\{f_1(x)-f_2(x)\}\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)\right\}}{f\ (x)}dx\right|$$

$$=\frac{1}{2\sigma^2}\left|\int_{-\infty}^{\infty}\frac{\{f_1(x)-f_2(x)\}\left\{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2(x)\right\}}{f\ (x)}dx\right.$$

$$\left.-\int_{-\infty}^{\infty}\{f_1(x)-f_2(x)\}dx\right|$$

$$=\frac{1}{2\sigma^2}\left|\int_{-\infty}^{\infty}\frac{p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1^2(x)-p\left(\frac{x-\theta_1}{\sigma}\right)^2 f_1(x)\quad(2)}{f(x)}\right.$$

$$\left.\frac{+(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_1(x)f_2(x)-(1-p)\left(\frac{x-\theta_2}{\sigma}\right)^2 f_2^2(x)}{}dx\right|$$

$$\leqq\frac{1}{2\sigma^2}\left[p\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{f_1^2(x)}{f(x)}dx\right|+p\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_1}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|\right.$$

$$\left.+(1-p)\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_1(x)f_2(x)}{f(x)}dx\right|+(1-p)\left|\int_{-\infty}^{\infty}\left(\frac{x-\theta_2}{\sigma}\right)^2\frac{f_2^2(x)}{f(x)}dx\right|\right]$$

$$=0\ (\Delta^2 e^{-\frac{\Delta^2}{2}})$$

(注) $\frac{f_1^2(x)}{f(x)}=\frac{f_1(x)\cdot pf_1(x)}{pf(x)}=\frac{f_1(x)\{pf_1(x)+(1-p)f_2(x)\}-(1-p)f_1(x)f_2(x)}{pf(x)}$

$$=\frac{f_1(x)}{p}-\frac{1-p}{p}\ \frac{f_1(x)f_2(x)}{f(x)}$$

同様にして

$$\frac{f_2^2(x)}{f(x)}=\frac{f_2(x)}{1-p}-\frac{p}{1-p}\ \frac{f_1(x)f_2(x)}{f(x)}$$

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