On the Minimum Length of Linear Codes over the Field of 9 Elements

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Abstract

We construct a lot of new $[n, 4, d]_9$ codes whose lengths are close to the Griesmer bound and prove the nonexistence of some linear codes attaining the Griesmer bound using some geometric techniques through projective geometries to determine the exact value of $n_9(4, d)$ or to improve the known bound on $n_9(4, d)$ for given values of d, where $n_q(k, d)$ is the minimum length n for which an $[n, k, d]_q$ code exists. We also give the updated table for $n_9(4, d)$ for all d except some known cases.

Keywords: optimal linear code; Griesmer bound; projective dual; geometric puncturing

1 Introduction

Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. An $[n, k, d]_q$ code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming weight $d = \min\{wt(c) \mid c \in$

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$\mathcal{C}, c \neq (0, \dots, 0)\}$, where $wt(c)$ is the number of non-zero entries in c. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. The weight distribution $(A_0, A_d, \dots) = (1, \alpha, \dots)$ is also expressed as $0^1 d^\alpha \cdots$. A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists ([8]). There is a natural lower bound on $n_q(k, d)$, the Griesmer bound: $n_q(k, d) \geqslant g_q(k, d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k, d)$ are determined for all d only for some small values of q and k. The problem to determine $n_q(4, d)$ for all d has been solved for $q = 2, 3, 4$ but not for $q \geqslant 5$. For $k = 3$, $n_q(3, d)$ is known for all d for $q \leqslant 9$. In this paper, we tackle the problem to determine $n_9(4, d)$ for all d. See [25] for the updated table of $n_q(k, d)$ for some small q and k. The following results are already known for $n_9(k, d)$ with $k = 3, 4$, see [5, 14, 15, 16, 17, 19, 21, 24, 25, 27].

Theorem 1.1. *$n_9(3, d) = g_9(3, d) + 1$ for $d = 9, 15$-$18, 25$-$27, 33$-$36, 43$-$45, 49$-$54, 58$-63 and $n_9(3, d) = g_9(3, d)$ for any other d.*

Theorem 1.2. (1) *$n_9(4, d) = g_9(4, d)$ for $d =$ 1-7, 10-12, 19, 28-30, 64-72, 568-576, 640-801, 1054-1080, and for $d \geqslant 1216$.*

(2) *$n_9(4, d) = g_9(4, d) + 1$ for $d = 8, 9, 13$-18, 25-27, 33, 34, 49, 58-63, 73-80, 559-562, 592-594, 601-603, 610-612, 622-639, 1036-1053, 1198-1215.*

(3) *$n_9(4, d) \leqslant g_9(4, d) + 1$ for $d =$ 20-24, 31, 32, 37-40, 46-48, 55-57, 82-88, 91-94, 100-102, 577-621, 1000-1035, 1135-1197.*

(4) *$g_9(4, d) + 1 \leqslant n_9(4, d) \leqslant g_9(4, d) + 2$ for $d = 35, 36$, 43-45, 50-54, 81, 514-558, 563-567.*

(5) *$n_9(4, d) \geqslant g_9(4, d) + 1$ for $d =$ 127-162, 217-243, 289-324, 379-405, 433-486, 964-972, 1108-1134.*

(6) *$n_9(4, d) \leqslant g_9(4, d) + 2$ for $d = 41, 42, 89, 90, 95$-99, 103-112, 487-513.*

A $[g_9(4, d), 4, d]_9$ code does not exist for $d \in \{$127-162, 217-243, 289-324, 379-405, 433-486, 514-567$\}$ since its residual $[g_9(4, d) - d, 3, \lceil d/9 \rceil]_5$ code does not exist by Theorem 1.1. It follows from the following result that $n_9(4, d) \leqslant g_9(4, d) + 1$ for $d \in \{$577-639, 1135-1215$\}$ and that $n_9(4, d) \leqslant g_9(4, d) + 2$ for $487 \leqslant d \leqslant 567$.

Theorem 1.3 ([15])**.** (1) *There exist $[\theta_3 - \theta_1 x, 4, q^3 - qx]_q$ codes for $0 \leqslant x \leqslant q^2 - 1$.*
(2) *There exist $[2q^3 - \theta_1 x, 4, 2q^3 - 2q^2 - qx]_q$ codes for $0 \leqslant x \leqslant q^2 - 1$.*

Corollary 1.4. *There exist $[g_q(4, d) + 1, 4, d]_q$ codes for*

(1) *$q^3 - 2q^2 + q + 1 \leqslant d \leqslant q^3 - q^2 - q$ for $q \geqslant 3$,*

(2) *$2q^3 - 4q^2 + 1 \leqslant d \leqslant 2q^3 - 3q^2$ for $q \geqslant 4$.*

As for the exact values of $n_q(4, d)$ for arbitrary q, the following results are known. Some results in Theorem 1.2 are obtained from these theorems.

Theorem 1.5 ([21, 24]). *$n_q(4,d) = g_q(4,d)$ for*

(1) $1 \leqslant d \leqslant q-2$; $q^2-2q+1 \leqslant d \leqslant q^2-q$; $q^3-2q^2+1 \leqslant d \leqslant q^3-2q^2+q$; $q^3-q^2-q+1 \leqslant d \leqslant q^3+q^2-q$; $d \geqslant 2q^3-3q^2+1$ *for all* q,

(2) $2q^3-5q^2+1 \leqslant d \leqslant 2q^3-5q^2+3q$ *for* $q \geqslant 7$.

Theorem 1.6 ([16, 17, 21, 24]). *$n_q(4,d) = g_q(4,d)+1$ for*

(1) $2q^3-3q^2-q+1 \leqslant d \leqslant 2q^3-3q^2$ *for* $q \geqslant 4$,

(2) $2q^3-3q^2-2q+1 \leqslant d \leqslant 2q^3-3q^2-q$ *for* $q \geqslant 5$,

(3) $2q^3-3q^2-3q+1 \leqslant d \leqslant 2q^3-3q^2-2q$ *for* $q \geqslant 11$,

(4) $2q^3-5q^2-2q+1 \leqslant d \leqslant 2q^3-5q^2$ *for* $q \geqslant 9$.

We prove that Theorem 1.6 (3) is also valid for $q = 9$. Our new results are summarized to the following.

Theorem 1.7. (1) $n_9(4,d) = g_9(4,d)$ *for* $d =$ 811-837, 892-918, 973-999, 1135-1152.

(2) $n_9(4,d) = g_9(4,d)+1$ *for* $d =$ 136-144, 585, 617-621, 810, 883-891, 964-972, 1108-1134, 1189-1197.

(3) $n_9(4,d) \leqslant g_9(4,d)+1$ *for* $d =$ 172-180, 802-809, 838-882, 919-963, 1081-1107.

(4) $g_9(4,d)+1 \leqslant n_9(4,d) \leqslant g_9(4,d)+2$ *for* $d =$ 127-135, 145-152, 154-158, 214-216, 367-369, 512, 513.

(5) $n_9(4,d) \geqslant g_9(4,d)+1$ *for* $d = 125, 126, 198, 206, 207, 370\text{-}378$.

(6) $n_9(4,d) \leqslant g_9(4,d)+2$ *for* $d = 113, 114, 118\text{-}121$, 163-171, 181-188, 208-216, 244-252, 361-369.

We also give the updated table for $n_9(4,d)$ as Table 2. We give the values and bounds of $g = g_9(4,d)$ and $n = n_9(4,d)$ for all d except for $640 \leqslant d \leqslant 801$ and for $d \geqslant 1216$ which are the cases satisfying $n_9(4,d) = g_9(4,d)$ by Theorem 1.5. In the table, "*s*-*t*" stands for $g_9(4,d)+s \leqslant n_9(4,d) \leqslant g_9(4,d)+t$.

2 Preliminary results

In this section, we give the geometric methods to construct new codes or to prove the nonexistence of codes with certain parameters.

We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A j-flat is a projective subspace of dimension j in $\mathrm{PG}(r,q)$. The 0-flats, 1-flats, 2-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, secundums* and *hyperplanes*, respectively. We denote by θ_j the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted by $\mathcal{M}_{\mathcal{C}}$. We see linear codes from this geometrical point of view. An *i-point* is a point of Σ which has multiplicity i in $\mathcal{M}_{\mathcal{C}}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_{\mathcal{C}}$ and let C_i be the set of i-points in Σ, $0 \leqslant i \leqslant \gamma_0$. We denote by $\Delta_1 + \cdots + \Delta_s$ the multiset consisting of the s sets $\Delta_1, \ldots, \Delta_s$ in Σ. We write $s\Delta$ for $\Delta_1 + \cdots + \Delta_s$ when $\Delta_1 = \cdots = \Delta_s$. Then, $\mathcal{M}_{\mathcal{C}} = \sum_{i=1}^{\gamma_0} iC_i$. For any subset S of Σ, we denote by $\mathcal{M}_{\mathcal{C}}(S)$ the multiset $\{P \in \mathcal{M}_{\mathcal{C}} \mid P \in S\}$. The *multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, is defined as the cardinality of $\mathcal{M}_{\mathcal{C}}(S)$, i.e., $m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|$, where $|T|$ denotes the number of elements in a set T. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and $n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}$, where $\mathcal{F}_j$ denotes the set of j-flats in Σ. Such a partition of Σ is called an *$(n, n-d)$-arc* of Σ. Conversely an $(n, n-d)$-arc of Σ gives an $[n,k,d]_q$ code in the natural manner. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane*, a *t-hyperplane* and so on are defined similarly. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \leqslant j \leqslant m.$$

Let $\lambda_s(\Pi)$ be the number of s-points in Π. We denote simply by γ_j and by λ_s instead of $\gamma_j(\Sigma)$ and $\lambda_s(\Sigma)$, respectively. It holds that $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$. When $\mathcal{C}$ is Griesmer, the values $\gamma_0, \gamma_1, \ldots, \gamma_{k-3}$ are also uniquely determined ([22]) as follows:

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \leqslant j \leqslant k-1. \tag{2.1}$$

When $\gamma_0 = 2$, we obtain

$$\lambda_2 = \lambda_0 + n - \theta_{k-1} \tag{2.2}$$

from $\lambda_0 + \lambda_1 + \lambda_2 = \theta_{k-1}$ and $\lambda_1 + 2\lambda_2 = n$. Denote by a_i the number of i-hyperplanes in Σ. Note that $a_i = A_{n-i}/(q-1)$ for $0 \leqslant i \leqslant n-d$. The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following:

$$\sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1}, \tag{2.3}$$

$$\sum_{i=1}^{\gamma_{k-2}} i a_i = n\theta_{k-2}, \tag{2.4}$$

$$\sum_{i=2}^{\gamma_{k-2}} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} s(s-1)\lambda_s. \tag{2.5}$$

When $\gamma_0 \leqslant 2$, we get the following from (2.3)-(2.5):

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \quad (2.6)$$

If $a_i = 0$ for all $i < n-d$, then every point in Σ is an s-point for some integer s. This fact is known as follows.

Lemma 2.1 ([1]). *Any linear code over a finite field with constant Hamming weight is a replication of simplex (i.e., dual Hamming) codes.*

See also Theorem 2.3 in [20] for a geometric proof of the above lemma.

Lemma 2.2 ([29]). *Let Π be an i-hyperplane through a t-secundum δ. Then*
(1) $t \leqslant \gamma_{k-2} - \dfrac{n-i}{q} = \dfrac{i+q\gamma_{k-2}-n}{q}$.
(2) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \geqslant i - \left\lfloor \dfrac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.
(3) $\gamma_{k-3}(\Pi) = \left\lfloor \dfrac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ *if an* $[i, k-1, d_1]_q$ *code with* $d_1 \geqslant i - \left\lfloor \dfrac{i+q\gamma_{k-2}-n}{q} \right\rfloor + 1$ *does not exist.*
(4) *Let c_j be the number of j-hyperplanes through δ other than Π. Then $\sum_j c_j = q$ and*

$$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \quad (2.7)$$

(5) *For a γ_{k-2}-hyperplane Π_0 with spectrum $(\tau_0, \dots, \tau_{\gamma_{k-3}})$, $\tau_t > 0$ holds if $i + q\gamma_{k-2} - n - qt < q$.*

Lemma 2.3. *Let Π be an i-hyperplane and let $\mathcal{C}_\Pi$ be an $[i, k-1, d_0]$ code generated by $\mathcal{M}_\mathcal{C}(\Pi)$. If any γ_{k-2}-hyperplane has no t-secundum with $t = \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$, then $d_0 \geqslant i - t + 1$.*

Proof. We have $d_0 \geqslant i - t$ by Lemma 2.2 (1). Suppose that Π has a t-secundum. Since $(i + q\gamma_{k-2} - n)/q < t + 1$, it follows from Lemma 2.2 (5) that a γ_{k-2}-hyperplane has a t-secundum, a contradiction. Hence, $m_\mathcal{C}(\Pi) \geqslant t + 1$ and our assertion follows. □

Next, we give a method to construct good codes by some orbits of a given projectivity in PG$(k-1, q)$. For a non-zero element $\alpha \in \mathbb{F}_q$, let $R = \mathbb{F}_q[x]/(x^N - \alpha)$ be the ring of polynomials over $\mathbb{F}_q$ modulo $x^N - \alpha$. We associate the vector $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{F}_q^N$ with the polynomial $a(x) = \sum_{i=0}^{N-1} a_i x^i \in R$. For $\mathbf{g} = (g_1(x), \dots, g_m(x)) \in R^m$,

$$C_\mathbf{g} = \{(r(x)g_1(x), \dots, r(x)g_m(x)) \mid r(x) \in R\}$$

is called the 1-*generator quasi-twisted* (QT) *code* with generator $\mathbf{g}$. $C_{\mathbf{g}}$ is usually called quasi-cyclic (QC) when $\alpha = 1$. When $m = 1$, $C_{\mathbf{g}}$ is called α-*cyclic* or *pseudo-cyclic* or *constacyclic*. All of these codes are generalizations of cyclic codes ($\alpha = 1$, $m = 1$). Take a monic polynomial $g(x) = x^k - \sum_{i=0}^{k-1} a_i x^i$ in $\mathbb{F}_q[x]$ dividing $x^N - \alpha$ with non-zero $\alpha \in \mathbb{F}_q$, and let T be the companion matrix of $g(x)$. Let τ be the projectivity of $\mathrm{PG}(k-1,q)$ defined by T. We denote by $[g^n]$ or by $[a_0 a_1 \cdots a_{k-1}^n]$ the $k \times n$ matrix $[P, TP, T^2P, \ldots, T^{n-1}P]$, where P is the column vector $(1,0,0,\ldots,0)^{\mathrm{T}}$ (h^{T} stands for the transpose of a row vector h). Then $[g^N]$ generates an α^{-1}-cyclic code. Hence one can construct a cyclic or pseudo-cyclic code from an orbit of τ. For non-zero vectors $P_2^{\mathrm{T}}, \ldots, P_m^{\mathrm{T}} \in \mathbb{F}_q^k$, we denote the matrix

$$[P, TP, T^2P, \ldots, T^{n_1-1}P; P_2, TP_2, \ldots, T^{n_2-1}P_2; \cdots ; P_m, TP_m, \ldots, T^{n_m-1}P_m]$$

by $[g^{n_1}] + P_2^{n_2} + \cdots + P_m^{n_m}$. Then, the matrix $[g^N] + P_2^N + \cdots + P_m^N$ defined from m orbits of τ of length N generates a QC or QT code, see [30]. It is shown in [30] that many good codes can be constructed from orbits of projectivities.

An $[n,k,d]_q$ code is called m-*divisible* if all codewords have weights divisible by an integer $m > 1$. It sometimes happens that QC or QT codes are divisible or can be extended to divisible codes.

Lemma 2.4 ([31]). *Let $\mathcal{C}$ be an m-divisible $[n,k,d]_q$ code with $q = p^h$, p prime, whose spectrum is*

$$(a_{n-d-(w-1)m}, a_{n-d-(w-2)m}, \ldots, a_{n-d-m}, a_{n-d}) = (\alpha_{w-1}, \alpha_{w-2}, \ldots, \alpha_1, \alpha_0),$$

where $m = p^r$ for some $1 \leqslant r < h(k-2)$ satisfying $\lambda_0 > 0$. Then there exists a t-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $t = q^{k-2}/m$, $n^* = \sum_{j=0}^{w-1} j\alpha_j = ntq - \frac{d}{m}\theta_{k-1}$, $d^* = ((n-d)q - n)t$ whose spectrum is

$$(a_{n^*-d^*-\gamma_0 t}, a_{n^*-d^*-(\gamma_0-1)t}, \ldots, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_{\gamma_0}, \lambda_{\gamma_0-1}, \ldots, \lambda_1, \lambda_0).$$

Note that a generator matrix for $\mathcal{C}^*$ is given by considering $(n-d-jm)$-hyperplanes as j-points in the dual space Σ^* of Σ for $0 \leqslant j \leqslant w-1$ [31]. So, $\mathcal{C}^*$ is uniquely determined up to equivalence. $\mathcal{C}^*$ is called the *projective dual* of $\mathcal{C}$, see also [3] and [9].

Lemma 2.5 ([28]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code and let $\cup_{i=0}^{\gamma_0} C_i$ be the partition of $\Sigma = \mathrm{PG}(k-1,q)$ obtained from $\mathcal{C}$. If $\cup_{i\geqslant 1} C_i$ contains a t-flat Δ and if $d > q^t$, then there exists an $[n-\theta_t, k, d']_q$ code $\mathcal{C}'$ with $d' \geqslant d - q^t$.*

The punctured code $\mathcal{C}'$ in Lemma 2.5 can be constructed from $\mathcal{C}$ by removing the t-flat Δ from the multiset $\mathcal{M}_{\mathcal{C}}$. We denote the resulting multiset by $\mathcal{M}_{\mathcal{C}} - \Delta$. The method to construct new codes from a given $[n,k,d]_q$ code by deleting the coordinates corresponding to some geometric object in $\mathrm{PG}(k-1,q)$ is called *geometric puncturing*, see [24].

Lemma 2.6 ([11]). *Let $\mathcal{C}_1$ be an $[n_1,k,d_1]_q$ code and $\mathcal{C}_2$ be an $[n_2,k,d_2]_q$ code. Then an $[n_1+n_2, k, d_1+d_2]_q$ code $\mathcal{C}$ exists.*

Lemma 2.7 ([2]). *Let $\mathcal{C}_1$ be an $[n_1, k, d_1]_q$ code containing a codeword of weight $d_1 + m$ with $m > 0$ and let $\mathcal{C}_2$ be an $[n_2, k-1, d_2]_q$ code. Then there exists an $[n_1+n_2, k, d]_q$ code with $d = d_1 + m$ if $m < d_2$ and $d = d_1 + d_2$ if $m \geqslant d_2$.*

An $[n, k, d]_q$ code with generator matrix G is called *extendable* if there exists a vector $h \in \mathbb{F}_q^k$ such that the extended matrix $[G, h^{\mathrm{T}}]$ generates an $[n+1, k, d+1]_q$ code. The following theorems will be applied to prove the nonexistence of codes with certain parameters in Section 4.

Theorem 2.8 ([7, 10]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $gcd(d, q) = 1$ whose weights are congruent to 0 or d modulo q. Then $\mathcal{C}$ is extendable.*

Theorem 2.9 ([23, 32]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $q \geqslant 5$, $d \equiv -2 \pmod q$, $k \geqslant 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod q$.*

Theorem 2.10 ([4]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $q \geqslant 4$, whose weights are congruent to $0, -1$ or $-2 \pmod q$ and $d \equiv -1 \pmod q$. Then $\mathcal{C}$ is extendable except the case $(\Phi_0, \Phi_1) = (\binom{q}{2}q^{k-3} + \theta_{k-3}, \binom{q}{2}q^{k-3})$ with q odd, where $\Phi_0 = \frac{1}{q-1}\sum_{q|i, i>0} A_i$, $\Phi_1 = \frac{1}{q-1}\sum_{i \not\equiv 0, d \pmod q} A_i$.*

Next, we give a survey of the known results on $n_q(4, d)$ apart from Theorems 1.5 and 1.6.

Theorem 2.11 ([17, 21]). *$n_q(4, d) = g_q(4, d) + 1$ for*
(1) $q^2 - q + 1 \leqslant d \leqslant q^2 - 1$ *for* $q \geqslant 3$,
(2) $d = q^2$ *for* $q = 2^h$, $h \geqslant 2$,
(3) $q^3 - q^2 - rq - 2 \leqslant d \leqslant q^3 - q^2 - rq$ *with* $4 \leqslant r \leqslant 6$ *for* $q \geqslant 9$.

Theorem 2.12 ([21]). *$n_q(4, d) \geqslant g_q(4, d) + 1$ for*
(1) $d = q - 1, q$ *for* $q \geqslant 4$,
(2) $d = 2q - 1, 2q$ *for* $q \geqslant 4$,
(3) $(v-1)q - 2 \leqslant d \leqslant (v-1)q$ *for* $4 \leqslant v < q$ *with v not dividing q*,
(4) $2q^2 - 2q + 1 \leqslant d \leqslant 2q^2$ *for* $q \geqslant 4$,
(5) $(v-1)q^2 - 3q + 1 \leqslant d \leqslant (v-1)q^2$ *for* $4 \leqslant v < q$ *with v not dividing q*.

Theorem 2.13 ([21]). *For $q > r$, $r = 3, 4$ and for $q > 2(r-1)$, $r \geqslant 5$, it holds that $n_q(4, d) \geqslant g_q(4, d) + 1$ for $2q^3 - rq^2 - q + 1 \leqslant d \leqslant 2q^3 - rq^2$.*

Theorem 2.14 ([16]). *$n_q(k, d) \geqslant g_q(k, d) + 1$ for $(k-2)q^{k-1} - rq^{k-2} - q + 1 \leqslant d \leqslant (k-2)q^{k-1} - rq^{k-2}$ for $3 \leqslant k - 1 \leqslant r \leqslant q - q/p$, $q = p^h$ with p prime.*

Theorem 2.15 ([17]). *$n_q(4, d) \geqslant g_q(4, d) + 1$ for*
(1) $2q^3 - rq^2 - 2q + 1 \leqslant d \leqslant 2q^3 - rq^2 - q$ *for* $3 \leqslant r \leqslant (q+1)/2$, $q \geqslant 5$,
(2) $2q^3 - 4q^2 - 3q + 1 \leqslant d \leqslant 2q^3 - 4q^2 - 2q$ *for* $q \geqslant 9$.

Theorem 2.16 ([27]). *$n_q(4, d) \geqslant g_q(4, d) + 1$ for $q^3 - q^2 - (s+1)q + 1 \leqslant d \leqslant q^3 - q^2 - sq$ for* (1) $s = 1, q \geqslant 3$; (2) $s = 2, q \geqslant 4$; (3) $s = 3, q \geqslant 7, q \neq 9$.

By Theorem 2.16 and Corollary 1.4, we get the following.

Theorem 2.17. *$n_q(4,d) = g_q(4,d)+1$ for $q^3-q^2-3q+1 \leqslant d \leqslant q^3-q^2-q$ for $q \geqslant 4$.*

Theorem 2.18 ([5])**.** *There exist codes with parameters $[10,3,8]_9$, $[17,3,14]_9$, $[16,4,12]_9$, $[24,4,19]_9$, $[36,4,30]_9$, $[41,4,34]_9$, $[48,4,40]_9$, $[58,4,49]_9$, $[92,4,80]_9$, $[102,4,88]_9$, $[109,4,94]_9$, $[118,4,102]_9$, $[125,4,108]_9$ and $[130,4,112]_9$.*

3 New codes

In this section, we construct several $[n,4,d]_9$ codes to give the upper bounds on $n_9(4,d)$. Let $\mathbb{F}_9 = \{0,1,\alpha,\cdots,\alpha^7\}$, with $\alpha^2 = \alpha+1$. For simplicity, we denote $\alpha,\cdots,\alpha^7$ by $2,3,\cdots,8$ so that $\mathbb{F}_9 = \{0,1,2,\cdots,8\}$.

Lemma 3.1. *There exists a $[132,4,114]_9$ code.*

Proof. Let $\mathcal{C}$ be the $[132,4,114]_9$ code with generator matrix $G = [5510^{13}] + 1017^{13} + 1215^{13} + 1031^{13} + 1167^{13} + 1211^{13} + 1705^{13} + 1382^{13} + 1706^{13} + 1334^{13} + 1051^{1} + 1051^{1}$. Then $\mathcal{C}$ has weight distribution $0^1 114^{2288} 117^{2288} 120^{1248} 123^{416} 126^{312} 132^{8}$. □

Lemma 3.2. *There exists a $[s(q^2+1),4,sq(q-1)]_q$ code for $1 \leqslant s \leqslant q-1$ with spectrum $(a_s, a_{s(q+1)}) = (q^2+1, q^3+q)$.*

Proof. Let $\mathcal{C}$ be a $[q^2+1,4,q^2-q]_q$ code. Recall that $\mathcal{M_C}$ is just an elliptic quadric in $\mathrm{PG}(3,q)$ with spectrum $(a_1,a_{q+1}) = (q^2+1,q^3+q)$, see [13]. Hence, the multiset $s\mathcal{M_C}$ consisting of the s copies of $\mathcal{M_C}$ gives the desired code. □

Corollary 3.3. *$n_q(4,d) \leqslant g_q(4,d)+s-1$ for $d = s(q^2-q)$ with $1 \leqslant s \leqslant q-1$.*

Lemma 3.4. *There exist $[140,4,121]_9$, $[174,4,152]_9$ and $[181,4,158]_9$ codes.*

Proof. By Theorem 2.18 and Lemma 3.2, there exist $[17,3,14]_9$, $[58,4,49]_9$, $[82,4,72]_9$, $[92,4,80]_9$ codes. There also exists a $[164,4,144]_9$ code containing a codeword of weight 162 by Lemma 3.2. Hence, there exist $[140,4,121]_9$ and $[174,4,152]_9$ codes by Lemmas 2.6 and a $[181,4,158]_9$ code by Lemma 2.7. □

Lemma 3.5. *There exist a QT $[205,4,180]_9$ code and a $[215,4,188]_9$ code.*

Proof. Let $\mathcal{C}$ be the $[205,4,180]_9$ code with generator matrix $G = [1218^{41}] + 6100^{41} + 3210^{41} + 3310^{41} + 7310^{41}$. Then $\mathcal{C}$ has weight distribution $0^1 180^{4920} 189^{1640}$. On the other hand, by Theorem 2.18, there exists $[10,3,8]_9$ code. Hence, by Lemma 2.7, there exists $[215,4,188]_9$ code. □

Lemma 3.6. *There exists a QT $[287,4,252]_9$ code.*

Proof. Let $\mathcal{C}$ be the $[287,4,252]_9$ code with generator matrix $G = [1218^{41}] + 1000^{41} + 6100^{41} + 3210^{41} + 3310^{41} + 7310^{41} + 1510^{41}$. Then $\mathcal{C}$ has weight distribution $0^1 252^{4592}$ $261^{1640} 270^{328}$. □

Lemma 3.7. *There exists a* $[418, 4, 369]_9$ *code.*

Proof. Let $\mathcal{C}$ be the $[418, 4, 369]_9$ code with generator matrix $G = [5510^{13}] + 1313^{13} + 1628^{13} + 1145^{13} + 1473^{13} + 1652^{13} + 1031^{13} + 1815^{13} + 1738^{13} + 1160^{13} + 1080^{13} + 1218^{13} + 1136^{13} + 1384^{13} + 1167^{13} + 1241^{13} + 1850^{13} + 1005^{13} + 1407^{13} + 1207^{13} + 1116^{13} + 1211^{13} + 1851^{13} + 1437^{13} + 1388^{13} + 1137^{13} + 1016^{13} + 1507^{13} + 1643^{13} + 1334^{13} + 1261^{13} + 1252^{13} + 1051^{1} + 1051^{1}$. Then $\mathcal{C}$ has weight distribution $0^1 369^{4784} 378^{1664} 387^{104} 405^8$. □

Lemma 3.8. *There exist* $[913, 4, 810]_9$*,* $[923, 4, 819]_9$*,* $[933, 4, 828]_9$ *and* $[943, 4, 837]_9$ *codes.*

Proof. Let $\mathcal{C}$ be the extended QC $[41, 4, 33]_9$ code with generator matrix $G = [1000^4] + 7211^4 + 1116^4 + 1574^4 + 1376^4 + 1507^4 + 1247^4 + 1426^4 + 1237^4 + 1860^4 + 1515^1$. Then $\mathcal{C}$ has weight distribution $0^1 33^{984} 36^{3608} 39^{1968}$. Applying Lemma 2.4, as the projective dual of $\mathcal{C}$, one can get a $[943, 4, 837]_9$ code $\mathcal{C}^*$ with weight distribution $0^1 837^{6232} 864^{328}$. It can be checked that the multiset for $\mathcal{C}^*$ has three mutually disjoint lines $\langle 1000, 1018\rangle$, $\langle 1002, 1102\rangle$, $\langle 1003, 1114\rangle$, where $x_0x_1 \cdots x_3$ stands for the point $\mathbf{P}(x_0, x_1 \cdots, x_3)$ of $\Sigma = \mathrm{PG}(3, 9)$ and $\langle P, Q\rangle$ stands for the line through the points P and Q in Σ. Hence, we get $[913, 4, 810]_9$, $[923, 4, 819]_9$ and $[933, 4, 828]_9$ codes by Lemma 2.5. □

Lemma 3.9. *There exist* $[954, 4, 846]_9$*,* $[964, 4, 855]_9$*,* $[974, 4, 864]_9$*,* $[984, 4, 873]_9$*,* $[994, 4, 882]_9$*,* $[1004, 4, 891]_9$*,* $[1014, 4, 900]_9$*,* $[1024, 4, 909]_9$ *and* $[1034, 4, 918]_9$ *codes.*

Proof. Let $\mathcal{C}$ be the $[38, 4, 30]_9$ code with generator matrix $G = [1000^4] + 1721^4 + 1215^4 + 1056^4 + 1574^4 + 1542^4 + 1761^4 + 1065^4 + 1168^4 + 1515^1 + 1357^1$, where $\mathbf{P}(1, 5, 1, 5)$ and $\mathbf{P}(1, 3, 5, 7)$ are fixed points under the projectivity defined by the companion matrix of $x^4 - 1$. Then $\mathcal{C}$ has weight distribution $0^1 30^{672} 33^{3504} 36^{2384}$. Applying Lemma 2.4, as the projective dual of $\mathcal{C}$, one can get a $[1034, 4, 918]_9$ code $\mathcal{C}^*$ with weight distribution $0^1 918^{6256} 945^{304}$. It can be checked that the multiset for $\mathcal{C}^*$ has eight mutually disjoint lines $\langle 1000, 1103\rangle$, $\langle 1002, 1111\rangle$, $\langle 1003, 1017\rangle$, $\langle 1005, 1121\rangle$, $\langle 1006, 1132\rangle$, $\langle 1007, 1140\rangle$, $\langle 1008, 1150\rangle$, $\langle 1010, 1105\rangle$. So, we get $[1034 - 10t, 4, 918 - 9t]_9$ codes for $1 \leqslant t \leqslant 8$ by Lemma 2.5. □

Lemma 3.10. *There exist* $[1045, 4, 927]_9$*,* $[1055, 4, 936]_9$*,* $[1065, 4, 945]_9$*,* $[1075, 4, 954]_9$*,* $[1085, 4, 963]_9$*,* $[1095, 4, 972]_9$*,* $[1105, 4, 981]_9$*,* $[1115, 4, 990]_9$ *and* $[1125, 4, 999]_9$ *codes.*

Proof. Let $\mathcal{C}$ be the $[35, 4, 27]_9$ code with generator matrix $G = 1018^4 + 1077^4 + 1220^4 + 1550^4 + 1034^4 + 1566^4 + 1356^4 + 1313^2 + 1652^2 + 1357^1 + 1111^1 + 1753^1$, where the columns of G consist of seven orbits of length 4, two orbits of length 2 and three fixed points under the projectivity defined by the companion matrix of $x^4 - 1$. Then $\mathcal{C}$ has weight distribution $0^1 27^{440} 30^{3240} 33^{2880}$. Applying Lemma 2.4, as the projective dual of $\mathcal{C}$, one can get a $[1125, 4, 999]_9$ code $\mathcal{C}^*$ with weight distribution $0^1 999^{6280} 1026^{280}$. It can be checked that the multiset for $\mathcal{C}^*$ has eight mutually disjoint lines $\langle 1000, 1001\rangle$, $\langle 1011, 1100\rangle$, $\langle 1012, 1114\rangle$, $\langle 1013, 1120\rangle$, $\langle 1014, 1130\rangle$, $\langle 1015, 1140\rangle$, $\langle 1016, 1150\rangle$, $\langle 1017, 1161\rangle$. Hence, applying Lemma 2.5, we get $[1045, 4, 927]_9$, $[1055, 4, 936]_9$, $[1065, 4, 945]_9$, $[1075, 4, 954]_9$, $[1085, 4, 963]_9$, $[1095, 4, 972]_9$, $[1105, 4, 981]_9$ and $[1115, 4, 990]_9$ codes. □

Lemma 3.11. *There exist* $[1257, 4, 1116]_9$*,* $[1267, 4, 1125]_9$ *and* $[1277, 4, 1134]_9$ *codes.*

Proof. Let $\mathcal{C}$ be the $[39, 4, 30]_9$ code with generator matrix $G = [1000^4] + 1721^4 + 1846^4 + 1473^4 + 1300^4 + 1851^4 + 1574^4 + 1281^4 + 1405^4 + 1256^2 + 1515^1$, where the columns of G consist of nine orbits of length 4, one orbit of length 2 and a fixed point under the projectivity defined by the companion matrix of $x^4 - 1$. Then $\mathcal{C}$ has weight distribution $0^1 30^{272} 33^{2616} 36^{3416} 39^{256}$. Applying Lemma 2.4, as the projective dual of $\mathcal{C}$, one can get a $[1277, 4, 1134]_9$ code $\mathcal{C}^*$ with weight distribution $0^1 1134^{6248} 1161^{312}$. It can be checked that the multiset for $\mathcal{C}^*$ has two mutually disjoint lines $\langle 1000, 1015 \rangle$ and $\langle 1002, 1102 \rangle$. Hence, we get $[1257, 4, 1116]_9$ and $[1267, 4, 1125]_9$ codes by Lemma 2.5. □

Lemma 3.12. *There exist* $[1227, 4, 1089]_9$*,* $[1237, 4, 1098]_9$ *and* $[1247, 4, 1107]_9$ *codes.*

Proof. Let $\mathcal{C}$ be the QT $[49, 4, 39]_9$ code with generator matrix $G = [1131^7] + 1000^7 + 1402^7 + 1846^7 + 1407^7 + 1445^7 + 1705^7$, giving weight distribution $0^1 39^{784} 42^{2136} 45^{3080} 48^{560}$. Applying Lemma 2.4, as the projective dual of $\mathcal{C}$, one can get a $[1247, 4, 1107]_9$ code $\mathcal{C}^*$ with weight distribution $0^1 1107^{6224} 1134^{280} 1161^{56}$. It can be checked that the multiset for $\mathcal{C}^*$ has two mutually disjoint lines $\langle 1000, 1111 \rangle$, $\langle 1003, 1126 \rangle$. Hence, we get $[1227, 4, 1089]_9$ and $[1237, 4, 1098]_9$ codes by Lemma 2.5. □

Lemma 3.13. *There exist* $[1287, 4, 1143]_9$ *and* $[1297, 4, 1152]_9$ *codes.*

Proof. There exist a $[1216, 4, 1080]_9$ code with weight distribution $0^1 1080^{6016} 1089^{512} 1152^{32}$ and a $[1206, 4, 1071]_9$ code with weight distribution $0^1 1071^{5936} 1080^{592} 1143^{32}$, see [24]. Since a $[81, 3, 72]_9$ code exists, one can apply Lemma 2.7 to obtain the desired codes. □

We have also constructed new codes for other values of d to make Table 2, e.g., $[274, 4, 240]_9$, $[312, 4, 273]_9$, $[378, 4, 332]_9$ and so on with the aid of a computer, but the upper bounds are still weak to be improved. So, we omit the proof of their constructions.

4 Nonexistence of some codes

In this section, we prove the nonexistence of some Griesmer codes to give the lower bounds on $n_9(4, d)$.

Lemma 4.1. *The spectrum of a* $[64, 3, 56]_9$ *code satisfies* $a_8 \leqslant 64$*.*

Proof. It follows from $(2.3)\times 6 - (2.4)\times 3 + (2.5)/2$ that $6a_0 + 3a_1 + a_2 + a_5 + 3a_6 + 6a_7 + 10a_8 = 642$. Hence $a_8 \leqslant 64$. □

The following can be obtained in the same way.

Lemma 4.2. *The spectrum of a* $[65, 3, 57]_9$ *code satisfies* $a_8 \leqslant 67$*.*

Lemma 4.3. *There exists no* $[1339, 4, 1189]_9$ *code.*

Table 1: The spectra of some $[n, 3, d]_9$ codes.

parameters	possible spectra	reference
$[9, 3, 7]_9$	$(a_0, a_1, a_2) = (37, 18, 36)$	[13]
$[10, 3, 8]_9$	$(a_0, a_1, a_2) = (36, 10, 45)$	[13]
$[17, 3, 14]_9$	$(a_0, a_1, a_2, a_3) = (18, 15, 19, 39)$	[18]
	$(a_0, a_1, a_2, a_3) = (17, 18, 16, 40)$	
	$(a_0, a_1, a_2, a_3) = (19, 12, 22, 38)$	
	$(a_0, a_1, a_2, a_3) = (18, 15, 19, 39)$	
$[48, 3, 42]_9$	$(a_0, a_3, a_6) = (3, 16, 72)$	[26]
$[78, 3, 69]_9$	$(a_0, a_6, a_8, a_9) = (1, 1, 27, 62)$	Lemma 4.7
	$(a_0, a_7, a_8, a_9) = (1, 3, 24, 63)$	
	$(a_6, a_9) = (13, 63)$	
$[81, 3, 72]_9$	$(a_0, a_9) = (1, 90)$	[6]
$[88, 3, 78]_9$	$(a_7, a_9, a_{10}) = (1, 27, 63)$	[6]
	$(a_8, a_9, a_{10}) = (3, 24, 64)$	
$[90, 3, 80]_9$	$(a_9, a_{10}) = (10, 81)$	[6]
$[91, 3, 81]_9$	$a_{10} = 91$	[6]
$[150, 3, 133]_9$	$(a_6, a_8, a_{16}, a_{17}) = (1, 2, 18, 70)$	[15]
	$(a_7, a_8, a_{15}, a_{16}, a_{17}) = (2, 1, 1, 16, 71)$	
	$(a_7, a_8, a_{15}, a_{16}, a_{17}) = (1, 2, 1, 17, 70)$	
	$(a_8, a_{15}, a_{16}, a_{17}) = (3, 1, 18, 69)$	
	$(a_8, a_{15}, a_{17}) = (3, 10, 78)$	

Proof. Let $\mathcal{C}$ be a putative Griesmer $[1339, 4, 1189]_9$ code. It follows from Lemma 2.1 that $\gamma_0 = 2$, $\gamma_1 = 17$, $\gamma_2 = 150$. From Table 1, the spectrum of a γ_2-plane Δ is one of (A) $(\tau_6, \tau_8, \tau_{16}, \tau_{17}) = (1, 2, 18, 70)$, (B) $(\tau_7, \tau_8, \tau_{15}, \tau_{16}, \tau_{17}) = (2, 1, 1, 16, 71)$, (C) $(\tau_7, \tau_8, \tau_{15}, \tau_{16}, \tau_{17}) = (1, 2, 1, 17, 70)$, (D) $(\tau_8, \tau_{15}, \tau_{16}, \tau_{17}) = (3, 1, 18, 69)$, (E) $(\tau_8, \tau_{15}, \tau_{17}) = (3, 10, 78)$. Thus, a j-line on Δ satisfies

$$j \in \{6, 7, 8, 15, 16, 17\}. \tag{4.1}$$

From (2.2) and (2.5), we have $\lambda_0(\Delta) = 5, 5, 4, 3, 4$ for the cases A,B,C,D,E, respectively. Since an i-plane δ can not meet Δ in a t-line with $t \in \{0, \dots, 5, 9, \dots, 14\}$, one can show $a_i = 0$ for all $i \notin \{43, \dots, 48, 52, \dots, 55, 61, \dots, 65, 79, 80, 81, 88, \dots, 91, 124, \dots, 150\}$ using Lemmas 2.2, 2.3, the Griesmer bound and Theorem 1.1. We refer to this procedure as the **first sieve** in the proofs of the nonexistence results in this section. For example, if there exists a 49-plane, then it corresponds to a $[49, 3, 43]_q$ code by Lemma 2.2 (1), which does not exist by Theorem 1.1. If there exists a 82-plane, then it corresponds to a $[82, 3, 73]_q$ code by Lemma 2.3, which does not exist by the Griesmer bound.

We also have $a_{81} = a_{90} = a_{91} = 0$ since the spectra of codes with parameters $[81, 3, 72]_9$, $[90, 3, 80]_9$, $[91, 3, 81]_9$ are $(a_0, a_9) = (1, 90)$, $(a_9, a_{10}) = (10, 81)$, $a_{10} = 91$, respectively, see Table 1. From (2.6), we get

$$\sum_{i=43}^{148} \binom{150 - i}{2} a_i = 81\lambda_2 - 34091. \tag{4.2}$$

For any w-plane through a t-line, (2.7) gives

$$\sum_{j} (150 - j) c_j = w + 11 - 9t. \tag{4.3}$$

with $\sum_j c_j = 9$. The equality (3.2) yields

$$\lambda_2 = 519 + \lambda_0. \tag{4.4}$$

Suppose $a_{43} > 0$. Since a t-line in a 43-plane δ satisfies $t \leqslant 6$, we may assume that every γ_2-plane has spectrum (A). Since the RHS of (4.3) is at most 54 and since the coefficient of c_{89} in (4.3) is 61, we get $a_{43} = 1$ and $a_j = 0$ for $j \leqslant 89$ with $j \neq 43$. Setting $w = 150$, the maximum possible contributions of c_j's to the LHS of (4.2) are $(c_{43}, c_{150}) = (1, 8)$ for $t = 6$; $(c_{124}, c_{139}, c_{150}) = (3, 1, 5)$ for $t = 8$; $(c_{133}, c_{150}) = (1, 8)$ for $t = 16$; $(c_{142}, c_{150}) = (1, 8)$ for $t = 17$. Estimating the LHS of (4.2) for the spectrum (A), we get

$$(\text{LHS of (4.2)}) \leqslant 5671\tau_6 + (3 \cdot 325 + 55)\tau_8 + 136\tau_{16} + 28\tau_{17} = 12139.$$

Hence $\lambda_2 \leqslant 570$. On the other hand, $(c_{43}, c_{150}) = (1, 8)$ is the unique solution of (4.3) for $(w, t) = (150, 6)$ since $a_{43} > 0$. Since each of the eight 150-planes through the 6-line in Δ and Δ itself contains a 0-point out of the 6-line $\delta \cap \Delta$, (4.4) yields $\lambda_2 \geqslant 519 + (\theta_2 - 43) + 9 = 576$, giving a contradiction. Hence $a_{43} = 0$. One can prove $a_{44} = a_{45} = a_{46} = a_{47} = a_{48} = 0$ similarly.

Suppose $a_{52} > 0$. Since a t-line in a 52-plane δ satisfies $\gamma_1(\delta) \leqslant 7$, we may assume that every γ_2-plane has spectrum (B) or (C). Since the RHS of (4.3) with $w = 52$ is at most 63 and since the coefficient of c_{89} in (4.3) is 61, we get $a_{52} = 1$ and $a_j = 0$ for $j \leqslant 89$ with $j \neq 52$. Setting $w = 150$, the maximum possible contributions of c_j's to the LHS of (4.2) are $(c_{52}, c_{150}) = (1, 8)$ for $t = 7$ if $c_{52} > 0$; $(c_{124}, c_{130}, c_{150}) = (3, 1, 5)$ for $t = 7$ if $c_{52} = 0$; $(c_{124}, c_{139}, c_{150}) = (3, 1, 5)$ for $t = 8$; $(c_{124}, c_{150}) = (1, 8)$ for $t = 15$; $(c_{133}, c_{150}) = (1, 8)$ for $t = 16$; $(c_{142}, c_{150}) = (1, 8)$ for $t = 17$. Estimating the LHS of (4.2), we get

$$(\text{LHS of (4.2)}) \leqslant 4753 + (3 \cdot 325 + 190)(\tau_7 - 1) + (3 \cdot 325 + 55)\tau_8 + 325\tau_{15} + 136\tau_{16} + 28\tau_{17}.$$

When Δ has spectrum (C), we have $81\lambda_2 \leqslant 45501$, i.e., $\lambda_2 \leqslant 561$. On the other hand, $(c_{52}, c_{150}) = (1, 8)$ is the unique solution of (4.3) for $(w, t) = (150, 7)$ when $a_{52} > 0$. Since each of the eight 150-planes through the 7-line in Δ and Δ itself contains one 0-point out of $\delta \cap \Delta$, (4.4) yields $\lambda_2 \geqslant 519 + (\theta_2 - 52) + 8 = 566$, giving a contradiction. We also get a contradiction similarly when Δ has spectrum (B). Hence $a_{52} = 0$. One can similarly prove that $a_{53} = a_{54} = a_{55} = 0$.

Suppose $a_i > 0$ with $i = 88$. Then, δ corresponds to a Griesmer $[88, 3, 78]_9$ code, whose spectrum is $(\tau_7, \tau_9, \tau_{10}) = (1, 27, 63)$ or $(\tau_8, \tau_9, \tau_{10}) = (3, 24, 64)$ from Table 1. Setting $w = 88$, the maximum possible contributions of c_j's to the LHS of (4.2) are $(c_{124}, c_{140}, c_{150}) = (1, 1, 7)$ for $t = 7$; $(c_{124}, c_{149}, c_{150}) = (1, 1, 7)$ for $t = 8$; $(c_{140}, c_{149}) = (1, 8)$ for $t = 9$; $c_{149} = 9$ for $t = 10$. Estimating the LHS of (4.2), we get

$$(\text{LHS of (4.2)}) \leqslant 1891 + (325 + 45)\tau_7 + 325\tau_8 + 45\tau_9.$$

From the possible spectra for δ, we have $81\lambda_2 \leqslant 38037$, i.e., $\lambda_2 \leqslant 469$. On the other hand, (4.4) implies $\lambda_2 \geqslant 519$, giving a contradiction. Hence $a_{88} = 0$. We can prove $a_{89} = a_{79} = a_{80} = 0$, similarly.

Suppose $a_{61} > 0$. A t-line in a 61-plane δ satisfies $t \leqslant 8$. Since the RHS of (4.3) is at most 72 and since the coefficient of c_{65} in (4.3) is 85, we get $a_{61} = 1$ and $a_j = 0$ for $j \leqslant 65$ with $j \neq 61$. Setting $w = 150$, the maximum possible contributions of c_j's to the LHS of (4.2) are $(c_{124}, c_{147}, c_{150}) = (4, 1, 4)$ for $t = 6$; $(c_{124}, c_{130}, c_{150}) = (3, 1, 5)$ for $t = 7$; $(c_{61}, c_{150}) = (1, 8)$ for $t = 8$ if $c_{61} > 0$; $(c_{124}, c_{139}, c_{150}) = (3, 1, 5)$ for $t = 8$ if $c_{61} = 0$; $(c_{124}, c_{150}) = (1, 8)$ for $t = 15$; $(c_{133}, c_{150}) = (1, 8)$ for $t = 16$; $(c_{142}, c_{150}) = (1, 8)$ for $t = 17$. We may assume that δ meets Δ in a 8-line. Then, estimating the LHS of (4.2), we get

$$\begin{aligned}(\text{LHS of (4.2)}) \leqslant\ & 3916 + (4 \cdot 325 + 3)\tau_6 + (3 \cdot 325 + 190)\tau_7 + (3 \cdot 325 + 55)(\tau_8 - 1) \\ & + 325\tau_{15} + 136\tau_{16} + 28\tau_{17}.\end{aligned}$$

When Δ has spectrum (D), we have $81\lambda_2 \leqslant 44772$, i.e., $\lambda_2 \leqslant 552$. On the other hand, we have $c_{150} \geqslant 8$ as the solution of (4.3) for $(w, t) = (150, 8)$ since $a_{61} > 0$. Since each of the eight 150-planes through the 8-line in Δ and Δ itself contains a 0-point out of $\delta \cap \Delta$, (4.4) yields $\lambda_2 \geqslant 519 + (\theta_2 - 61) + 1 \cdot 9 = 558$, giving a contradiction. One can get a contradiction similarly when Δ has any other spectrum. Hence $a_{61} = 0$. We can prove $a_{62} = a_{63} = 0$ similarly.

Suppose $a_{64} > 0$. A t-line in a 64-plane δ satisfies $t \leqslant 8$. Since the RHS of (4.3) is at most 75 and since the coefficient of c_{65} in (4.3) is 85, we get $a_{64} = 1$ and $a_{65} = 0$. Setting $w = 150$, the maximum possible contributions of c_j's to the LHS of (4.2) are $(c_{64}, c_{147}, c_{150}) = (1, 1, 7)$ for $t = 8$ if $c_{64} > 0$; $(c_{124}, c_{139}, c_{150}) = (3, 1, 5)$ for $t = 8$ if $c_{64} = 0$; and the same c_j's for other t with the case assuming $a_{61} > 0$. Estimating the LHS of (4.2), we get

$$\begin{aligned}(\text{LHS of (4.2)}) \leqslant\ & (3655 + 3) + (4 \cdot 325 + 3)\tau_6 + (3 \cdot 325 + 190)\tau_7 \\ & + (3 \cdot 325 + 55)(\tau_8 - 1) + 325\tau_{15} + 136\tau_{16} + 28\tau_{17}. \qquad (4.5)\end{aligned}$$

When Δ has spectrum (D), (4.5) gives $81\lambda_2 \leqslant 44514$, i.e., $\lambda_2 \leqslant 549$. On the other hand, we have $c_{150} \geqslant 5$ as the solution of (4.3) for $(w, t) = (150, 8)$ since $a_{64} > 0$. Since each of the five 150-planes through the 8-line in Δ and Δ itself contains a 0-point out of $\delta \cap \Delta$, (4.4) yields $\lambda_2 \geqslant 519 + (\theta_2 - 64) + 1 \cdot 6 = 552$, giving a contradiction. One can get a contradiction similarly when Δ has spectrum (C), (A) or (B). Now, we may assume that every γ_2-plane has spectrum (E). Then, (4.5) gives $81\lambda_2 \leqslant 45243$, i.e., $\lambda_2 \leqslant 558$. To calculate the RHS of (4.5), the number of γ_2-plane is estimated as $7 + 5 \cdot 2 + 8 \cdot 10 + 8 \cdot 78 = 721$. This contradicts that $a_{150} \leqslant 64 \cdot 8 = 512$ since δ meets a 150-plane in an 8-line and δ contains at most 64 8-lines by Lemma 4.1. So, we need to reduce the estimated number of γ_2-planes from 721 to at most 512, which yields the RHS of (4.5) to $45243 - (721 - 512) \cdot 2$, since we set $c_{148} = 0$ to maximize the LHS of (4.2). Hence $\lambda_2 \leqslant 553$. On the other hand, we have $c_{150} \geqslant 5$ as the solution of (4.3) for $(w, t) = (150, 8)$ since $a_{64} > 0$. Since each of the five 150-planes through the 8-line in Δ and Δ itself contains two 0-points out of $\delta \cap \Delta$, (4.4) yields $\lambda_2 \geqslant 519 + (\theta_2 - 64) + 2 \cdot 6 = 558$, giving a contradiction. Hence $a_{64} = 0$. We can prove $a_{65} = 0$ similarly using Lemma 4.2.

Therefore, $a_i > 0$ implies $124 \leqslant i \leqslant 150$. Setting $w = 150$, the maximum possible contributions of c_j's to the LHS of (4.3) are $(c_{124}, c_{147}, c_{150}) = (4, 1, 4)$ for $t = 6$;

$(c_{124}, c_{130}, c_{150}) = (3, 1, 5)$ for $t = 7$; $(c_{124}, c_{139}, c_{150}) = (3, 1, 5)$ for $t = 8$; $(c_{124}, c_{150}) = (1, 8)$ for $t = 15$; $(c_{133}, c_{150}) = (1, 8)$ for $t = 16$; $(c_{142}, c_{150}) = (1, 8)$ for $t = 17$. Estimating the LHS of (4.3), we get

$$(\text{LHS of (4.2)}) \leqslant (4\cdot 325+3)\tau_6+(3\cdot 325+190)\tau_7+(3\cdot 325+55)\tau_8+325\tau_{15}+136\tau_{16}+28\tau_{17}.$$

When Δ has spectrum (D), we have $81\lambda_2 \leqslant 41886$, i.e., $\lambda_2 \leqslant 517$. On the other hand, we have $\lambda_2 \geqslant 519$, giving a contradiction. One can get a contradiction similarly when Δ has spectrum (A), (B) or (C). Hence, we may assume that every 150-plane has spectrum (E). When Δ has spectrum (E), we have $81\lambda_2 \leqslant 42615$, i.e., $\lambda_2 \leqslant 526$. Since the ten 15-lines on Δ are passing through a fixed 0-point (see the proof of Lemma 4.4 in [15]), one can take a 17-line ℓ on Δ containing no 0-point. Then, there is another 150-plane Δ' through ℓ since we have $c_{150} \geqslant 1$ as the solution of (4.3) for $(w, t) = (150, 17)$. Counting the number of 0-points in $\Delta \cup \Delta'$, (4.4) yields $\lambda_2 \geqslant 519 + 2 \cdot 4 = 527$, giving a contradiction. This completes the proof. □

Lemma 4.4. *There exists no* $[995, 4, 883]_9$ *code.*

Proof. Let $\mathcal{C}$ be an $[995, 4, 883]_9$ code. By Lemma 2.1, $\gamma_0 = 2$, $\gamma_1 = 13$, $\gamma_2 = 112$. Let Δ be a γ_2-plane. Then, $\mathcal{M}_{\mathcal{C}}(\Delta)$ is just two copies of Δ with a 7-arc of lines deleted by Theorem 43 in [12] since the multiset $2\Delta - \mathcal{M}_{\mathcal{C}}(\Delta)$ forms a $(70, 7)$-minihyper whose point multiplicity is at most 2. Hence, the spectrum of Δ is $(\tau_4, \tau_{13}) = (7, 84)$. By the first sieve, we have $a_i = 0$ for all $i \notin \{$23-28, 77-81, 86-91, 104-112$\}$. If a 77-plane δ exists, then it meets Δ in a 4-line. On the other hand, δ corresponds to a $[77, 3, 68]_9$ code by Lemma 2.2 containing no 4-line, a contradiction. Hence $a_{77} = 0$. Similarly, we have $a_i = 0$ for $i = 78, \ldots, 81, 86, \ldots, 91$. For any w-plane through a t-line, (2.7) gives

$$\sum_j (112 - j)c_j = w + 5 - 9t \tag{4.6}$$

with $\sum_j c_j = 9$. Suppose $a_{23} > 0$ and let δ be a 23-plane. Then, we have $a_{23} = 1$ and $a_j = 0$ for $24 \leqslant j \leqslant 28$ from (4.6). Take a 4-line l on Δ which is not $\delta \cap \Delta$ and consider the planes through l from (4.6) with $(w, t) = (112, 4)$. Then, the equation (4.6) has no solution. Hence $a_{23} = 0$. We get $a_{24} = \cdots = a_{28} = 0$ similarly. Now, $a_i > 0$ implies $104 \leqslant i \leqslant 112$. Setting $(w, t) = (112, 4)$, the equation (4.16) has no solution, a contradiction. This completes the proof. □

Lemma 4.5. *There exists no* $[912, 4, 810]_9$ *code.*

Proof. Let $\mathcal{C}$ be an $[n = 912, 4, d = 810]_9$ code. By Lemma 2.1, $\gamma_0 = 2$, $\gamma_1 = 12$, $\gamma_2 = 102$. Let Δ be a γ_2-plane. Let l be an i-line with $i > 0$ containing a 1-point P. Counting the 1-points on the lines through P, we get $\gamma_2 = 102 \leqslant (12 - 1) \cdot 9 + i$, hence $3 \leqslant i$. So a j-line with $j \geqslant 1$ on Δ satisfies $3 \leqslant j \leqslant 12$. We have $a_i = 0$ for all $i \notin \{48, 75, 76, 77, 78, 79, 80, 81, 102\}$ by the first sieve. From (2.6), we get

$$891a_{48} + 81a_{75} + 65a_{76} + 50a_{77} + 36a_{78} + 23a_{79} + 11a_{80} = 81\lambda_2 - 10692. \tag{4.7}$$

For any w-plane through a t-line, (2.7) gives

$$\sum_j (102 - j)c_j = w + 6 - 9t \tag{4.8}$$

with $\sum_j c_j = 9$. Suppose $a_{48} > 0$. The spectrum of a 48-plane is $(\tau_0, \tau_3, \tau_6) = (3, 16, 72)$ from Table 1. Setting $w = 48$ and $t = 0$, the equation (4.8) has no solution. Hence $a_{48} = 0$. Suppose $a_i > 0$ for $76 \leqslant i \leqslant 81$. Setting $w = i$ and $t = 9$, the equation (4.8) has no solution. Hence $a_i = 0$ for $76 \leqslant i \leqslant 81$.

Suppose $a_{75} > 0$. Then, we have $c_{75} = 3 - t/3$ from (4.8). Setting $w = i$ and $t \notin \{0, 3, 6, 9\}$, the equation (4.8) has no solution. Let δ be a 75-plane and l' be a line on δ. We have $|l' \cap \delta \cap C_0| = 1, 4, 7, 10$. Then, $|\delta \cap C_0| = 91 - 75 = 16$. Considering the lines through a fixed 0-point of δ not on l', we have $|\delta \cap C_0| \geqslant 3 \cdot 10 + 1 = 31$ if l' is a 0-line, a contradiction. If l' is a 3-line, we have $|\delta \cap C_0| \geqslant 3 \cdot 7 + 1 = 22$, a contradiction. Let the spectrum of a $[75, 3, 66]_9$ code corresponding to δ be (τ_6, τ_9). Since the code is Griesmer, we have $\tau_6 + \tau_9 = 91$ and $6\tau_6 + 9\tau_9 = 750$ from (2.3), (2.4). Hence $(\tau_6, \tau_9) = (23, 68)$, a contradiction to (2.5). Hence $a_{75} = 0$.

Setting $w = 102$, we have $0 = 108 - 9t$ from (4.8). Let the spectrum of a $[102, 3, 90]_9$ code corresponding to δ be τ_{12}. Then, we have $\tau_{12} = 91$ and $12\tau_{12} = 1020$ from (2.3) and (2.4), giving a contradiction. This completes the proof. □

If a 3-divisible $[24, 4, 18]_9$ code exists, then so does a 27-divisible $[912, 4, 810]_9$ code as a projective dual, which is impossible by the above lemma. Hence we get the following.

Corollary 4.6. *There exists no 3-divisible* $[24, 4, 18]_9$ *code.*

Let $\mathcal{C}$ be a $[78, 3, 69]_9$ code. Since $\mathcal{C}$ is Griesmer, the set C_0 of 0-points for $\mathcal{C}$ forms a $(13, 1)$-blocking set in $\mathrm{PG}(2, 9)$. If C_0 contains a line l, then C_0 consists of l and three points, say Q_1, Q_2, Q_3. In this case, there are two possibilities according to the condition if the three points Q_1, Q_2, Q_3 are collinear or not. If C_0 contains no line, C_0 forms a non-trivial blocking set (see [13]) and is a subgeometry $\mathrm{PG}(2, 3)$ by Theorem 13.11 in [13]. Hence we get the following.

Lemma 4.7. *The spectrum of a* $[78, 3, 69]_9$ *code is one of the following:*
(a) $(a_0, a_6, a_8, a_9) = (1, 1, 27, 62)$,
(b) $(a_0, a_7, a_8, a_9) = (1, 3, 24, 63)$,
(c) $(a_6, a_9) = (13, 78)$.

Lemma 4.8. *There exists no* $[695, 4, 617]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[695, 4, 617]_9$ code. Let Δ be a γ_2-plane in $\Sigma = \mathrm{PG}(3, 9)$. Then the spectrum of Δ is one of (a), (b), (c) in Lemma 4.7.

Let δ_i be an i-plane in Σ and let l be a t-line in δ_i. Then, by Lemma 2.2 (1), we have $t \leqslant (i + 7)/9$. By the first sieve, we get

$$a_i = 0 \text{ for all } i \notin \{0, 47, 48, 65, 74, 75, 76, 77, 78\}.$$

Suppose $a_0 > 0$ and let δ_0 be a 0-plane. If there exists an i-plane δ ($\neq \delta_0$) with $i \leqslant 65$, then, considering the planes through $\delta_0 \cap \delta$, we have

$$695 \leqslant 65 + 78 \times 8 = 689,$$

a contradiction. Thus, we have $a_0 = 1$ and $a_{47} = a_{48} = a_{65} = 0$. Hence, from (2.6), we get

$$6a_{74} + 3a_{75} + a_{76} = 1242. \tag{4.9}$$

Setting $i = t = 0$, the maximum possible contributions of c_j's in (2.7) to the LHS of (4.9) are $(c_{74}, c_{75}, c_{78}) = (1, 1, 7)$. Estimating the LHS of (4.9) we get

$$1242 \leqslant (6 + 3) \times 91 = 819,$$

a contradiction. Hence $a_0 = 0$. From (2.6), we have

$$465a_{47} + 435a_{48} + 78a_{65} + 6a_{74} + 3a_{75} + a_{76} = 4245. \tag{4.10}$$

Next, we show that a 78-plane in Σ has spectrum of type (c) in Lemma 4.7. Setting $i = 78$, the maximum possible contributions of c_j's in (2.7) to the LHS of (4.10) are $(c_{47}, c_{65}, c_{74}, c_{76}, c_{78}) = (2, 1, 2, 1, 3)$ for $t = 0$; $(c_{47}, c_{78}) = (1, 8)$ for $t = 6$; $(c_{65}, c_{74}, c_{77}, c_{78}) = (1, 2, 1, 5)$ for $t = 7$; $(c_{65}, c_{78}) = (1, 8)$ for $t = 8$ and $(c_{74}, c_{78}) = (1, 8)$ for $t = 9$. Estimating the LHS of (4.10) for the spectrum (a) in Lemma 4.7, we get

$$4245 \leqslant (465 \times 2 + 78 + 6 \times 2 + 1) + 465 + 78 \times 27 + 6 \times 62 = 3964,$$

a contradiction. Similarly for the spectrum (b) in Lemma 4.7, we get

$$4245 \leqslant (465 \times 2 + 78 + 6 \times 2 + 1) + (78 + 6 \times 2) \times 3 + 78 \times 24 + 6 \times 63 = 3541,$$

a contradiction again. Hence, every 78-plane has spectrum of type (c) in Lemma 4.7. Using this fact, we rule out a 65-plane.

Suppose $a_{65} > 0$. Then $\delta_{65} \cap C_1$ forms a $(65, 8)$-arc by Lemma 2.2(1). Setting $i = 65$ and $t = 8$, (2.7) has the unique solution $c_{78} = 9$, which contradicts to the fact that a 78-plane has no 8-line. Hence $a_{65} = 0$.

Suppose $a_{48} > 0$. Then $\delta_{48} \cap C_1$ forms a $(48, 6)$-arc by Lemma 2.2(1). Setting $i = 48$, the maximum possible contributions of c_j's in (2.7) to the LHS of (4.10) are $(c_{47}, c_{74}, c_{76}, c_{77}) = (1, 5, 1, 2)$ for $t = 0$; $(c_{74}, c_{76}, c_{77}) = (6, 1, 2)$ for $t = 3$ and $(c_{77}, c_{78}) = (1, 8)$ for $t = 6$, since a 78-plane has only 6-lines or 9-lines. Recall from Table 1 that the spectrum of a 48-plane is $(\tau_0, \tau_3, \tau_6) = (3, 16, 72)$. Estimating the LHS of (4.10) we get

$$4245 \leqslant (465 + 6 \times 5 + 1) \times 3 + (6 \times 6 + 1) \times 16 + 435 = 2515,$$

a contradiction. Hence $a_{48} = 0$. Now, we have

$$a_i = 0 \text{ for all } i \notin \{47, 74, 75, 76, 77, 78\}.$$

One can obtain the following two equalities

$$31a_{47} + 4a_{74} + 3a_{75} + 2a_{76} + a_{77} = 715, \tag{4.11}$$

$$1922a_{47} + 302a_{74} + 228a_{75} + 153a_{76} + 77a_{77} = 50810 \tag{4.12}$$

from (2.3)-(2.5). So, (4.11) $\times$ 151 $-$ (4.12) $\times$ 2 gives

$$837a_{47} = 6345 + 3a_{75} + 4a_{76} + 3a_{77} \geqslant 6345,$$

whence we have

$$a_{47} \geqslant 8. \tag{4.13}$$

On the other hand, (4.12) $-$ (4.11) $\times$ 62 gives

$$54a_{74} + 42a_{75} + 29a_{76} + 15a_{77} = 6480. \tag{4.14}$$

Setting $i = 78$, the maximum possible contributions of c_j's in (2.7) to the LHS of (4.14) are $(c_{74}, c_{75}, c_{78}) = (7, 1, 1)$ for $t = 6$ and $(c_{74}, c_{78}) = (1, 8)$ for $t = 9$. It follows from (4.13) that c_j's in (2.7) must be $(c_{47}, c_{78}) = (1, 8)$ for at least eight 6-lines in δ_{78}. Hence, from the spectrum (c) in Lemma 4.7, estimating the LHS of (4.14) yields

$$6480 \leqslant (54 \times 7 + 42) \times (13 - 8) + 54 \times 78 = 6312,$$

a contradiction. This completes the proof. □

The above theorem implies $n_9(4, d) \geqslant g_9(4, d) + 1$ for $617 \leqslant d \leqslant 621$, but we do not know whether $[g_9(4, d), 4, d]_9$ codes exist or not for $613 \leqslant d \leqslant 616$. So, Theorem 2.16 (3) is still open for $q = 9$.

Lemma 4.9. *There exists no* $[659, 4, 585]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[n = 659, 4, d = 585]_9$ code. By Lemma 2.1, $\gamma_0 = 1$, $\gamma_1 = 9$, $\gamma_2 = 74$. Let Δ be a γ_2-plane. Let l be an i-line with $i > 0$ containing a 1-point P. Counting the 1-points on the lines through P, we get $\gamma_2 = 74 \leqslant (9 - 1) \cdot 9 + i$, hence $2 \leqslant i$. So a j-line with $j \geqslant 1$ on Δ satisfies $j \in \{2, 3, 4, 5, 6, 7, 8, 9\}$. We have $a_i = 0$ for all $i \notin \{0, 47, 48, 65, 74\}$ by the first sieve. From (2.3)-(2.5), we get

$$2405a_0 + 243a_{47} + 221a_{48} = 2349. \tag{4.15}$$

For any w-plane through a t-line, (2.7) gives

$$\sum_j (74 - j)c_j = w + 7 - 9t \tag{4.16}$$

with $\sum_j c_j = 9$. Suppose $a_0 > 0$. Setting $w = t = 0$, the RHS of (4.16) is 7, and the equation (4.16) has no solution. Hence $a_0 = 0$. Suppose $a_{48} > 0$. The spectrum of a $[48, 3, 42]_9$ code is $(\tau_0, \tau_3, \tau_6) = (3, 16, 72)$ from Table 1. Setting $w = 48$ and $t = 6$, the equation (4.16) has no solution. Hence $a_{48} = 0$. Then, we have $243a_{47} = 2349$ from (4.15), i.e., a_{47} is not integer, a contradiction. This completes the proof. □

Lemma 4.10. *There exists no* $[578, 4, 513]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[578, 4, 513]_9$ code. Then, we have $\gamma_0 = 1$ from (2.1), and an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i + 7)/9$. Hence, we have $a_i = 0$ for all $i \notin \{0, 47, 48, 65\}$ by the first sieve. Since (2.7) has no solution for $(i, t) = (0, 0)$ and $(48, 6)$, we obtain $a_0 = a_{48} = 0$. Now, (2.3) and (2.4) yield $(a_{47}, a_{65}) = (39, 781)$, giving a contradiction in (2.5) with $\gamma_0 = 1$. □

Lemma 4.11. *There exists no* $[577, 4, 512]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[577, 4, 512]_9$ code. Then, an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i + 8)/9$, and we have $a_i = 0$ for all $i \notin \{$0, 1, 10, 28, 37, 46, 47, 48, 55, 64, 65$\}$ by the first sieve. By Lemma 4.10, we may assume that $\mathcal{C}$ is not extendable. It follows from Theorem 2.10 that $\Phi_1 = a_{48} = 324$. Let δ be a 48-plane, which has spectrum $(\tau_0, \tau_3, \tau_6) = (3, 16, 72)$ from Table 1. Let c_{48} be the number of 48-planes ($\neq \delta$) through a fixed t-line on δ. Then, we have $c_{48} \leqslant 3$ for $t = 0$; $c_{48} \leqslant 1$ for $t = 3$; $c_{48} = 0$ for $t = 6$. Hence, $a_{48} \leqslant 3\tau_0 + \tau_3 + 1 = 26$, a contradiction. This completes the proof. □

Lemma 4.12. *There exists no* $[418, 4, 370]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[n = 418, 4, d = 370]_9$ code. By Lemma 2.1, $\gamma_0 = 1$, $\gamma_1 = 6$, $\gamma_2 = 48$. Let Δ be a γ_2-plane, which has spectrum $(\tau_0, \tau_3, \tau_6) = (3, 16, 72)$ from Table 1. Then, we have $a_i = 0$ for all $i \notin \{$0, 13-17, 40-48$\}$ by the first sieve. From (2.6), we get

$$\begin{aligned}1128a_0 + 595a_{13} + 561a_{14} + 528a_{15} + 496a_{16} + 465a_{17} + 28a_{40}\\ +21a_{41} + 15a_{42} + 10a_{43} + 6a_{42} + 3a_{45} + a_{46} = 8704.\end{aligned} \tag{4.17}$$

For any w-plane through a t-line, (2.7) gives

$$\sum_j (48 - j)c_j = w + 14 - 9t \tag{4.18}$$

with $\sum_j c_j = 9$. Suppose $a_0 > 0$. Setting $w = t = 0$, the maximum possible contribution of c_j's in (4.18) to the LHS of (4.17) is $(c_{40}, c_{42}, c_{48}) = (1, 1, 7)$. Estimating the LHS of (4.17) we get

$$8704 \leqslant 1128 + (28 + 15) \times \theta_2 = 5041,$$

a contradiction. Hence $a_0 = 0$.

Next, suppose $a_{13} > 0$. Then, from (4.18) with $w = 13$, we have $a_{13} = 1$ and $a_j = 0$ for $14 \leqslant j \leqslant 17$. Setting $w = 48$, the maximum possible contributions of c_j's in (4.18) to the LHS of (4.17) are $(c_{13}, c_{40}, c_{45}, c_{48}) = (1, 3, 1, 4)$ for $t = 0$ if $c_{13} > 0$; $(c_{40}, c_{42}, c_{48}) = (7, 1, 1)$ for $t = 0$ if $c_{13} = 0$; $(c_{13}, c_{48}) = (1, 8)$ for $t = 3$ if $c_{13} > 0$; $(c_{40}, c_{44}, c_{48}) = (4, 1, 4)$ for $t = 3$ if $c_{13} = 0$; $(c_{40}, c_{48}) = (1, 8)$ for $t = 6$. Since $a_{13} = 1$, estimating the LHS of (4.17) we get

$$8704 \leqslant (595 + 3 \times 28 + 3) + (7 \times 28 + 15)(\tau_0 - 1) + (4 \times 28 + 6)\tau_3 + 28\tau_6 = 5008,$$

a contradiction. Hence $a_{13} = 0$. One can prove $a_{14} = a_{15} = a_{16} = a_{17} = 0$ similarly.

Now, we have $a_j = 0$ for all $j \leqslant 39$. Considering the maximum possible contributions of c_j's in (4.18) with $w = 48$ to the LHS of (4.17), we get a contradiction similarly as above. This completes the proof. □

Lemma 4.13. *There exists no* $[416, 4, 369]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[416, 4, 369]_9$ code. Then, an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i + 7)/9$, and we have $a_i = 0$ for all $i \notin \{0, 47\}$ by the first sieve. Since (2.7) has no solution for $(i, t) = (0, 0)$, we obtain $a_0 = 0$. Hence, $\mathcal{C}$ is one-weight, which is contradictory to Lemma 2.1. □

Lemma 4.14. *There exists no* $[415, 4, 368]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[415, 4, 368]_9$ code. Then, an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i+8)/9$, and we have $a_i = 0$ for all $i \notin \{0, 1, 10, 28, 37, 46, 47\}$ by the first sieve. Hence $\mathcal{C}$ is extendable by Theorem 2.8, which contradicts Lemma 4.13. □

Lemma 4.15. *There exists no* $[414, 4, 367]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[414, 4, 367]_9$ code. Then, an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i+9)/9$, and we have $a_i = 0$ for all $i \notin \{0, 1, 9, 10, 27, 28, 36, 37, 45, 46, 47\}$ by the first sieve. Hence $\mathcal{C}$ is extendable by Theorem 2.9, which contradicts Lemma 4.14. □

The following three lemmas can be proved similarly to Lemmas 4.13-4.15.

Lemma 4.16. *There exists no* $[244, 4, 216]_9$ *code.*

Lemma 4.17. *There exists no* $[243, 4, 215]_9$ *code.*

Lemma 4.18. *There exists no* $[242, 4, 214]_9$ *code.*

Lemma 4.19. *There exists no* $[234, 4, 207]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[234, 4, 207]_9$ code. Then, an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i + 9)/9$, and we have $a_i = 0$ for all $i \notin \{0, 1, 9, 10, 27\}$ by the first sieve. Since (2.7) has no solution for $(i, t) = (0, 0)$, $(1, 0)$, $(9, 1)$ and $(10, 1)$, we obtain $a_0 = a_1 = a_9 = a_{10} = 0$. Thus, $\mathcal{C}$ is one-weight, which is contradictory to Lemma 2.1. □

Lemma 4.20. *There exists no* $[233, 4, 206]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[233, 4, 206]_9$ code. Then, an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i+10)/9$, and we have $a_i = 0$ for all $i \notin \{0, 1, 8, 9, 10, 17, 26, 27\}$ by the first sieve. By Lemma 4.19, we may assume that $\mathcal{C}$ is not extendable. It follows from Theorem 2.10 that $\Phi_1 = a_1 + a_{10} = 324$. Suppose $a_1 > 0$. Then, (2.7) with $i = 1$,

$t \leqslant 1$ has no solution of $c_j > 0$ for $j \leqslant 10$. This means that $a_1 = 1$ and $a_{10} = 0$, giving a contradiction. Hence $a_1 = 0$ and $a_{10} = 324$. Let δ be a 10-plane, which has spectrum $(\tau_0, \tau_1, \tau_2) = (36, 10, 45)$ from Table 1. Let c_{10} be the number of 10-planes ($\neq \delta$) through a fixed t-line on δ. Then, we have $c_{10} \leqslant 1$ for $t = 0$ and $c_{10} = 0$ for $t = 1, 2$. Hence, $a_{10} \leqslant \tau_0 + 1 = 37$, a contradiction again. This completes the proof. □

Lemma 4.21. *There exists no* $[224, 4, 198]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[224, 4, 198]_9$ code. Then, $\gamma_0 = 1$ from (2.1), and we have $a_i = 0$ for all $i \notin \{0, 1, 8, 9, 10, 17, 26\}$ by the first sieve. Since (2.7) has no solution for $(i, t) = (0, 0)$, $(1, 0)$, $(9, 1)$ and $(10, 1)$, we obtain $a_0 = a_1 = a_9 = a_{10} = 0$. It follows from (2.3)-(2.5) that $\mathcal{C}$ has spectrum $(a_8, a_{17}, a_{26}) = (36, 32, 752)$. Let δ be a 17-plane, whose spectrum $(\tau_0, \tau_1, \tau_2, \tau_3)$ is one of the four possible spectra for $[17, 3, 14]_9$ codes in Table 1. Let c_{17} be the number of 17-planes ($\neq \delta$) through a fixed t-line on δ. Then, we have $c_{17} = 1$ or 3 for $t = 0$; $c_{17} = 0$ or 2 for $t = 1$; $c_{17} = 1$ for $t = 2$; $c_{17} = 0$ for $t = 3$. Hence, $a_{17} \geqslant \tau_0 + \tau_2 + 1 \geqslant 34 > 32$, a contradiction. This completes the proof. □

Lemma 4.22. *There exists no* $[143, 4, 126]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[143, 4, 126]_9$ code. Then, we have $\gamma_0 = 1$ from (2.1), and an i-plane corresponds to an $[i, 3, d_i]_9$ code with $i - d_i \leqslant (i + 10)/9$. Hence, we have $a_i = 0$ for all $i \notin \{0, 1, 8, 9, 10, 17\}$ by the first sieve. Since (2.7) has no solution for $(i, t) = (0, 0)$, $(1, 1)$, $(9, 2)$, and $(10, 2)$, we obtain $a_0 = a_1 = a_9 = a_{10} = 0$. Now, (2.3) and (2.4) yield $(a_8, a_{17}) = (103, 717)$, giving a contradiction in (2.5) with $\gamma_0 = 1$. □

Lemma 4.23. *There exists no* $[142, 4, 125]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[142, 4, 125]_9$ code. We have $\gamma_0 = 1$ by Lemma 2.1. Let Δ be a γ_2-plane, whose spectrum is one of the four spectra for $[17, 3, 14]_9$ codes in Table 1. We have $a_i = 0$ for all $i \notin \{0, 1, 7, 8, 9, 10, 16, 17\}$ by the first sieve. From (2.3)-(2.5), we get

$$136a_0 + 120a_1 + 45a_7 + 36a_8 + 28a_9 + 21a_{10} = 4878. \tag{4.19}$$

For any w-plane through a t-line, (2.7) gives

$$\sum_j (17 - j)c_j = w + 11 - 9t \tag{4.20}$$

with $\sum_j c_j = 9$. Suppose $a_0 > 0$. Setting $w = t = 0$, the maximum possible contribution of c_j's in (4.18) to the LHS of (4.17) is $(c_7, c_{16}, c_{17}) = (1, 1, 7)$. Estimating the LHS of (4.19) we get $4878 \leqslant 45\theta_2 + 136 = 4231$, a contradiction. Hence $a_0 = 0$. Similarly, one can prove that $a_1 = a_9 = a_{10} = 0$ using the spectra in Table 1. Now, applying Theorem 2.8, $\mathcal{C}$ is extendable, which contradicts Lemma 4.22. This completes the proof. □

Proof of Theorem 1.7. We first note that one can get an $[n-1, k, d-1]_q$ code from a given $[n, k, d]_q$ code by puncturing and that the nonexistence of an $[n-1, k, d-1]_q$ code implies the nonexistence of an $[n, k, d]_q$ code. For example, the existence of a $[82, 4, 72]_9$ code implies $n_9(4, d) = g_9(4, d)$ for $64 \leqslant d \leqslant 72$, and the nonexistence of a $[84, 4, 73]_9$ code implies $n_9(4, d) \geqslant g_9(4, d) + 1$ for $73 \leqslant d \leqslant 81$. The part (5) follows from Lemmas 4.12, 4.19-4.23. See Lemmas 3.8, 3.9, 3.10, 3.13 for (1), and Lemmas 3.5, 3.8, 3.9, 3.10, 3.12 for (3).

(2) For the nonexistence of Griesmer codes, see Theorem 1.2 (5), Lemmas 4.9, 4.8, 4.5, 4.4, 2.14, 2.15, 2.15, 2.14, 4.3 for $d = 136$, 585, 617, 810, 883, 964, 1108, 1117, 1126, 1189, respectively. The existence of $[g_9(4, d)+1, 4, d]_9$ codes follows from Theorem 1.4 for $d = 585$, 621, 1189-1197 and Lemmas 3.2, 3.8, 3.9, 3.10, 3.11, for $d = 144$, 810, 891, 972, 1108-1134, respectively.

(4) The nonexistence of Griesmer codes for $d = 127$, 214, 367, 512 follows from Theorem 1.2 and Lemmas 4.18, 4.15, 4.11, respectively. It follows from Theorem 2.12 (4) that Griesmer codes do not exist for $d = 145$, 154. On the other hand, there exist $[g_9(4, d) + 2, 4, d]_9$ codes for $d = 135$ by puncturing from a $[164, 4, 144]_9$ code and for $d = 152$, 158 by Lemma 3.4. As for the existence of $[g_9(4, d) + 2, 4, d]_9$ codes for other d, see Theorem 1.2 for $d = 513$ and the next (6) for $d = 216, 369$.

(6) See Lemmas 3.1, 3.4, 3.5, 3.2, 3.6, 3.7 $d = 114$, 121, 188, 216, 252, 369, respectively. A $[g_9(4, d)+2, 4, d]_9$ code for $d = 171$ is obtained by puncturing from a $[205, 4, 180]_9$ code.

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Table 2. Values and bounds for $n = n_9(4, d)$ with $g = g_9(4, d)$

d	g	n	d	g	n	d	g	n	d	g	n	d	g	n
1	4	4	64	74	74	127	145	1-2	190	216	0-3	253	287	0-3
2	5	5	65	75	75	128	146	1-2	191	217	0-3	254	288	0-3
3	6	6	66	76	76	129	147	1-2	192	218	0-3	255	289	0-3
4	7	7	67	77	77	130	148	1-2	193	219	0-3	256	290	0-3
5	8	8	68	78	78	131	149	1-2	194	220	0-4	257	291	0-3
6	9	9	69	79	79	132	150	1-2	195	221	0-4	258	292	0-3
7	10	10	70	80	80	133	151	1-2	196	222	0-4	259	293	0-3
8	11	12	71	81	81	134	152	1-2	197	223	0-4	260	294	0-3
9	12	13	72	82	82	135	153	1-2	198	224	1-4	261	295	0-3
10	14	14	73	84	85	136	155	156	199	226	0-3	262	297	0-3
11	15	15	74	85	86	137	156	157	200	227	0-3	263	298	0-3
12	16	16	75	86	87	138	157	158	201	228	0-3	264	299	0-3
13	17	18	76	87	88	139	158	159	202	229	0-3	265	300	0-3
14	18	19	77	88	89	140	159	160	203	230	0-3	266	301	0-3
15	19	20	78	89	90	141	160	161	204	231	0-3	267	302	0-4
16	20	21	79	90	91	142	161	162	205	232	0-3	268	303	0-4
17	21	22	80	91	92	143	162	163	206	233	1-3	269	304	0-4
18	22	23	81	92	1-2	144	163	164	207	234	1-3	270	305	0-4
19	24	24	82	95	0-1	145	165	1-2	208	236	0-2	271	307	0-3
20	25	0-1	83	96	0-1	146	166	1-2	209	237	0-2	272	308	0-3
21	26	0-1	84	97	0-1	147	167	1-2	210	238	0-2	273	309	0-3
22	27	0-1	85	98	0-1	148	168	1-2	211	239	0-2	274	310	0-4
23	28	0-1	86	99	0-1	149	169	1-2	212	240	0-2	275	311	0-4
24	29	0-1	87	100	0-1	150	170	1-2	213	241	0-2	276	312	0-4
25	30	31	88	101	0-1	151	171	1-2	214	242	1-2	277	313	0-4
26	31	32	89	102	0-2	152	172	1-2	215	243	1-2	278	314	0-4
27	32	33	90	103	0-2	153	173	1-3	216	244	1-2	279	315	0-4
28	34	34	91	105	0-1	154	175	1-2	217	246	1-3	280	317	0-3
29	35	35	92	106	0-1	155	176	1-2	218	247	1-3	281	318	0-3
30	36	36	93	107	0-1	156	177	1-2	219	248	1-3	282	319	0-3
31	37	0-1	94	108	0-1	157	178	1-2	220	249	1-3	283	320	0-3
32	38	0-1	95	109	0-2	158	179	1-2	221	250	1-3	284	321	0-3
33	39	40	96	110	0-2	159	180	1-3	222	251	1-3	285	322	0-3
34	40	41	97	111	0-2	160	181	1-3	223	252	1-3	286	323	0-3
35	41	1-2	98	112	0-2	161	182	1-3	224	253	1-3	287	324	0-3
36	42	1-2	99	113	0-2	162	183	1-3	225	254	1-3	288	325	0-3
37	44	0-1	100	115	0-1	163	186	0-2	226	256	1-3	289	327	1-4
38	45	0-1	101	116	0-1	164	187	0-2	227	257	1-3	290	328	1-4
39	46	0-1	102	117	0-1	165	188	0-2	228	258	1-3	291	329	1-4
40	47	0-1	103	118	0-2	166	189	0-2	229	259	1-3	292	330	1-4
41	48	0-2	104	119	0-2	167	190	0-2	230	260	1-3	293	331	1-4
42	49	0-2	105	120	0-2	168	191	0-2	231	261	1-4	294	332	1-4
43	50	1-2	106	121	0-2	169	192	0-2	232	262	1-4	295	333	1-4
44	51	1-2	107	122	0-2	170	193	0-2	233	263	1-4	296	334	1-4
45	52	1-2	108	123	0-2	171	194	0-2	234	264	1-4	297	335	1-4
46	54	0-1	109	125	0-2	172	196	0-1	235	266	1-3	298	337	1-4
47	55	0-1	110	126	0-2	173	197	0-1	236	267	1-3	299	338	1-4
48	56	0-1	111	127	0-2	174	198	0-1	237	268	1-3	300	339	1-4
49	57	58	112	128	0-2	175	199	0-1	238	269	1-3	301	340	1-4
50	58	1-2	113	129	0-2	176	200	0-1	239	270	1-3	302	341	1-4
51	59	1-2	114	130	0-2	177	201	0-1	240	271	1-3	303	342	1-5
52	60	1-2	115	131	0-3	178	202	0-1	241	272	1-4	304	343	1-5
53	61	1-2	116	132	0-3	179	203	0-1	242	273	1-4	305	344	1-5
54	62	1-2	117	133	0-3	180	204	0-1	243	274	1-4	306	345	1-5
55	64	0-1	118	135	0-2	181	206	0-2	244	277	0-2	307	347	1-4
56	65	0-1	119	136	0-2	182	207	0-2	245	278	0-2	308	348	1-4
57	66	0-1	120	137	0-2	183	208	0-2	246	279	0-2	309	349	1-4
58	67	68	121	138	0-2	184	209	0-2	247	280	0-2	310	350	1-4
59	68	69	122	139	0-3	185	210	0-2	248	281	0-2	311	351	1-4
60	69	70	123	140	0-3	186	211	0-2	249	282	0-2	312	352	1-4
61	70	71	124	141	0-3	187	212	0-2	250	283	0-2	313	353	1-4
62	71	72	125	142	1-3	188	213	0-2	251	284	0-2	314	354	1-4
63	72	73	126	143	1-3	189	214	0-3	252	285	0-2	315	355	1-4

Table 2 (continued)

d	g	n	d	g	n	d	g	n	d	g	n	d	g	n
316	357	1-3	379	428	1-4	442	499	1-3	505	570	0-2	568	641	641
317	358	1-3	380	429	1-4	443	500	1-3	506	571	0-2	569	642	642
318	359	1-3	381	430	1-4	444	501	1-3	507	572	0-2	570	643	643
319	360	1-3	382	431	1-4	445	502	1-3	508	573	0-2	571	644	644
320	361	1-3	383	432	1-4	446	503	1-3	509	574	0-2	572	645	645
321	362	1-3	384	433	1-4	447	504	1-3	510	575	0-2	573	646	646
322	363	1-3	385	434	1-4	448	505	1-3	511	576	0-2	574	647	647
323	364	1-3	386	435	1-4	449	506	1-3	512	577	1-2	575	648	648
324	365	1-3	387	436	1-4	450	507	1-3	513	578	1-2	576	649	649
325	368	0-3	388	438	1-3	451	509	1-3	514	580	1-2	577	651	0-1
326	369	0-3	389	439	1-3	452	510	1-3	515	581	1-2	578	652	0-1
327	370	0-3	390	440	1-3	453	511	1-3	516	582	1-2	579	653	0-1
328	371	0-3	391	441	1-4	454	512	1-3	517	583	1-2	580	654	0-1
329	372	0-3	392	442	1-4	455	513	1-3	518	584	1-2	581	655	0-1
330	373	0-3	393	443	1-4	456	514	1-3	519	585	1-2	582	656	0-1
331	374	0-3	394	444	1-4	457	515	1-3	520	586	1-2	583	657	0-1
332	375	0-3	395	445	1-4	458	516	1-3	521	587	1-2	584	658	0-1
333	376	0-4	396	446	1-4	459	517	1-3	522	588	1-2	585	659	660
334	378	0-4	397	448	1-4	460	519	1-3	523	590	1-2	586	661	0-1
335	379	0-4	398	449	1-4	461	520	1-3	524	591	1-2	587	662	0-1
336	380	0-4	399	450	1-4	462	521	1-3	525	592	1-2	588	663	0-1
337	381	0-4	400	451	1-4	463	522	1-3	526	593	1-2	589	664	0-1
338	382	0-4	401	452	1-4	464	523	1-3	527	594	1-2	590	665	0-1
339	383	0-4	402	453	1-4	465	524	1-3	528	595	1-2	591	666	0-1
340	384	0-4	403	454	1-4	466	525	1-3	529	596	1-2	592	667	668
341	385	0-4	404	455	1-4	467	526	1-3	530	597	1-2	593	668	669
342	386	0-4	405	456	1-4	468	527	1-3	531	598	1-2	594	669	670
343	388	0-4	406	459	0-3	469	529	1-3	532	600	1-2	595	671	0-1
344	389	0-4	407	460	0-3	470	530	1-3	533	601	1-2	596	672	0-1
345	390	0-4	408	461	0-3	471	531	1-3	534	602	1-2	597	673	0-1
346	391	0-4	409	462	0-3	472	532	1-3	535	603	1-2	598	674	0-1
347	392	0-4	410	463	0-3	473	533	1-3	536	604	1-2	599	675	0-1
348	393	0-4	411	464	0-3	474	534	1-3	537	605	1-2	600	676	0-1
349	394	0-4	412	465	0-3	475	535	1-3	538	606	1-2	601	677	678
350	395	0-4	413	466	0-3	476	536	1-3	539	607	1-2	602	678	679
351	396	0-4	414	467	0-3	477	537	1-3	540	608	1-2	603	679	680
352	398	0-3	415	469	0-3	478	539	1-3	541	610	1-2	604	681	0-1
353	399	0-3	416	470	0-3	479	540	1-3	542	611	1-2	605	682	0-1
354	400	0-3	417	471	0-3	480	541	1-3	543	612	1-2	606	683	0-1
355	401	0-3	418	472	0-3	481	542	1-3	544	613	1-2	607	684	0-1
356	402	0-3	419	473	0-3	482	543	1-3	545	614	1-2	608	685	0-1
357	403	0-3	420	474	0-3	483	544	1-3	546	615	1-2	609	686	0-1
358	404	0-3	421	475	0-3	484	545	1-3	547	616	1-2	610	687	688
359	405	0-3	422	476	0-3	485	546	1-3	548	617	1-2	611	688	689
360	406	0-3	423	477	0-3	486	547	1-3	549	618	1-2	612	689	690
361	408	0-2	424	479	0-3	487	550	0-2	550	620	1-2	613	691	0-1
362	409	0-2	425	480	0-3	488	551	0-2	551	621	1-2	614	692	0-1
363	410	0-2	426	481	0-3	489	552	0-2	552	622	1-2	615	693	0-1
364	411	0-2	427	482	0-3	490	553	0-2	553	623	1-2	616	694	0-1
365	412	0-2	428	483	0-3	491	554	0-2	554	624	1-2	617	695	696
366	413	0-2	429	484	0-3	492	555	0-2	555	625	1-2	618	696	697
367	414	1-2	430	485	0-3	493	556	0-2	556	626	1-2	619	697	698
368	415	1-2	431	486	0-3	494	557	0-2	557	627	1-2	620	698	699
369	416	1-2	432	487	0-3	495	558	0-2	558	628	1-2	621	699	700
370	418	1-4	433	489	1-3	496	560	0-2	559	630	631	622	701	702
371	419	1-4	434	490	1-3	497	561	0-2	560	631	632	623	702	703
372	420	1-4	435	491	1-3	498	562	0-2	561	632	633	624	703	704
373	421	1-4	436	492	1-3	499	563	0-2	562	633	634	625	704	705
374	422	1-4	437	493	1-3	500	564	0-2	563	634	1-2	626	705	706
375	423	1-4	438	494	1-3	501	565	0-2	564	635	1-2	627	706	707
376	424	1-4	439	495	1-3	502	566	0-2	565	636	1-2	628	707	708
377	425	1-4	440	496	1-3	503	567	0-2	566	637	1-2	629	708	709
378	426	1-4	441	497	1-3	504	568	0-2	567	638	1-2	630	709	710

Table 2 (continued)

d	g	n	d	g	n	d	g	n	d	g	n	d	g	n
631	711	712	856	965	0-1	919	1036	0-1	982	1107	1107	1045	1177	1178
632	712	713	857	966	0-1	920	1037	0-1	983	1108	1108	1046	1178	1179
633	713	714	858	967	0-1	921	1038	0-1	984	1109	1109	1047	1179	1180
634	714	715	859	968	0-1	922	1039	0-1	985	1110	1110	1048	1180	1181
635	715	716	860	969	0-1	923	1040	0-1	986	1111	1111	1049	1181	1182
636	716	717	861	970	0-1	924	1041	0-1	987	1112	1112	1050	1182	1183
637	717	718	862	971	0-1	925	1042	0-1	988	1113	1113	1051	1183	1184
638	718	719	863	972	0-1	926	1043	0-1	989	1114	1114	1052	1184	1185
639	719	720	864	973	0-1	927	1044	0-1	990	1115	1115	1053	1185	1186
802	904	0-1	865	975	0-1	928	1046	0-1	991	1117	1117	1054	1188	1188
803	905	0-1	866	976	0-1	929	1047	0-1	992	1118	1118	1055	1189	1189
804	906	0-1	867	977	0-1	930	1048	0-1	993	1119	1119	1056	1190	1190
805	907	0-1	868	978	0-1	931	1049	0-1	994	1120	1120	1057	1191	1191
806	908	0-1	869	979	0-1	932	1050	0-1	995	1121	1121	1058	1192	1192
807	909	0-1	870	980	0-1	933	1051	0-1	996	1122	1122	1059	1193	1193
808	910	0-1	871	981	0-1	934	1052	0-1	997	1123	1123	1060	1194	1194
809	911	0-1	872	982	0-1	935	1053	0-1	998	1124	1124	1061	1195	1195
810	912	913	873	983	0-1	936	1054	0-1	999	1125	1125	1062	1196	1196
811	915	915	874	985	0-1	937	1056	0-1	1000	1127	0-1	1063	1198	1198
812	916	916	875	986	0-1	938	1057	0-1	1001	1128	0-1	1064	1199	1199
813	917	917	876	987	0-1	939	1058	0-1	1002	1129	0-1	1065	1200	1200
814	918	918	877	988	0-1	940	1059	0-1	1003	1130	0-1	1066	1201	1201
815	919	919	878	989	0-1	941	1060	0-1	1004	1131	0-1	1067	1202	1202
816	920	920	879	990	0-1	942	1061	0-1	1005	1132	0-1	1068	1203	1203
817	921	921	880	991	0-1	943	1062	0-1	1006	1133	0-1	1069	1204	1204
818	922	922	881	992	0-1	944	1063	0-1	1007	1134	0-1	1070	1205	1205
819	923	923	882	993	0-1	945	1064	0-1	1008	1135	0-1	1071	1206	1206
820	925	925	883	995	996	946	1066	0-1	1009	1137	0-1	1072	1208	1208
821	926	926	884	996	997	947	1067	0-1	1010	1138	0-1	1073	1209	1209
822	927	927	885	997	998	948	1068	0-1	1011	1139	0-1	1074	1210	1210
823	928	928	886	998	999	949	1069	0-1	1012	1140	0-1	1075	1211	1211
824	929	929	887	999	1000	950	1070	0-1	1013	1141	0-1	1076	1212	1212
825	930	930	888	1000	1001	951	1071	0-1	1014	1142	0-1	1077	1213	1213
826	931	931	889	1001	1002	952	1072	0-1	1015	1143	0-1	1078	1214	1214
827	932	932	890	1002	1003	953	1073	0-1	1016	1144	0-1	1079	1215	1215
828	933	933	891	1003	1004	954	1074	0-1	1017	1145	0-1	1080	1216	1216
829	935	935	892	1006	1006	955	1076	0-1	1018	1147	0-1	1081	1218	0-1
830	936	936	893	1007	1007	956	1077	0-1	1019	1148	0-1	1082	1219	0-1
831	937	937	894	1008	1008	957	1078	0-1	1020	1149	0-1	1083	1220	0-1
832	938	938	895	1009	1009	958	1079	0-1	1021	1150	0-1	1084	1221	0-1
833	939	939	896	1010	1010	959	1080	0-1	1022	1151	0-1	1085	1222	0-1
834	940	940	897	1011	1011	960	1081	0-1	1023	1152	0-1	1086	1223	0-1
835	941	941	898	1012	1012	961	1082	0-1	1024	1153	0-1	1087	1224	0-1
836	942	942	899	1013	1013	962	1083	0-1	1025	1154	0-1	1088	1225	0-1
837	943	943	900	1014	1014	963	1084	0-1	1026	1155	0-1	1089	1226	0-1
838	945	0-1	901	1016	1016	964	1086	1087	1027	1157	0-1	1090	1228	0-1
839	946	0-1	902	1017	1017	965	1087	1088	1028	1158	0-1	1091	1229	0-1
840	947	0-1	903	1018	1018	966	1088	1089	1029	1159	0-1	1092	1230	0-1
841	948	0-1	904	1019	1019	967	1089	1090	1030	1160	0-1	1093	1231	0-1
842	949	0-1	905	1020	1020	968	1090	1091	1031	1161	0-1	1094	1232	0-1
843	950	0-1	906	1021	1021	969	1091	1092	1032	1162	0-1	1095	1233	0-1
844	951	0-1	907	1022	1022	970	1092	1093	1033	1163	0-1	1096	1234	0-1
845	952	0-1	908	1023	1023	971	1093	1094	1034	1164	0-1	1097	1235	0-1
846	953	0-1	909	1024	1024	972	1094	1095	1035	1165	0-1	1098	1236	0-1
847	955	0-1	910	1026	1026	973	1097	1097	1036	1167	1168	1099	1238	0-1
848	956	0-1	911	1027	1027	974	1098	1098	1037	1168	1169	1100	1239	0-1
849	957	0-1	912	1028	1028	975	1099	1099	1038	1169	1170	1101	1240	0-1
850	958	0-1	913	1029	1029	976	1100	1100	1039	1170	1171	1102	1241	0-1
851	959	0-1	914	1030	1030	977	1101	1101	1040	1171	1172	1103	1242	0-1
852	960	0-1	915	1031	1031	978	1102	1102	1041	1172	1173	1104	1243	0-1
853	961	0-1	916	1032	1032	979	1103	1103	1042	1173	1174	1105	1244	0-1
854	962	0-1	917	1033	1033	980	1104	1104	1043	1174	1175	1106	1245	0-1
855	963	0-1	918	1034	1034	981	1105	1105	1044	1175	1176	1107	1246	0-1

Table 2 (continued)

d	g	n	d	g	n
1108	1248	1249	1171	1319	0-1
1109	1249	1250	1172	1320	0-1
1110	1250	1251	1173	1321	0-1
1111	1251	1252	1174	1322	0-1
1112	1252	1253	1175	1323	0-1
1113	1253	1254	1176	1324	0-1
1114	1254	1255	1177	1325	0-1
1115	1255	1256	1178	1326	0-1
1116	1256	1257	1179	1327	0-1
1117	1258	1259	1180	1329	0-1
1118	1259	1260	1181	1330	0-1
1119	1260	1261	1182	1331	0-1
1120	1261	1262	1183	1332	0-1
1121	1262	1263	1184	1333	0-1
1122	1263	1264	1185	1334	0-1
1123	1264	1265	1186	1335	0-1
1124	1265	1266	1187	1336	0-1
1125	1266	1267	1188	1337	0-1
1126	1268	1269	1189	1339	1340
1127	1269	1270	1190	1340	1341
1128	1270	1271	1191	1341	1342
1129	1271	1272	1192	1342	1343
1130	1272	1273	1193	1343	1344
1131	1273	1274	1194	1344	1345
1132	1274	1275	1195	1345	1346
1133	1275	1276	1196	1346	1347
1134	1276	1277	1197	1347	1348
1135	1279	1279	1198	1349	1350
1136	1280	1280	1199	1350	1351
1137	1281	1281	1200	1351	1352
1138	1282	1282	1201	1352	1353
1139	1283	1283	1202	1353	1354
1140	1284	1284	1203	1354	1355
1141	1285	1285	1204	1355	1356
1142	1286	1286	1205	1356	1357
1143	1287	1287	1206	1357	1358
1144	1289	1289	1207	1359	1360
1145	1290	1290	1208	1360	1361
1146	1291	1291	1209	1361	1362
1147	1292	1292	1210	1362	1363
1148	1293	1293	1211	1363	1364
1149	1294	1294	1212	1364	1365
1150	1295	1295	1213	1365	1366
1151	1296	1296	1214	1366	1367
1152	1297	1297	1215	1367	1368
1153	1299	0-1			
1154	1300	0-1			
1155	1301	0-1			
1156	1302	0-1			
1157	1303	0-1			
1158	1304	0-1			
1159	1305	0-1			
1160	1306	0-1			
1161	1307	0-1			
1162	1309	0-1			
1163	1310	0-1			
1164	1311	0-1			
1165	1312	0-1			
1166	1313	0-1			
1167	1314	0-1			
1168	1315	0-1			
1169	1316	0-1			
1170	1317	0-1			