Effects of the Number of Energy Levels on Radiative Collapse of Hydrogen Plasmas

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Effects of the Number of Energy Levels on Radiative Collapse of Hydrogen Plasmas

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The evolution of cascading systems consisting of an arbitrary number of energy levels was investigated analytically. Solving a system of differential equations under a given initial condition, the expressions for the population of each level were derived, which include the transition probability between excited states. Applying these results to hydrogen plasmas, radiative collapse in which excited atoms emit radiation and cascade to the ground state was discussed. In particular, effects of the number of energy levels on radiative power were studied.

1. Introduction

When the electron density varies rapidly in partially ionized gases, the population distribution of excited atoms evolves accordingly, which is seriously dependent on the characteristic time of electron density variation as well as the electron density itself.[1-3] When the electron density is large, the processes of excitation and de-excitation by electron collisions are important. On the contrary, when the electron density becomes small, the radiative processes become dominant.

In this paper, we will consider a limiting case in which the electron density vanishes abruptly and, after that, the collisional processes are negligible and only the radiative decay processes are important. At first, we will deal with the evolution of cascading systems, in which only the transition from upper excited states to lower excited states are allowed. It will be shown that for these systems an analytical solution will be obtainable under arbitrary initial conditions. Then we will apply these results to radiative collapse of hydrogen plasmas and effects of the number of excited levels will be discussed.

2. Time Evolution of Cascading Systems

We will consider a cascading system consisting of different states of energy, the number of which is n. Each state is numbered from 1 to n in the increasing order of energy. The transition probability from state p to state q is $A_{p,q}$. We will assume that the transition always takes place from upper states to lower states and the inverse processes, i.e., excitation, never occur. Thus, $A_{p,q}=0$ for $p<q$. The life-time of state p is given by $1/A(p)$, where $A(p)$ is the decay constant of the excited state p and is expressed as

$$A(p)=\sum_{q<p} A_{p,q} \tag{1}$$

It should be noted that $A(1)=0$ since the state 1 corresponds to the ground state, i.e., the state of the lowest energy, and is stable. The evolution of the population N_p of excited state p is described by the following system of differential equations:

$$\frac{dN_p}{dt}=\sum_{q>p}^{n} A_{q,p}N_q - A(p)N_p \tag{2}$$

In this paper we will solve this system of differential equations under arbitrary initial conditions: $N_p=N_p(0)$ at $t=0$. For convenience, we will use the coefficients $c_{p,q}$ defined by

$$c_{p,q}=A_{q,p} \quad (p\neq q)$$
$$c_{p,p}=A(p)$$

where $A(p)$ is defined by Eq.(1). Then Eqs.(2) are rewritten as

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$$\frac{dN_p}{dt} = \sum_{q>p}^{n} c_{p,q} N_q - c_{p,p} N_p \tag{3}$$

It should be noted that since $c_{p,q}=0$ for $p>q$, Eqs.(3) can be solved analytically. When the initial population $N_p(0)$ of state p is assigned, the result is given as follows:

$$N_p = \sum_{q \geq p}^{n} B_{p,q} \exp(-c_{q,q} t) \tag{4}$$

where the coefficients $B_{p,q}$ are given by

$$B_{p,q} = \sum_{i=p+1}^{q} \frac{c_{p,i} B_{i,q}}{c_{p,p} - c_{q,q}} \tag{5}$$

for $p<q$,

$$B_{p,p} = N_p(0) - \sum_{q>p}^{n} B_{p,q} \tag{6}$$

for $p=q$ and

$$B_{p,q} = 0 \tag{7}$$

for $p>q$. The coefficients $B_{p,q}$ can be calculated from Eqs.(5)−(7) in the descending order of subscripts p and q. At first $B_{n,n}$ is determined from Eq.(6), i.e., $B_{n,n}=N_n(0)$. Then $B_{n-1,n}$ is obtained from Eq.(5), and $B_{n-1,n-1}$ is obtained from Eq.(6) in terms of $N_{n-1}(0)$, $B_{n-1,n}$ and $B_{n,n}$, and so on.

It should be noted that $c_{1,1}=0$ since the ground state is stable, and thus the asymptotic population of the ground state is $N_1(\infty)=B_{1,1}$ by virtue of Eq.(4). This asymptotic population is readily obtainable from physical consideration as

$$N_1(\infty) = \sum_{p=1}^{n} N_p(0)$$

Thus, the following relation is obtained:

$$B_{1,1} = \sum_{p=1}^{n} N_p(0)$$

In the next section, we will apply these results to a radiative collapse of hydrogen plasmas.

3. Radiative Collapse of Hydrogen Plasmas

If a partially-ionized plasma is in thermal equilibrium at temperature T, the number density of the excited atoms N_p^E at state p is given by the Saha equation

$$N_p^E = N_e^2 / F_p(T) \tag{8}$$

where N_e is the electron density and $F_p(T)$ is the thermal equilibrium constant defined by

$$F_p(T) = \frac{2g_i}{g_p} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp\left(-\frac{E_p}{kT} \right) \tag{9}$$

In the above relation, g_i and g_p are respectively the statistical weight of the ion and the excited atom of state p, and E_p is the ionization energy of the excited atom of state p. The other nomenclature is the standard one.

When the electron density becomes abruptly so small that atomic processes induced by electron impact may be neglected compared to radiative transition, then excited atoms cascade to the ground state by spontaneous emission of radiation. If the reabsorption of radiation is neglected, the equations describing the evolution of the number density N_p of excited atoms at state p are given by Eqs.(2), where, in this case, $A_{p,q}$ is the probability of radiative transition from state p to state q. We will introduce nondimensional quantities $x_p=N_p/N_p^E$. Then the governing equations are

$$\frac{dx_p}{dt} = \sum_{q>p}^{n} c_{p,q} x_q - c_{p,p} x_p \tag{10}$$

In this case,

$$c_{p,q} = A_{q,p} \frac{F_p}{F_q} \qquad (p \neq q)$$

$$c_{p,p} = \sum_{q<p} A_{p,q} \equiv A(p)$$

It should be noted that if the plasma is in thermal equilibrium at $t=0$, the initial condition is $x_p(0)=1$

for p=1, ..., n, where n is the uppermost level of excited states considered. The solution of Eqs.(10) is expressed as follows:

$$x_p = \sum_{q \geq p}^{n} B_{p,q} \exp(-c_{q,q}\, t) \qquad (11)$$

Here the coefficients $B_{p,q}$ are given by Eqs.(5)−(7) in which $N_p(0)$ should be replaced with $x_p(0)=1$.

4. Results and Discussion

In this section we will investigate the evolution of the population of excited atoms utilizing Eq.(11). At first, we will calculate the transition probabilities $A_{p,q}$ by making use of Johnson's formula.[4] In particular, the decay constant $A(p)$ of excited state p is given approximately by

$$A(p) \simeq 1.59\times10^{10} p^{-4.52}\ [s^{-1}]$$

which is readily obtained for p=2∼40 by the least square method. This result agrees quite well with the formula given by Griem.[1]

A typical example of numerical calculation by making use of Eq.(11) is shown in Fig.1 for the evolution of the population of excited atoms for n=40 and T =10000 K. It is seen from this figure that the population of an excited state p, for example, evolves in three stages. In the first stage, the population decreases exponentially with time, the decay constant of which is equal to $c_{p,p}=A(p)$, i.e.,

$$x_p = \exp(-c_{p,p}t) \qquad (12)$$

The higher the excited state, the longer the life-time. Thus, after the first stage, the population decreases according to a power law rather than exponentially. This is due to the atoms of higher excited level decaying into this level. The transition from the first stage to the second occurs earlier as the temperature increases. This is shown in Fig.2, in which the evolution of the population of excited level p=2 is plotted for n=40 and T=10000∼100000 K. From Figs.1 and 2 the population evolution is approximately expressed as

$$x_p(t) \propto t^{-\alpha}$$

In this equation the exponent α depends not only on the considered state itself but also on the initial temperature T and the number of excited states n. In the case of n=40, $\alpha \simeq 0.4$ for lower excited states (see Figs.1−3). In the last stage, the population decreases exponentially again. In this stage, however, the decay constant is equal to that of the uppermost excited state $c_{n,n}$.

The duration of the second stage is seriously dependent on the uppermost level. As the number of the considered levels n increases, the duration of the second stage becomes longer. This is shown in Fig.3, in which

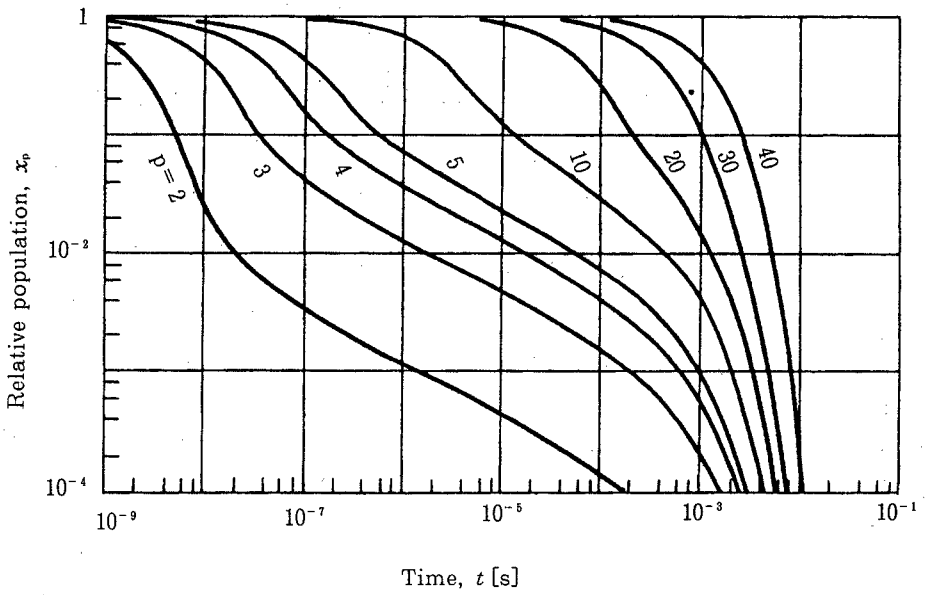

Fig. 1. Evolution of the population of excited atoms for n=40 and T=10000 K.

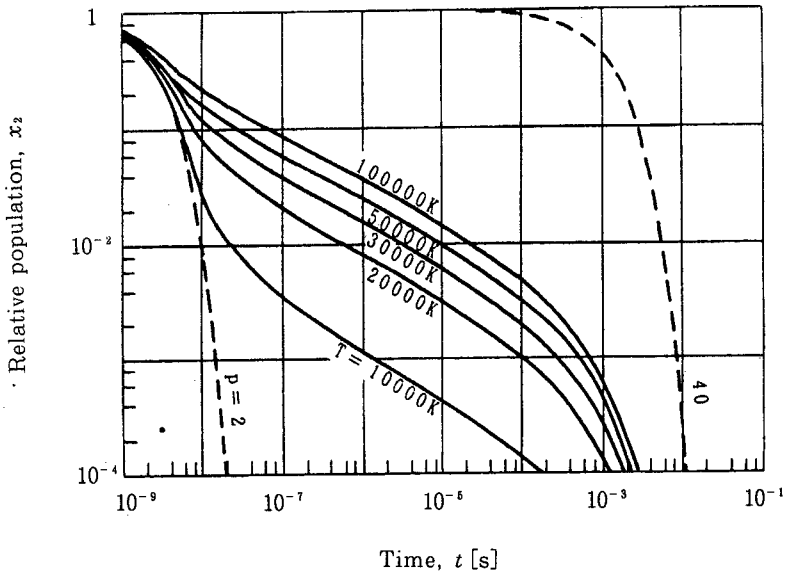

Fig. 2. Influence of the temperature on the evolution of the population of atoms in the first excited state for n=40. Dashed lines are Eq.(12) for p=2 and 40.

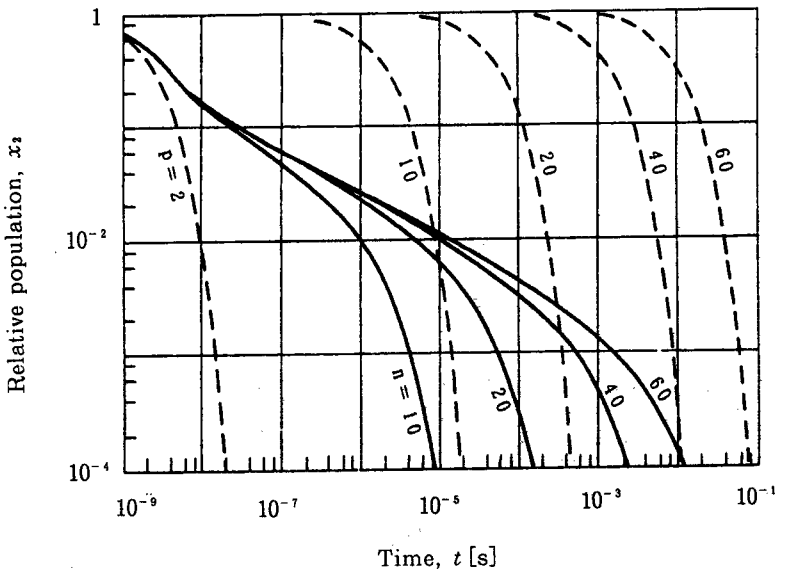

Fig. 3. Influence of the number of excited states on the evolution of the population of atoms in the first excited state for T=50000 K. Dashed lines are Eq.(12) for p=2, 10, 20, 40 and 60.

the population of level p=2 is plotted for T=50000 K and n=10, 20, 40 and 60.

The asymptotic population of the ground state $N_1(\infty)$ is

$$N_1(\infty)=N_1^E B_{1,1}=\sum_{p=1}^{n} N_p^E$$

This means that all the excited atoms decay into the ground state. Thus the asymptotic population $N_1(\infty)$ depends on the temperature as well as the number of considered levels. This is shown in Fig.4 where $x_1(\infty)=N_1(\infty)/N_1^E$ is plotted as a function of temperature T.

The evolution of radiative power is shown in Fig.5 for the temperature T=50000 K. It is seen from this figure that in order to obtain the appropriate result, the number of energy levels should be made to in-

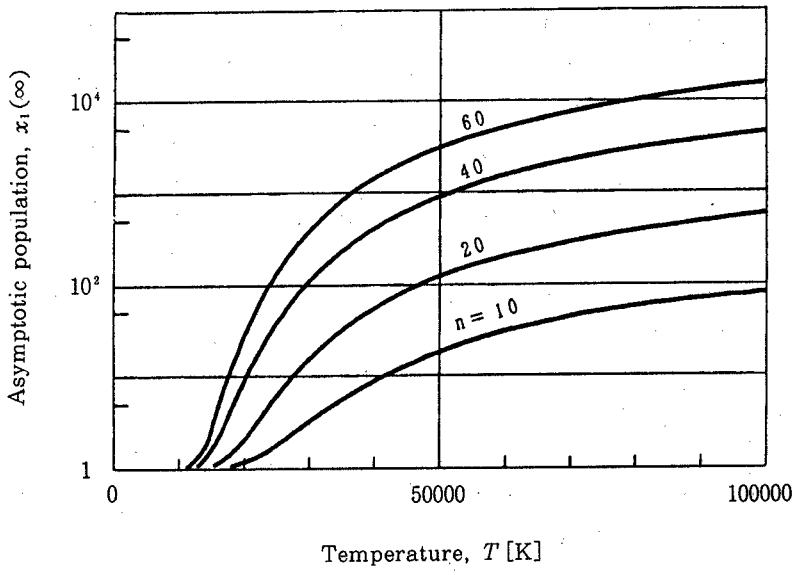

Fig. 4. Asymptotic non-dimensional population of atoms in the ground state $x_1(\infty)$ vs temperature T for n=10, 20, 40 and 60.

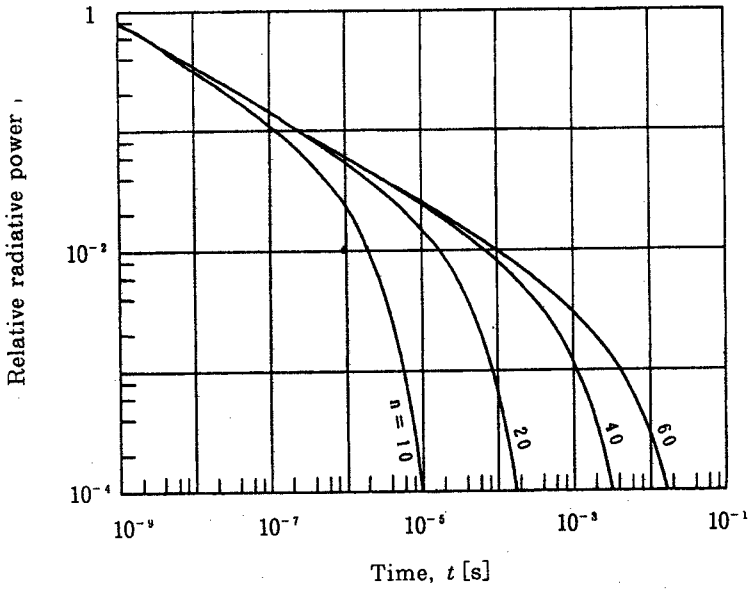

Fig. 5. Influence of the number of excited states on the evolution of radiative power for T=50000 K.

crease with time. For $t<10^{-6}$s, it is sufficient to consider 10 energy levels. We must take, however, at least 20 energy levels for $t\simeq10^{-5}$s and 40 energy levels for $t\simeq10^{-3}$s. The same result is obtained for other temperatures. This result is predicted from the fact that the life-time of the uppermost energy level n, given by $1/A(n)$, increases with n. In fact, the life-time of the level n=10, 20, and 40 is approximately equal to 1.9×10^{-6}s, 4.7×10^{-5}s and 1.2×10^{-3}s, respectively. This implies that the excited atoms at level n=20, for example, decay and emit radiation at $t\simeq10^{-5}$s.

5. Conclusions

The evolution of cascading systems was dealt with

theoretically, and analytical expressions for the evolution of the population of each state were derived, which include the transition probability between excited states. Applying these results to hydrogen plasmas, radiative collapse was investigated. Assuming that the population distribution of excited atoms was initially in thermal equilibrium at a temperature, we calculated the evolution of the population of excited atoms. It was found that the evolution can be divided into three stages. In the first stage, the population of an excited state decreases exponentially with the decay constant of the state. Then it decreases following a power law with time. After this second stage, it decreases again exponentially in the third stage with the decay constant of the uppermost excited state. We found that the number of excited states is an important parameter. As this number increases, the transition from the second stage to the third is delayed and the duration of the second stage becomes long. It was shown that when the initial equilibrium temperature is high, the transition from the first stage to the second occurs earlier. The evolution of radiative power was also investigated and it was found that, in order to obtain the appropriate result, the number of energy levels should be made to increase correspondingly to the time considered.

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