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Frozen Region Formed around a Circular Cylinder Immersed in a Darcy Flow at Low Peclet Number

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The frozen region formed around a cold circular cylinder in a Darcy flow is investigated. The Peclet number is assumed to be small and the method of matched asymptotic expansions is used to obtain the asymptotic solution of the energy equation. The equation which predicts the diameter of the frozen region at the steady state is obtained.

1. Introduction

The present paper is concerned with a frozen region formed around a cold circular cylinder immersed in a porous medium through which a liquid is flowing according to Darcy's law. Yosinobu[1] already investigated the similar problem. He considers a frozen region formed around a constant line heat sink, instead of a freezing tube of finite radius, immersed in a Darcy flow and obtained the diameter and the geometrical shape of the frozen region at small Peclet number in the ultimate steady state that the system will reach. His theory is valid only when the diameter of the frozen region is much larger than that of the freezing tube.

The purpose of the present paper is to extend the problem considered by Yosinobu to the case of a freezing tube of finite radius. The Peclet number is assumed to be small and the method of matched asymptotic expansions[2] is used to obtain the temperature field outside the frozen region. Although the analysis is developed for the case when the cylinder is immersed in a porous medium, it is shown in the discussion that the results obtained are valid also for the case of pure fluids up to the order ε, ε being a perturbation parameter defined later.

2. Formulation of the Problem

Consider a circular cylinder of radius $r_0(=D/2)$ immersed in a porous medeum through which a liquid is flowing according to Darcy's law. If we cool the surface of the cylinder below the freezing temperature T_0 of the liquid, a frozen region will be formed around it. We shall restrict our interest only to the ultimate steady state that the system will reach. The governing equations for the superficial velocity and the locally averaged temperature can be written, in non-dimensional form, as

$$\Delta\psi=0, \tag{1}$$

$$Pe\left(u_r\frac{\partial t}{\partial r}+\frac{u_\theta}{r}\frac{\partial t}{\partial \theta}\right)=\Delta t, \tag{2}$$

$$\Delta=\partial^2/\partial r^2+(1/r)\partial/\partial r+(1/r^2)\partial^2/\partial\theta^2, \tag{3}$$

where radial coordinate r is referred to L/2, L being the length of the diameter of the frozen region between the front and the rear stagnation points, the velocities u_r and u_θ to U_∞ and

$$t=\frac{t'-T_\infty}{T_0-T_\infty}, \tag{4}$$

t' being the dimensional temperature outside the frozen region. The stream function ψ is defined as

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\ u_\theta=-\frac{\partial\psi}{\partial r} \tag{5}$$

The governing equation for the temperature

$$t^*=\frac{t^{*\prime}-T_\infty}{T_0-T_\infty}, \tag{6}$$

in the frozen region, $t^{*\prime}$ being the dimensional temperature in the frozen region, is

$$\Delta t^*=0, \tag{7}$$

The boundary conditions are

$$t^*=\theta_w \qquad \text{at } r=1/m, \tag{8a}$$

$$t^*=t=1,\ k\frac{\partial t}{\partial n}=k^*\frac{\partial t^*}{\partial n},\ \psi=0 \quad \text{at } r=S(\theta), \tag{8b}$$

$$t\to 0,\ \psi\to r\sin\theta \qquad \text{as } r\to\infty, \tag{8c}$$

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where $\theta_w=(T_w-T_\infty)/(T_0-T_\infty)$.

3. Asymptotic Solutions for Small *Pe*

We now assume that the Peclet number Pe is small and shall obtain solutions of the above equations in forms of expansions in terms of Pe. For obtaining solutions outside the frozen region, we use the method of matched asymptotic expansions. In the inner region near the surface of the frozen region, we assume expansionsfor ψ and t of the forms,

$$\psi=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots+O(Pe), \tag{9}$$

$$t=t_0+\varepsilon t_1+\varepsilon^2 t_2+\cdots+O(Pe), \tag{10}$$

where

$$\varepsilon=(-\ln Pe+\ln 4-\gamma)^{-1}=(-\ln Pe^*+\ln\frac{4}{m}-\gamma)^{-1} \tag{11}$$

These assumptions are similar to those for other two-dimensional flow problems at low Peclet number. In the outer region far from the frozen region, we introduce the following outer variables,

$$R=Pe\cdot r,\ \Psi(R,\theta)=Pe\psi(r,\theta),\ T(R,\theta)=t(r,\theta)/\delta(Pe), \tag{12}$$

where $\delta(Pe)$ is the still unknown order of the temperature in the outer region. In terms of these variables, Eq. (2) can be written as

$$U_R\frac{\partial T}{\partial R}+\frac{U_\theta}{R}\frac{\partial T}{\partial\theta}=\Delta_R T \tag{13}$$

where Δ_R is the same differential operator as Δ but with r replaced by R and

$$U_R=\frac{1}{R}\frac{\partial\Psi}{\partial\theta},\ U_\theta=-\frac{\partial\Psi}{\partial R}, \tag{14}$$

The expansion for T is assumed to be of the form

$$T=T_0+\varepsilon T_1+\varepsilon^2 T_2+\cdots+O(Pe), \tag{15}$$

This expansion is required to satisfy Eq. (13) with the boundary condition at infinity and to match the inner expansions (10). The matching condition can be written as

$$\lim_{r\to\infty} t(r,\theta)=\frac{1}{\delta(Pe)}\lim_{R\to 0} T(R,\theta). \tag{16}$$

The expansion for Ψ is obtained by rewriting Eq. (9) in the variable R instead of r.

The temperature in the frozen region t^*, the temperature on the surface of the frozen region θ_w and the function denoting the boundary of the frozen region $S(\theta)$ are also expanded as

$$t^*=t_0^*+\varepsilon t_1^*+\varepsilon^2 t_2^*+\cdots+O(Pe), \tag{17}$$

$$\theta_w=\theta_0+\varepsilon\theta_1+\varepsilon^2\theta_2+\cdots+O(Pe), \tag{18}$$

$$S(\theta)=S_0(\theta)+\varepsilon S_1(\theta)+\varepsilon^2 S_2(\theta)+\cdots+O(Pe). \tag{19}$$

respectively.

Differential equation for t_n is the heat conduction equation

$$\Delta t_n=0. \tag{20}$$

Since the thermal convection term is neglected in Eq. (20), it is reasonable to assume that, to the order smaller than Pe, the shape of the frozen region remains a circle with the center at the origin, so that we have

$$S_n(\theta)=\begin{cases}1 & \text{for}\quad n=0,\\ 0 & \text{for}\quad n\geq 1\end{cases}. \tag{21}$$

Hence the stream function ψ_n is given by

$$\psi_n=\begin{cases}(r-r^{-1})\sin\theta & \text{for } n=0,\\ 0 & \text{for } n\geq 1.\end{cases} \tag{22}$$

From this, the stream function in the outer region Ψ may be expressed as

$$\Psi=R\sin\theta+O(Pe), \tag{23}$$

and the equation for T_n becomes

$$\cos\theta\frac{\partial T_n}{\partial R}-\frac{\sin\theta}{R}\frac{\partial T_n}{\partial\theta}=\Delta_R T_n \tag{24}$$

The solutions of Eqs. (20) and (24) satisfying the boundary conditions and the matching condition (16) can be obtained after a straightforward manner as

$$t=1-\varepsilon\ln r+O(Pe), \tag{25}$$

$$T=(t/\varepsilon)=\exp\left(\frac{1}{2}R\cos\theta\right)K_0\left(\frac{1}{2}R\right)+O(Pe), \tag{26}$$

where K_0 denotes the modified Bessel function.

The solution for t^* and the still unknown parameter m can be determined by applying the boundary conditions, $(t_0^*)_{r=1}=1$, $(t_n^*)_{r=1}=0$ for $n\geq 1$, $(t_n^*)_{r=\frac{1}{m}}=\theta_n$ and $k^*(\partial t_n^*/\partial r)_{r=1}=k(\partial t_n/\partial r)_{r=1}$. The final results are

$$t^*=1-\varepsilon\frac{k}{k^*}\ln r+O(Pe), \tag{27}$$

$$\theta_w=1+\varepsilon\frac{k}{k^*}\ln m+O(Pe). \tag{28}$$

Equation (28) predicts m for a given value of θ_w.

In the analysis presented so far, the thermal convection is taken into account only in the outer region and it does not contribute to deformation of the frozen region from a circle with the center at the origin. It is clear that the deformation appears when the analysis is continued up to the term of $O(Pe)$. The calculation obtaining the term is, however, difficult and is beyond the scope of the present paper.

4. Discussions

When the diameter and the surface temperature of the freezing tube, the velocity and the temperature of the uniform stream and some necessary physical constants of the saturated porous medium and the frozen region are given, Eq. (28) predicts the diameter L of the frozen region. Fig. 1 shows the curves of $(k^*/k)(1-\theta_w)=(k^*/k)(T_w-T_0)/(T_\infty-T_0)$ plotted as a function of m. From Eq. (27), the heat flux at the surface of the freezing tube is obtained, in non-dimensional form, as

$$Q=r_0Q'/k^*(T_0-T_\infty)=(k/k^*)\ \varepsilon+O(Pe). \tag{29}$$

It is seen that, to the order ε, the heat flux is uniform over the surface of the freezing tube. Therefore, when constant heat flux Q' is given as a boundary condition at the surface of the tube inatead of a constant temperature, Eq. (29) is valid to predict L. Figure 2 shows the curves of $(k^*/k)\ Q$ plotted as a function of mPe^*. It is to be noted that, for a given value of $(k^*/k)Q$, m is in inverse proportion to Pe^*.

Finally, it should be noted that, in the present analysis, we have used, as the velocity in the energy equation, only the leading term in Eq. (23), namely, uniform stream. If we consider the flow of pure fluid instead of the flow in porous medium, the velocity in the outer region can be written as (Proudman and Pearson[2])

$$\Psi=R\ \sin\theta+O(\varepsilon)+O(Pe). \tag{30}$$

This suggests that Eqs. (25)−(28) are valid up to $O(\varepsilon)$ for pure fluids also; namely, we have for pure fluids

$$t=1-\varepsilon\ln r+O(\varepsilon^2)+O(Pe), \tag{31}$$

$$T=\exp\left(\frac{1}{2}R\ \cos\theta\right)\ K_0\left(\frac{1}{2}R\right)+O(\varepsilon)+O(Pe), \tag{32}$$

$$t^*=1-\varepsilon\ (k/k^*)\ \ln r+O(\varepsilon^2)+O(Pe), \tag{33}$$

$$\theta_w=1+\varepsilon\ (k/k^*)\ \ln m+O(\varepsilon^2)+O(Pe). \tag{34}$$

The deformation of the frozen region from a circle is, of course, of the order Pe similarly to the case of Darcy flow.

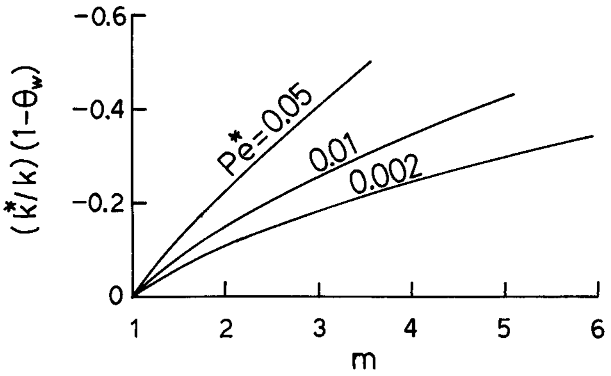

Fig. 1 Variation in $(k^*/k)(1-\theta_w)$ as a function of m

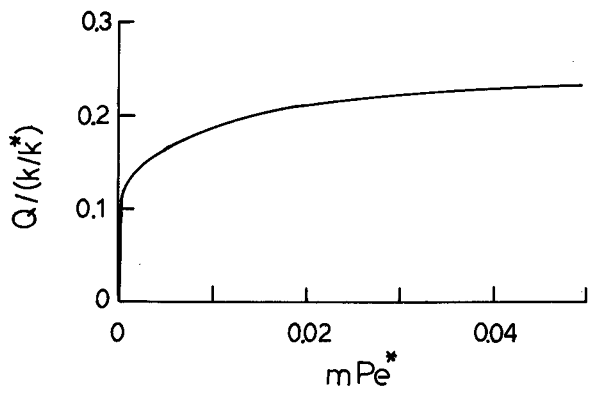

Fig. 2 Variation in $(k^*/k)\ Q$ as a function of mPe^*

Nomenclature

c	specific heat of liquid
D	diameter of the cylinder
k	effective thermal conductivity
k^*	thermal conductivity of the frozen region
L	diameter of the frozen region
m	ratio L/D
Pe	Peclet number based on the radius of the frozen region $(=\rho c LU_\infty/2k=mPe^*)$
Pe^*	Peclet number based on the radius of the cylinder
Q	heat flux at the surface of the cylinder
r	radial coordinate
r_0	radius of the cylinder (=D/2)
$S(\theta)$	function denoting the shape of the boundary of the frozen region
t	non-dimensional temperature outside the frozen region
t^*	non-dimensional temperature in the frozen region
T_0	freezing temperature of liquid
T_w	temperature on the surface of the cylinder
T_∞	temperature at infinity
u_r	non-dimensional velocity in r-direction
u_θ	non-dimensional velocity in θ-direction
U_∞	velocity at infinity

Greek symbols

ε	small parameter defined by Eq. (11)
θ	tangential coordinate
θ_w	non-dimensional temperature on the surface of the cylinder
ρ	density of liquid
ψ	stream function

References

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