Bifurcations and Chaos Phenomena in Piecewise Linear Systems with Unsymmetrical Restoring Force

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This is a study of period-doubling bifurcations and chaos phenomena in the nonautonomous piecewise linear dissipative systems with unsymmetrical restoring force. In this paper we have clarified the period-doubling bifurcation condition analytically and various period-doubling sequences and chaotic behavior are examined numerically. And we have the results that complex features of period-doubling and various forms of strange attractors may depend on the nonlinearity, that is, the ratio of the slope, the loss factor, and the amplitude of external force.

1. Introduction

In the nonlinear systems such as mechanical systems, astromy, hydrodynamics, meteorology, plasma physics, electric oscillators, and solid state systems, etc., the many phenomena that are caused by the inherent nonlinear nature that under some conditions leads to strong irregular behavior, chaos and at other time to characteristic ordering phenomena are known.

That is to say, whenever dynamical chaos is found, it is accompanied by nonlinearity[(1)~(4)].

In many papers nonlinear phenomena have been discussed in curvilinear restoring force systems[(5)~(9)]. The number of paper dealing wich piecewise linear systems is fairly low[(10)~(13)]. We have discussed the periodic solutions in piecewise linear systems with unsymmetrical restoring force[(14)~(17)]. In these systems,we have merits;

i. systems can be described in each of the intervals by linear differential equations,
ii. the small parameters are not needed,
iii. the results of the piecewise linear systems appears to be sufficiently general to be able to account for a considerable number of phenomena encountered in curvilinear systems.

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Therefore, we have investigated the nonlinear phenomena in piecewise linear dissipative systems with two straight line segments in the case of unsymmetrical restoring force. Even though in such a simple case, we have complex features of period doubling and various forms of strange attractors by means of the nonlinearity, that is, the ratio of the slope, the loss factor, and the amplitude of the external force, etc..
In this paper, we discuss the period-doubling bifurcation condition analytically and numerical analysis of period-doubling sequences and chaos phenomena under the condition of various stiffness and loss factors.

2. Periodicity Conditions

In this section the system with restoring force (see Fig.1) expressed in Eq.(1) will be considered:

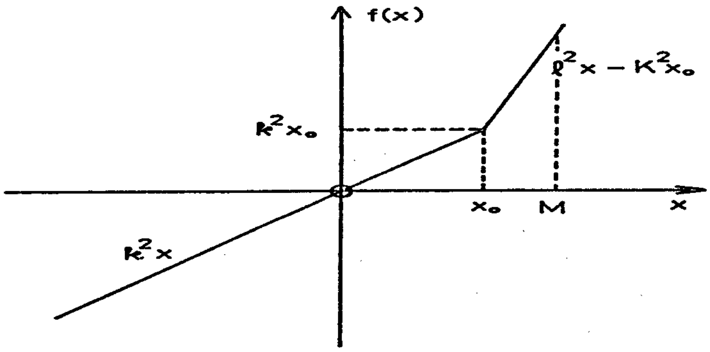


Fig. 1 Restoring force characteristic

$$\ddot{x}+2\alpha\dot{x}+f(x) = E\cos\omega t \tag{1}$$

$$f(x) = \begin{cases} \ell^2x - K^2x_0 & (x \geqq x_0) \\ k^2x & (x \leqq x_0) \end{cases} \tag{2}$$

where $\ell^2=k^2+K^2$, and k, ℓ, K, and x_0 are positive constants. The initial conditions are expressed by

$$\left.\begin{aligned} x(0)&=M \\ \dot{x}(0)&=N \end{aligned}\right\} \tag{3}$$

In this paper dots over a quantity refer to differentiation with respect to time t. Let the period of external force be T_0 and the period of the solution be T. Then,

$$T_0=\frac{2\pi}{\omega},\ T=mT_0\,(m=1,\ 2,\ 3,\ \cdots) \tag{4}$$

Here, we have the following periodicity conditions (5).

$$\left.\begin{array}{l} x(T)=M \\ \dot{x}(T)=N \end{array}\right\} \tag{5}$$

In this paper the periodic solutions are classified according to the number of the times the solution reaches the deflection point, x_0, during the period[5]. For $2n$ times the solution is designated ${}_nA$ type solution in case $M>x_0$, or ${}_nB$ ($M<x_0$).

Here, we have derived the periodicity conditions for ${}_1B$ type ($M<x_0$) periodic solution with period T, shown in Fig.2. Other types of periodic solutions can be analyzed similarly and not discussed here. From Fig.2 we have following equations.

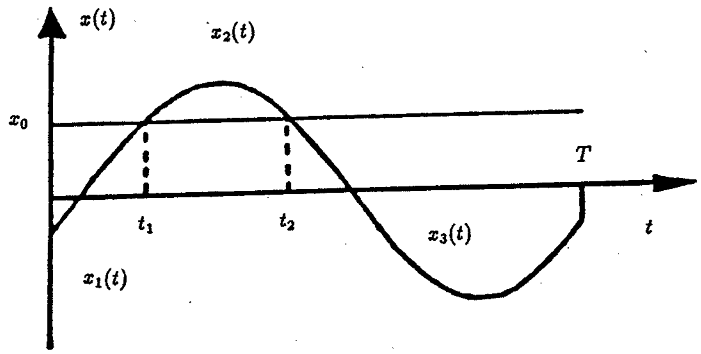


Fig. 2 Periodic solution of type ${}_1B$

$$\left.\begin{array}{l} x_1(t_1)=x_0 \\ x_2(t_2)=x_0 \\ x_3(T)=M \\ \dot{x}_3(T)=N \end{array}\right\} \tag{6}$$

where $x_i(t)$ means the solution for the interval $t_{i-1}\leqq t\leqq t_i$ ($i=1,\ 2,\ 3$, and $t_0=0,\ t_3=T=mT_0$) and the solution $x_i(t)$ which reaches the deflection point, x_0 at $t=t_i$ ($i=1,\ 2$) is connecting the solution $x_{i+1}(t)$ smoothly at every deflection point.

Given the system, that is, for given ℓ, k, K, and x_0, equations (6) are the periodicity conditions for obtaining ${}_1B$ type solution with period T. Then, if initial values M and N, loss factor α, and amplitude E of the external force are known, the remaining elements are obtained, that is to say, basic frequency ω of the external force, and transition time t_1 and t_2, which lead to periodic solution, will be found.

Finally, we write down the concrete form of equation (6) under the conditions (3) . In the following solutions, we set

$$\omega_{a1}=\sqrt{k^2-\alpha^2} \text{ and } \omega_{a2}=\sqrt{\ell^2-\alpha^2} \tag{7}$$

Thus

$$\begin{aligned} x_1(t_1)=&e^{-\alpha t_1}\{A_1\cos\omega_{a1}t_1+\frac{1}{\omega_{a1}}(\alpha A_1+B_1)\sin\omega_{a1}t_1\}\\ &+C_1\cos\omega t_1+D_1\sin\omega t_1=x_0 \end{aligned} \tag{8}$$

$$\begin{aligned} x_2(t_2)=&e^{-\alpha(t_2-t_1)}\{A_2\cos\omega_{a2}(t_2-t_1)\\ &+\frac{1}{\omega_{a2}}(\alpha A_2+B_2)\sin\omega_{a2}(t_2-t_1)\}+\frac{K^2}{\ell^2}x_0\\ &+C_2\cos\omega t_2+D_2\sin\omega t_2=x_0 \end{aligned} \tag{9}$$

$$\begin{aligned} x_3(T)=&e^{-\alpha(T-t_2)}\{A_3\cos\omega_{a1}(T-t_2)\\ &+\frac{1}{\omega_{a1}}(\alpha A_3+B_3)\sin\omega_{a1}(T-t_2)\}\\ &+C_1\cos\omega T+D_1\sin\omega T=M \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{x}_3(T)=&e^{-\alpha(T-t_2)}\{B_3\cos\omega_{a1}(T-t_2)\\ &+\frac{1}{\omega_{a1}}(k^2A_3+\alpha B_3)\sin\omega_{a1}(T-t_2)\}\\ &+\omega D_1\cos\omega T-\omega C_1\sin\omega T=N \end{aligned} \tag{11}$$

where

$$\begin{aligned} &C_1=\frac{(k^2-\omega^2)E}{(k^2-\omega^2)^2+4\alpha^2\omega^2},\quad D_1=\frac{2\alpha\omega E}{(k^2-\omega^2)^2+4\alpha^2\omega^2}.\\ &C_2=\frac{(\ell^2-\omega^2)E}{(\ell^2-\omega^2)^2+4\alpha^2\omega^2},\quad D_2=\frac{2\alpha\omega E}{(\ell^2-\omega^2)^2+4\alpha^2\omega^2}\\ &A_1=M-C_1\\ &A_2=x_0-C_2\cos\omega t_1-D_2\sin\omega t_1,\\ &A_3=x_0-C_1\cos\omega t_2-D_1\sin\omega t_2,\\ &B_2=\dot{x}_1(t_1)-\omega D_2\cos\omega t_1+\omega C_2\sin\omega t_1,\\ &B_3=\dot{x}_2(t_2)-\omega D_1\cos\omega t_2+\omega C_1\sin\omega t_2,\\ &\dot{x}_1(t_1)=e^{-\alpha t_1}\{B_1\cos\omega_{a1}t_1\\ &\qquad-\frac{1}{\omega_{a1}}(k^2A_1+\alpha B_1)\sin\omega_{a1}t_1\} \end{aligned} \tag{12}$$

$$+\omega D_1\cos\omega t_1-\omega C_1\sin\omega t_1,$$

$$\dot{x}_2(t_2)=e^{-\alpha(t_2-t_1)}\{B_2\cos\omega_{\alpha}(t_2-t_1)-\frac{1}{\omega_{\alpha}}(\ell^2A_2+B_2)\sin\omega_{\alpha}(t_2-t_1)\}+\omega D_2\cos\omega t_2-\omega C_2\sin\omega t_2$$

$$T=m\frac{2\pi}{\omega}\ (m: \text{positive integer})$$

3. Stability and Period-Doubling Bifurcation Condition

If $x^p(t)$ is the periodic solution obtainable by using the periodicity conditions in section 2, the stability of $x^p(t)$ can be studied by the first-order variation equation as to the solution of Eq.(1). Now, if y is the variation, then the first-order variation equation is given as

$$\ddot{y}+2\alpha\dot{y}+a(t)y=0 \tag{13}$$

where

$$a(t)\equiv\frac{\partial f(x)}{\partial x}\bigg|_{x=x^p(t)} \tag{14}$$

Also, $a(t)$ clearly has the following characteristics ;

$$\left.\begin{aligned}a(t)&=a(t+T)\\&=\begin{cases}\ell^2\ (x^p(t)>x_0)\\k^2\ (x^p(t)<x_0)\end{cases}\end{aligned}\right\} \tag{15}$$

Let the independent solutions of equation (13) be denoted by $\phi(t)$ and $\psi(t)$ where $\phi(0)=\dot{\psi}(0)=1$, $\dot{\phi}(0)=\psi(0)=0$.
Let the two characteristic roots of equation (13) be ρ_1 and ρ_2. Then

$$\left.\begin{aligned}\rho_1\rho_2&=e^{-2\alpha T}\\\rho_1+\rho_2&=\phi(T)+\dot{\psi}(T)\end{aligned}\right\} \tag{16}$$

From equations (16), we have three cases of the stability of periodic solutions as follows [18].

(1) completely stable (C.S) if $|\rho_1|<1$, $|\rho_2|<1$
(2) directly unstable (D.U) if $\rho_1>1>\rho_2>0$
(3) inversely unstable (I.U) if $\rho_1<-1<\rho_2<0$

From equations (16), it is clear that the conditions for the stable and unstable region boundary are followed :

$$\text{(i)}\quad \phi(T)+\dot{\phi}(T)=-1-e^{-2\alpha T} \tag{17}$$

$$\text{(ii)}\quad \phi(T)+\dot{\phi}(T)=1+e^{-2\alpha T} \tag{18}$$

The equation (17) means the period doubling bifurcation condition, and Eq.(18) means the jump phenomenon condition.

Finally, we give the concrete form of equation (17) in terms of, α, t_1, t_2, and ω when the periodic solution is the T-periodic solution type ${}_1B$ as shown in Fig.2 (or Fig.3)

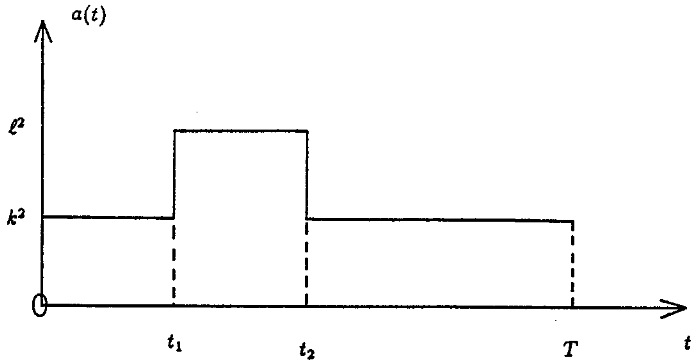


Fig. 3 Coefficient $a(t)$ for periodic solution type ${}_1B$

$$\begin{aligned}&\cos\omega_{02}(t_2-t_1)\cos\omega_{01}\{T-(t_2-t_1)\}\\&-\frac{1}{2}\left(\frac{\omega_{02}}{\omega_{01}}+\frac{\omega_{01}}{\omega_{02}}\right)\sin\omega_{02}(t_2-t_1)\sin\omega_{01}\{T-(t_2-t_1)\}\\&=-\cosh(\alpha T)\end{aligned} \tag{19}$$

4. Branching Phenomena [19]

By investigation of the results in the preceding two sections, it is well known that the systems have stable solutions under the specific values of parameters. The stable solutions tend to the unstable solutions by changing the parameter values little by little. Then the unstable solutions disappear physically and another stable solutions will appear, that is, branching has occurred.

The essential aim of this section is the identification of the branching behavior near bifurcation point of equation (1). In the neighborhood of the T-period solution obtained in section 2, there exist two situations according to equations (17) and (18) in section 3, which we have reported in the preceding articles[15],[16]. The solution of equation (1) with $x=M$ and $\dot{x}=N$ at $t=0$ is written by $x\,(t\,;M,\,N,\,E)$ and the functions F and G are defined as follows.

$$\left.\begin{aligned} F(M, N, E) &\equiv x(nT; M, N, E) - x(0; M, N, E) \\ G(M, N, E) &\equiv \dot{x}(nT; M, N, E) - \dot{x}(0; M, N, E) \end{aligned}\right\} \tag{20}$$

where $n=1$ and 2.

It is now clear that solution $x(t; M, N, E)$ has a period nT if and only if

$$F(M, N, E) = G(M, N, E) = 0 \tag{21}$$

Analysis of branching at an endpoint of an unstable arc of the curve $F=G=0$ involves the computations of several partial derivatives of the functions $F\ (M, N, E)$ and $G\ (M, N, E)$ at the point.

Since these computations are long and tedious, we shall omit most of them in what follows. When the equation (17) or (18) is satisfied, let the point satisfying equations (5) be denoted by (M_0, N_0, E_0) and then we have following results in the analysis of $(M-E)$ plane.

In case $n=2$, (equation (17) satisfied) at the bifurcation point (M_0, E_0) we have two branches, one is tangent to

$$E = E_0 \tag{22}$$

and the other

$$M - M_0 = C_1 (E - E_0), \ (C_1 ; \text{constant}) \tag{23}$$

On the second branch, the solution satisfies

$$x^p(t) = x^p(t+T) \tag{24}$$

The first branch tangent to $E=E_0$ behaves like

$$E - E_0 = C_2 (M - M_0)^2 \ (C_2 : \text{constant}) \tag{25}$$

and we have

$$x^p(t+T) = -x^p(t) \tag{26}$$

Thus the first branch corresponds to the $2T$—period solution, existing only on one side of $E=E_0$. This means period doubling bifurcation and this period-doubling mechanism is one route to chaos.

In the case of $n=1$, (equation (18)), we have

$$E = E_0, \ M - M_0 + C_3 (N - N_0) = 0 \ (C_3 : \text{constant})$$

The branching which occurs in this case is the well known jump phenomena for which a stable periodic solution coalesces with an unstable solution and disappears.

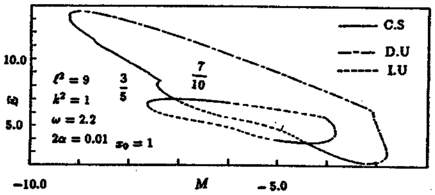
5. Numerical Analysis

In this section we analyse the behavior of the various period doubling bifurcation and chaotic behavior numerically by using the results obtained in sections 2, 3. Here we set the parameters as follows : $k^2=1$, $x_0=1$, and $\omega=2.2$. As to the slope ℓ^2 (stiffness ratio), we adopted next three cases ; (1) $\ell^2=9$, (2) $\ell^2=25$, (3) $\ell^2=49$ and as to damping coefficient 2α, we examined in the range $2\alpha=0.002\sim0.3$.
Here, also, we discriminate the solution $x(t)$ as the periodic solution with period T under the following conditions.

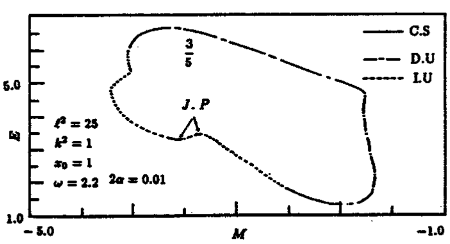
$$|x(T)-x(0)|<10^{-10},\quad |\dot{x}(T)-\dot{x}(0)|<10^{-10}$$

5. 1. Regions in which periodic solutions are sustained and Branching phenomena

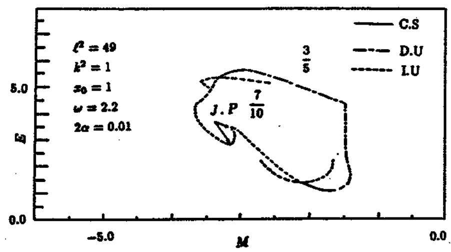
In Fig.4, we show M-E plane behavior of the periodic solutions with the period $T=5T_0$ ($T_0=\dfrac{2\pi}{\omega}$) in what follows we say 5-periodic group in three cases: (1) $\ell^2=9$, (2) $\ell^2=25$, (3) $\ell^2=49$ when loss factor 2α is kept constant, $2\alpha=0.01$. Increasing the value of ℓ^2, for example, for $\ell^2=25$ or 49, we have the jumping points (J.P), which do not exist when $\ell^2=9$, on the side of inversely unstable regions of periodic solutions of order 3/5 and features of behavior of periodic solutions are more complex according as the magnitude of nonlinearity.



(a) $\ell^2=9$



(b) $\ell^2=25$



(c) $\ell^2=49$

Fig. 4 5-periodic group and bifurcation diagrams ($k^2=1$, $\omega=2.2$, $x_0=1$, $2\alpha=0.01$)

These characteristics are familiar with other periodic solutions, such as $T=3T_0$ (3-periodic group) and $T=T_0$ (2-periodic group). Bifurcation diagrams of fundamental solutions (2-periodic group) in case $\ell^2=49$, $2\alpha=0.01$ are shown in Fig.5. Fig.6 ($2\alpha=0.01$) and Fig.7 ($2\alpha=0.1$) present the bifurcation diagram of the periodic solution of order 2/3 (3-periodic group) in both cases $\ell^2=49$, we see that in these cases these diagrams became more complex and especially in Fig.7 when the amplitude of external force, E, is increased from below, stable periodic solution of order 5/6 appear about E=3.75 and then stable solutions of order 7/12 bifurcate from periodic solutions of order 5/6 about E=3.9. As increasing E through E=4.5, we have again stable periodic solutions of order 5/6, which corresponds to window in chaos.

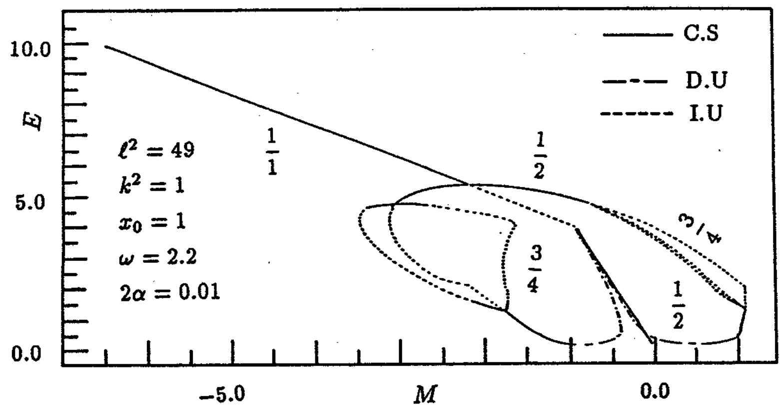


Fig. 5 Bifurcation diagram of 2-periodic group ($2\alpha=0.01$)

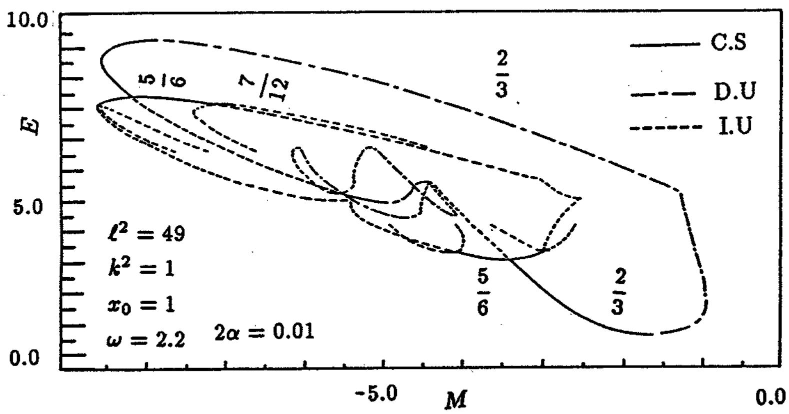


Fig. 6 Bifurcation diagram of 3-periodic group ($2\alpha=0.01$)

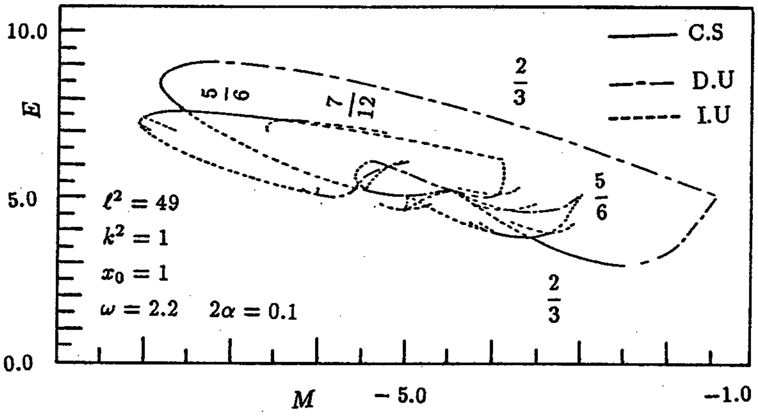


Fig. 7 Bifurcation diagram of 3-periodic group (2α=0.1)

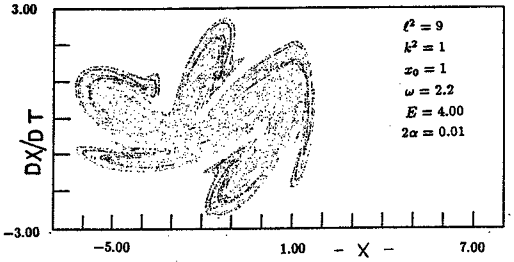
5. 2. Chaos Phenomena

It is well known that there is no universal route from regular behavior to chaos but some routes to chaos exist. A very popular model to chaos is said to be a route based on a sequence of period doublings. We have showed a sequence of period doublings depending on the systems parameters, 2α and E, in the system expressed in equation (1). In the numerical calculations we observed next conditions (1) ~ (8), as deciding the chaotic behavior.

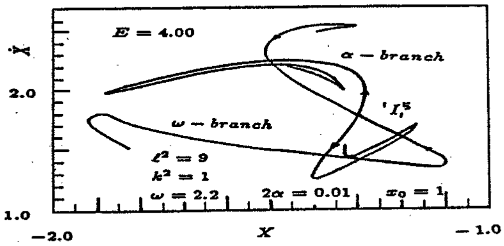
(1) existence of process of period-doubling bifurcation.
(2) non-existence of stable periodic solutions.
(3) Poincaré map becomes a set of one dimensional manifolds.
(4) existence of a continuous spectrum.
(5) existence of homoclinic orbits.
(6) autocorrelation functions decay.
(7) one of the Lyapunov exponents is positive [20].
(8) fractal dimension is not integer [20].

Especially as to conditions (2), (3), we calculated the mapping points by running the program for 100,000 forcing cycles and checked the periodic points within 5,000T_0. In case the stiffness ratio is small ($\ell^2=9$), the chaotic phenomena appear in a narrow zone of 2α and E, and the results are shown in Fig.8 (2α=0.01 and E=4.0) and in Fig.9 (2α=0.1, E=3.875). When ℓ^2=25, the chaotic zone is also limited because of the broad regions of the stable 3-periodic group. One of the results is shown in Fig.10. From Fig.11 to Fig.13 we showed the various chaotic phenomena in case ℓ^2=49.

We show Poincaré map in (a), enlargement of a part of Poincaré map in (b), wave form in (c), power spectrum (d), and autocorrelation function (e) in Fig. 11 in case $2\alpha=0.01$, $E=4.55$. In Fig. 12, we present Poincaré maps on both sides of 3-periodic window in the case of $2\alpha=0.1$. In Fig. 12 (a) Lyapunov exponents of chaotic attractor are $\mu_1=0.127$ and $\mu_2=-0.227$. In Fig. 13 we have the Poincarē map and homoclinic orbit in the case of comparatively large loss factor, $2\alpha=0.3$, when $E=4.0$.



(a) Poincaré map



(b) homoclinic orbit

Fig. 8 Chaotic motion for $\ell^2=9$ ($k^2=1$, $\omega=2.2$, $x_0=1$ $2\alpha=0.01$, $E=4.0$)

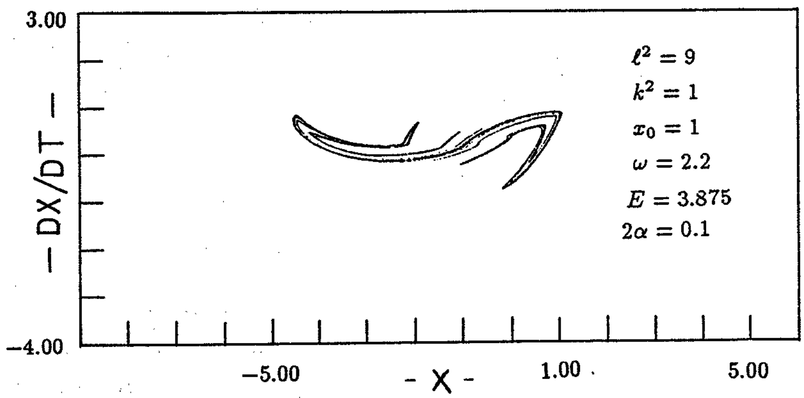


Fig. 9 Poincaré map in case $\ell^2=25$, $k^2=1$, $\omega=2.2$, $x_0=1$ $2\alpha=0.1$, $E=3.895$

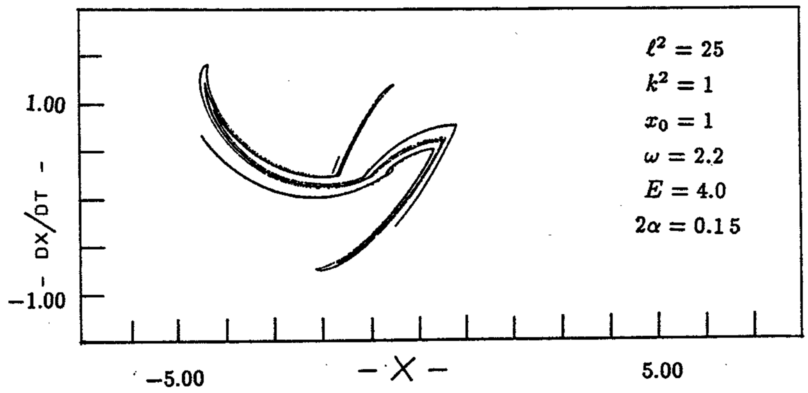
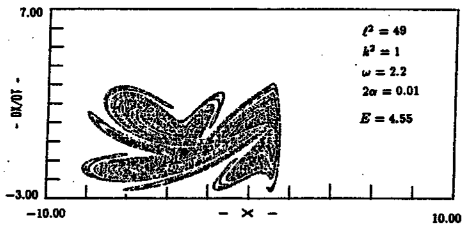
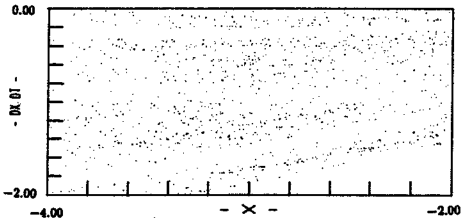


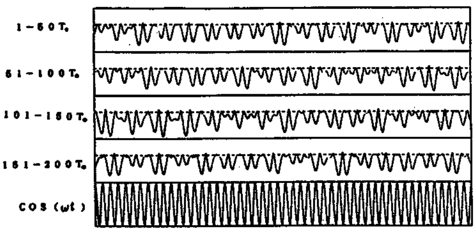
Fig. 10 Poincaré map in case
ℓ^2=25, k^2=1, ω=2.2, x_0=1
2α=0.15, E=4.0



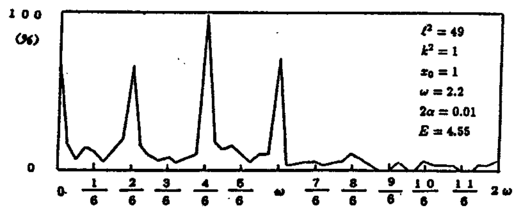
(a) Poincaré map



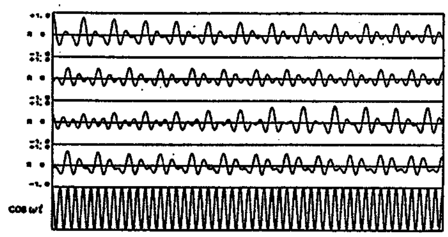
(b) enlargement of a part of (a)



(c) wave form (time history)

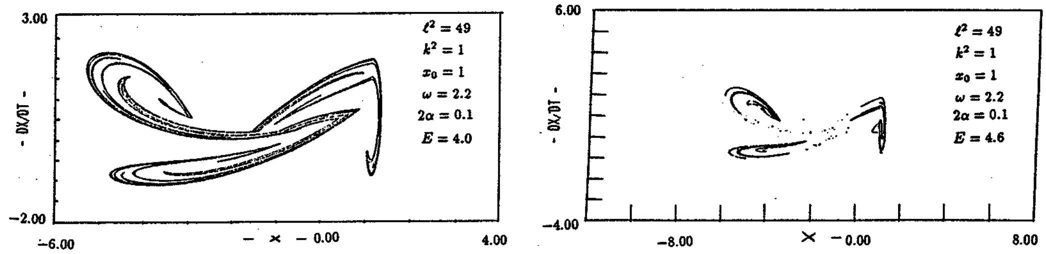


(d) power spectrum



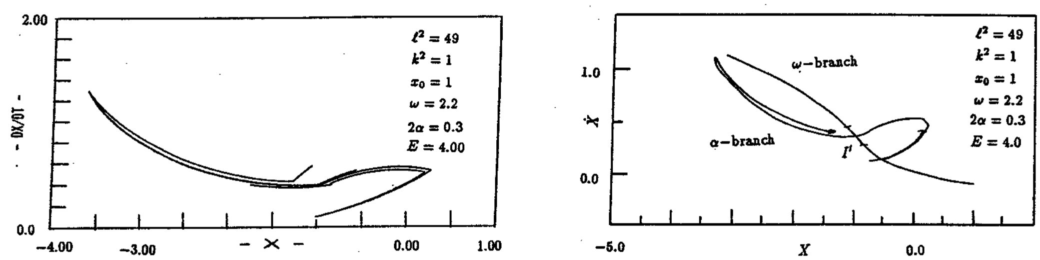
(e) autocorrelation function

Fig. 11 Chaotic phenomena in case
ℓ^2=49, k^2=1, ω=2.2, x_0=1
2α=0.01, E=4.55



(a) E=4.0 (b) E=4.6

Fig. 12 Poincaré map in case
ℓ^2=49, k^2=1, ω=2.2, x_0=1
2α=0.1



(a) Poincaré map (b) homoclinic orbit

Fig. 13 Chaotic motion in case
ℓ^2=49, k^2=1, ω=2.2, x_0=1
2α=0.3

6. Conclusions

In the present paper, we describe that the period-doubling bifurcations develop and in result the chaotic motions appear for the simplest piecewise linear systems with unsymmetrical restoring force represented by two straight-line segments.

The results are summarized as follows:

i. For the piecewise linear system, the period-doubling bifurcation condition is clarified.

ii. The period-doubling bifurcation phenomena have complex features of branching in small E when the nonlinearity factor ℓ^2 (stiffness ratio) is increased while the loss factor 2α is kept constant.

iii. As to the chaos attractors, the region of the chaos is in general broad in E, the amplitude of external force, when the stiffness ratio ℓ^2 is large. But, the chaotic motions are prevented by stable 3-period solutions in the case of small loss factor in magnitude when $\ell^2=25$.

iv. The form of chaos attractors has relation to the fractal dimension.

The future work includes investigation on the differences of the stiffness ratio and coexistence of multiple steady state and the appearance of chaos.

Finally, it is noted that numerical calculations were performed by using ACOS-930 at the computer center, University of Osaka Prefecture.

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