A study of Period-Doubling Bifurcations of 4/7-Harmonics in the Nonautonomous Piecewise Linear Systems with Unsymmetrical Restoring Force

A Study of Period-Doubling Bifurcations of 4/7-Harmonics in the Nonautonomous Piecewise Linear Systems with Unsymmetrical Restoring Force

Yoshiaki SHIRAO*, Hiroaki KAWABATA*,
Toshikuni NAGAHARA* and Yoshio INAGAKI*

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This is a study of period-doubling bifurcations of 4/7−harmonics in the nonautonomous piecewise linear systems with unsymmetrical restoring force in the case of the damped systems.

In this report we have examined part of a sequence of period doublings, and the homoclinic orbits by making use of numerical calculations.

1. Introduction

In the nonlinear systems such as mechanical systems, astronomy, hydrodynamics, meterology, plasma physics, electronic oscillators, and solid state systems, etc., the many phenomena that are caused by the inherent nonlinear nature that under some conditions leads to strong irregular behavior, chaos, and at other time to characteristic ordering phenomena are known[1].

It is well-known that there is no universal route from regular behavior to chaos but some routes to chaos exist. A very popular model to chaos is said to be a route based on a sequence of period doublings[2], in which the period of oscillations undergoes a doubling at specific values with increasing (decreasing) value of the parameter.

There are famous period doubling bifurcations as to 1/2-harmonics[3], and 3/5-harmonics[2]. In this paper we deal with period doubling bifurcation in 4/7-harmonics in the neighborhood of 1/2-harmonic solutions in the case of non-autonomous piecewise linear system with unsymmetrical restoring force[4),5),6)].

We have studied periodic solutions in piecewise linear systems because certain calculations can be made explicitly and a certain of the results will carry over to more general equations.

This paper will give the period doubling bifurcation condition and some results as to the bifurcation diagrams both qualitatively and numerically.

* Department of Electrical Engineering, College of Engineering

2. Periodic Conditions

In this section the system with restoring force (see Fig. 1) expressed in equation (1) will be considered.

$$\ddot{x}+2\alpha\dot{x}+f(x)=E\cos\omega t \tag{1}$$

$$f(x)=\begin{cases} l^2x-K^2x_0 & (x\geqq x_0) \\ k^2x & (x\leqq x_0) \end{cases} \tag{2}$$

There $l^2=k^2+K^2$, and $k, l, K,$ *and* x_0 are positive constants. In this paper dots over a quantity refer to differentiation with respect to time t.

Assume the initial conditions as follows;

$$\left.\begin{aligned} x(0)&=M \\ \dot{x}(0)&=N \end{aligned}\right\} \tag{3}$$

In this paper the periodic solutions are classified according to the number of times the solution reaches the corner point, x_0, during the period[4]. For $2n$ times the solution is designated ${}_nA$ type solution in case $M>x_0$, or ${}_nB(M<x_0)$. Here we have derived the periodicity conditions for ${}_1B$ type $(M<x_0)$ periodic solution with period T, shown in Fig. 2.
From Fig. 2 we have following equations.

$$\left.\begin{aligned} x_1(t_1)&=x_0 \\ x_2(t_2)&=x_0 \\ x_3(T)&=M \\ \dot{x}_3(T)&=N \end{aligned}\right\} \tag{4}$$

where $x_i(t)$ means the solution for the interval $t_{i-1}\leqq t\leqq t_i(i=1,2,3,$ and $t_0=0,\ t_3=T=\frac{2p\pi}{\omega}$;p:positive integer) and the solution $x_i(t)$ which reaches the corner point x_0 at $t=t_i$(i=1,2) is connecting the solution $x_{i+1}(t)$ smoothly at every corner point.

Given the system, that is, for given $l, k, K,$ and x_0 equations (4) are the periodicity conditions for obtaining ${}_1B$ type solution with period T. Then, if initial values M and

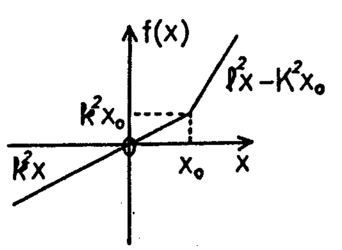

Fig. 1 Restoring force characteristics

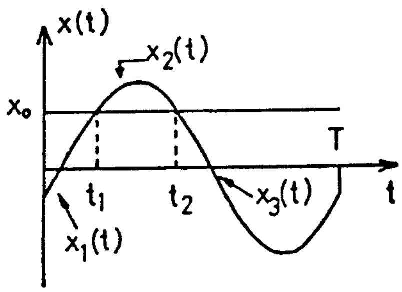

Fig. 2 Periodic solution of type ${}_1B$

N, loss factor α, and amplitude E of the external force are known, the remaining elements are obtained, that is to say, basic frequency ω of the external force, and transition time t_1 and t_2, which lead to periodic solution, will be found.

Finally, we write down the concrete form of equation (4) under the conditions (3). In the following solutions, we set

$$\omega_{01}=\sqrt{k^2-\alpha^2} \text{ and } \omega_{02}=\sqrt{l^2-\alpha^2} \tag{5}$$

Thus

$$\begin{aligned} x_1(t_1)=e^{-\alpha t_1}\ &\{A_1\cos\omega_{01}t_1+\frac{1}{\omega_{01}}(\alpha A_1+B_1)\sin\omega_{01}t_1\} \\ &+C_1\cos\omega t_1+D_1\sin\omega t_1=x_0 \end{aligned} \tag{6}$$

$$\begin{aligned} x_2(t_2)=e^{-\alpha(t_2-t_1)}\ &\{A_2\cos\omega_{02}(t_2-t_1) \\ &+\frac{1}{\omega_{02}}(\alpha A_2+B_2)\sin\omega_{02}(t_2-t_1)\}+\frac{K^2}{l^2}x_0 \\ &+C_2\cos\omega t_2+D_2\sin\omega t_2=x_0 \end{aligned} \tag{7}$$

$$\begin{aligned} x_3(T)=e^{-\alpha(T-t_2)}\ &\{A_3\cos\omega_{01}(T-t_2) \\ &+\frac{1}{\omega_{01}}(\alpha A_3+B_3)\sin\omega_{01}(T-t_2)\} \\ &+C_1\cos\omega T+D_1\sin\omega T=M \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{x}_3(T)=e^{-\alpha(T-t_2)}\ &\{B_3\cos\omega_{01}(T-t_2) \\ &+\frac{1}{\omega_{01}}(k^2A_3+\alpha B_3)\sin\omega_{01}(T-t_2)\} \\ &+\omega D_1\cos\omega T-\omega C_1\sin\omega T=N \end{aligned} \tag{9}$$

where

$$C_1=\frac{(k^2-\omega^2)E}{(k^2-\omega^2)^2+4\alpha^2\omega^2}\ , \quad D_1=\frac{2\alpha\omega E}{(k^2-\omega^2)^2+4\alpha^2\omega^2}$$

$$C_2=\frac{(l^2-\omega^2)E}{(l^2-\omega^2)^2+4\alpha^2\omega^2}\ , \quad D_2=\frac{2\alpha\omega E}{(l^2-\omega^2)^2+4\alpha^2\omega^2}$$

$$A_1=M-C_1\ , \quad B_1=N-\omega D_1$$

$$A_2=x_0-C_2\cos\omega t_1-D_2\sin\omega t_1\ ,$$

$$A_3=x_0-C_1\cos\omega t_2-D_1\sin\omega t_2\ ,$$

$$B_2=\dot{x}_1(t_1)-\omega D_2\cos\omega t_1+\omega C_2\sin\omega t_1\ ,$$

$$B_3=\dot{x}_2(t_2)-\omega D_1\cos\omega t_2+\omega C_1\sin\omega t_2\ ,$$

$$\dot{x}_1(t_1)=e^{-\alpha t_1}\{B_1\cos\omega_{01}t_1$$
$$-\frac{1}{\omega_{01}}(k^2A_1+\alpha B_1)\sin\omega_{01}t_1\}$$
$$+\omega D_1\cos\omega t_1-\omega C_1\sin\omega t_1 ,$$
$$\dot{x}_2(t_2)=e^{-\alpha(t_2-t_1)}\{B_2\cos\omega_{02}(t_2-t_1)$$
$$-\frac{1}{\omega_{02}}(l^2A_2+B_2)\sin\omega_{02}(t_2-t_1)\}$$
$$+\omega D_2\cos\omega t_2-\omega C_2\sin\omega t_2 , \qquad (10)$$
$$T=p\frac{2\pi}{\omega} \quad (p\text{: positive integer})$$

3. Stability and Period-doubling Bifurcation Condition

The problem of the stability of the periodic solutions of our nonlinear systems always leads to the equation of Hill's type.

Let the ${}_1B$ type T-period solution obtained from the periodicity conditions given in section 2 be denoted by $x^0(t)$. The stability of $x^0(t)$ can be determined by the first variational equation of the solution of equation (1)[7].

Let the variation be y.

The first variational equation is given by

$$\ddot{y}+2\alpha\dot{y}+a(t)y=0 \qquad (11)$$

$$a(t)\equiv\left.\frac{\partial f}{\partial x}\right|_{x=x^0(t)} \qquad (12)$$

Also, $a(t)$ is understood to have the following properties:

$$\left.\begin{array}{l} a(t)=a(t+T) \\ a(t)=\begin{cases} l^2 & (x^0(t)>x_0) \\ k^2 & (x^0(t)<x_0) \end{cases} \end{array}\right\} \qquad (13)$$

Let the independent solutions of equation (11) be denoted by $\phi(t)$ and $\psi(t)$ where $\phi(0)=\dot{\psi}(0)=1$, $\dot{\phi}(0)=\psi(0)=0$.

Let the two characteristic roots of equation (11) be ρ_1 and ρ_2. Then

$$\left.\begin{array}{l} \rho_1\rho_2=e^{-2\alpha T} \\ \rho_1+\rho_2=\phi(T)+\dot{\psi}(T) \end{array}\right\} \qquad (14)$$

From equations (14), we have three cases of the stability of periodic solutions as follows[8].

(1) completely stable (C.S) if $|\rho_1|<1$, $|\rho_2|<1$

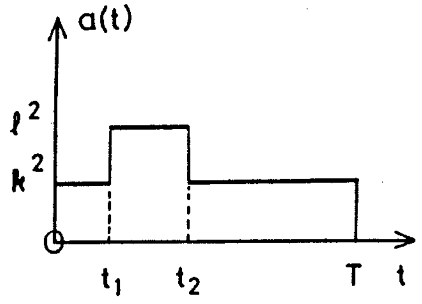

Fig. 3 Coefficient $a(a)$ of periodic solution type ${}_1B$

(2) directly unstable (D.U) if $\rho_1>1>\rho_2>0$
(3) inversely unstable (I.U) if $\rho_1<-1<\rho_2<0$

From equations (14), it is clear that the conditions for the stable and unstable region boundary are followed:

$$\text{(i)}\quad \phi(T)+\dot{\psi}(T)=-1-e^{-2\alpha T} \tag{15}$$

$$\text{(ii)}\quad \phi(T)+\dot{\psi}(T)=1+e^{-2\alpha T} \tag{16}$$

The equation (15) means the period doubling bifurcation condition. Finally, we give the concrete form of equation (15) in terms of α, t_1, t_2, and ω when the periodic solution is the T-periodic solution type ${}_1B$ as shown in Fig. 2 (or Fig. 3)

$$\begin{aligned} &e^{-\alpha T}[2\cos\omega_{02}(t_2-t_1)\cos\omega_{01}\ \{T-(t_2-t_1)\} \\ &-(\frac{\omega_{02}}{\omega_{01}}+\frac{\omega_{01}}{\omega_{02}})\sin\omega_{02}(t_2-t_1)\sin\omega_{01}\{T-(t_2-t_1)\}] \\ &=-1-e^{-2\alpha T} \end{aligned} \tag{17}$$

4. Branching Phenomena

It is well known that bifurcation theory concerns the solution $x(t;\lambda)$of a problem which depends upon a scalar or vector parameter λ.[9] The solution is said to bifurcate from the solution $x^0(t;\ \lambda_0)$ at the parameter value $\lambda=\lambda_0$ if there are two or more distinct solutions which approach $x^0(t;\ \lambda_0)$ as λ tends to λ_0.

The first problem of bifurcation theory is to determine the solution $x^0(t;\ \lambda_0)$ and parameter value λ_0 at which bifurcation occurs (see section 2,3). The second problem is to find the number of solution which bifurcate from $x^0(t;\ \lambda_0)$. A third problem is to determine the behavior of these solutions for λ near λ_0. The behavior of the solutions for λ outside a small neighborhood of λ_0 is also important, but is not considered here.

Though the essential aim of this section is the identification of the branching behavior near bifurcation point of equation (1), we have reported some results in the preceding articles.[4),5)]

The solution of equation (1) with $x=M$ and $\dot{x}=N$ at $t=0$ is written by $x(t;M,N,E)$ and the functions F and G are defined as follows.

$$\left.\begin{aligned} F(M,N,E) &\equiv x(nT;M,N,E)-x(0;M,N,E) \\ G(M,N,E) &\equiv \dot{x}(nT;M,N,E)-\dot{x}(0;M,N,E) \end{aligned}\right\} \tag{18}$$

where $n = 1$ and 2.
It is now clear that solution $x(t;\ M,\ N,\ E)$ has a period nT if and only if

$$F(M,N,E)=G(M,N,E)=0 \tag{19}$$

Analysis of branching at an endpoint of an unstable are of the curve $F=G=0$ involves the computations of several partial derivatives of the functions $F(M,\ N,\ E)$ and $G(M,\ N,\ E)$ at the point.

Since these computations are long and tedious, we shall omit most of them in what follows. When the equation (15) or (16) is satisfied, let the point satisfying equations (4) be denoted by $(M_0,\ N_0,\ E_0)$ and then we have following results in the analysis of $(M-E)$ plane.

In case $n = 2$, (equation (15) satisfied) at the bifurcation point $(M_0,\ E_0)$ we have two branches, one is tangent to $E=E_0$ and the other $M-M_0=C_1(E-E_0)$, (C_1; constant).

On the second branch, the solution curve, satisfies $x^0(t)=x^0(t+T)$. The first branch tangent to $E=E_0$ behaves like

$$E-E_0=C_2(M-M_0)^2 \quad (C_2\text{: } constant)$$

and

$$x^0(t+T)=-x^0(t)$$

Thus the first branch becomes the solution period $2T$ existing only on one side of $E=E_0$. This is means period doubling bifurcation.
In the case of $n=1$, (equation (16)), we have

$$E=E_0,\ M-M_0+C_3(N-N_0) \quad (C_3\text{: } constant)$$

The branching which occurs in this case is the well known jump phenomena for which a stable periodic solution coalesces with an unstable solution and disappears.

5. Numerical Analysis

In this section we analyse the behavior of the period doubling bifurcation of 4/

7-harmonics numerically by using the results obtained in sections 2, 3. Here we set the parameters as follows: $k^2=1.0$, $l^2=9.0$, $x_0=1.0$, and $\omega=2.2$. As to damping coefficient 2α, we use $2\alpha=0.01$ and 0.02.
Numerical results in $M-E$ plane are shown in Fig. 4(a), (b), (c) in case $2\alpha=0.02$ and in Fig. 4(d) in case $2\alpha=0.01$, where we discriminate the solution $x(t)$ as the periodic solution with period T under the following conditions.

$$|x(T)-x(0)| < 10^{-10}, |\dot{x}(T)-\dot{x}(0)| < 10^{-10}$$

In these figures the branching phenomena are very much in agreement with analysis obtained in section 4.
Also, homoclinic orbits in $E=6.93$ are shown in Fig. 5 and Poincaré section in $M-N$ plane is shown in Fig. 6, which was obtained by running the program for 20000 forcing cycles.

6. Conclusions

In the previous sections, analysis of period doubling bifurcations of 4/7-harmonics for the simplest piecewise linear system with unsymmetrical restoring force are present in the case of a single harmonic excitation with loss.
The results are summarized as follows:

(1) The results shown in Fig. 4, present two different phenomena, that is to say, one is the jump phenomena, the other period doubling bifurcation phenomena, giving the exact explanation of qualitative analysis in section 4.

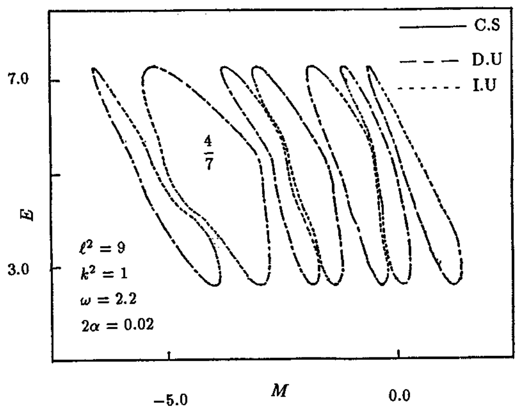

Fig. 4(a) Regions of 4/7-harmonics in M-E plane ($2\alpha=0.02$)

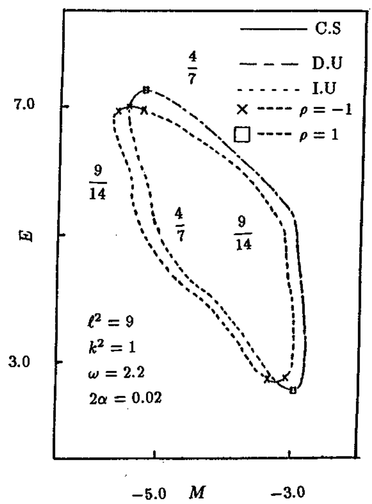

Fig. 4(b) Bifurcation diagram of 9/4-harmonics ($2\alpha=0.02$)

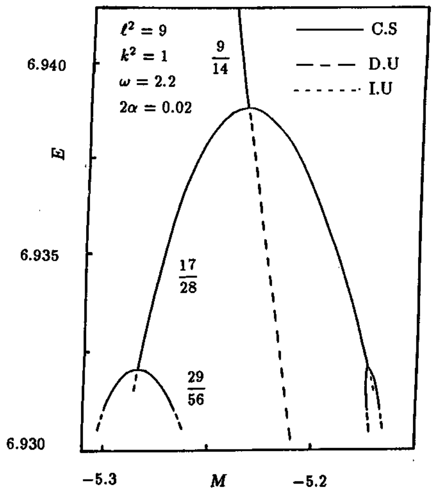

Fig. 4(c) Bifurcation diagram of 29/56-harmonics from 9/14-harmonics ($2\alpha=0.02$)

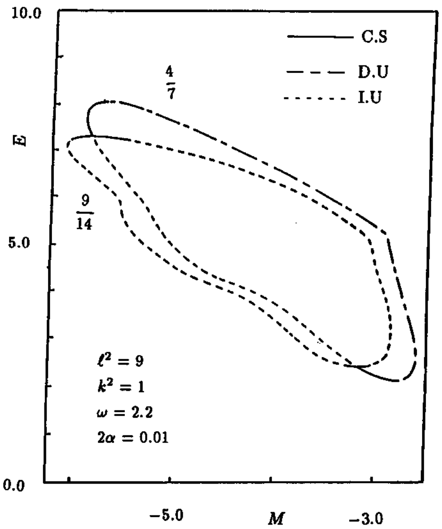

Fig. 4(d) Bifurcation diagram of 9/14-harmonics from 4/7-harmonics ($2\alpha=0.01$)

Fig. 4 Branching phenomena of 4/7-harmocs in case $k^2=1$, $l^2=9$, $\omega=2.2$ and $x_0=1$

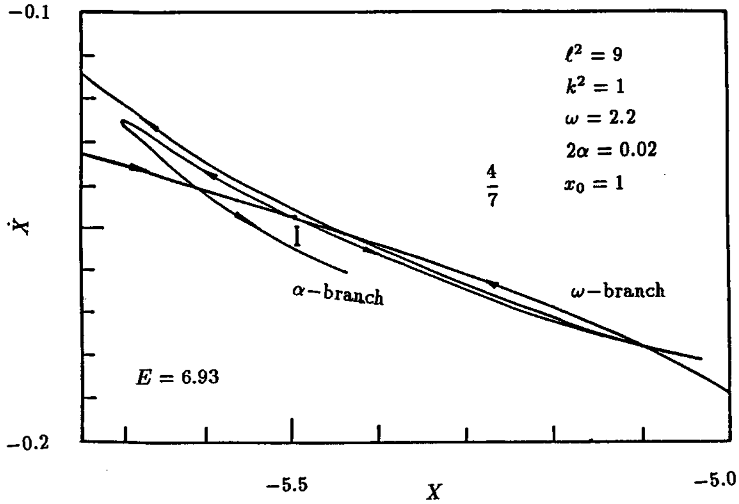

Fig. 5 Homoclinic orbits of 4/7-harmonics in $E=6.93$ from fixed point of I.U

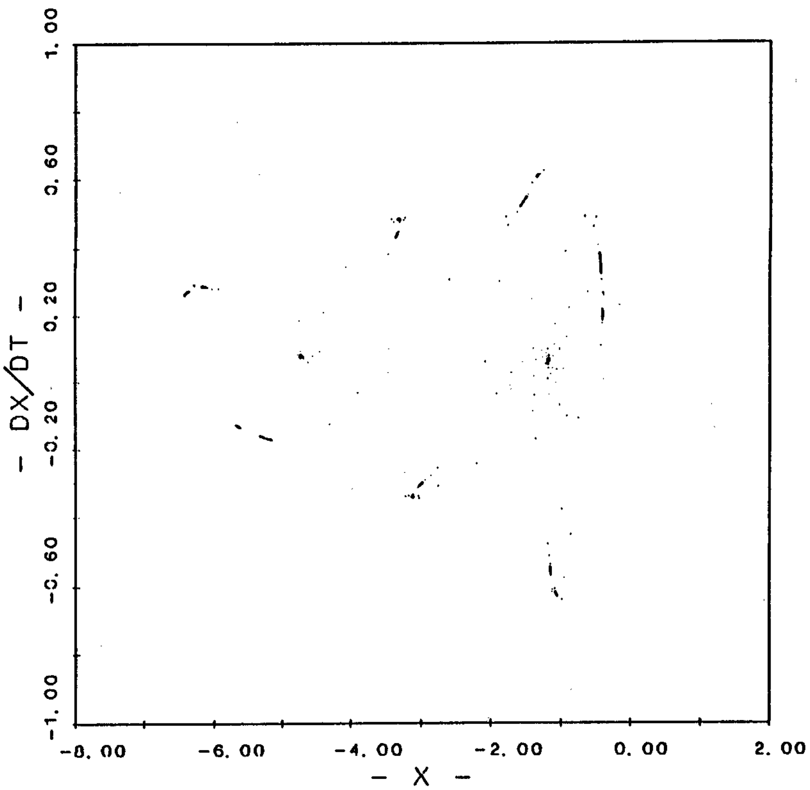

Fig. 6 Poincaré section in case
$k^2=1$, $l^2=9$, $\omega=2.2$, $x_0=1$, and $E=6.93$

(2) From Fig. 4(c) we can assume a sequence of period doublings of 4/7-harmonics infinitely.
(3) Homoclinic orbits are obtained in Fig. 5.
(4) Poincaré section, shown in Fig. 6, means that the mapping points of 4/7-harmonics seem to grow up to strange attractor, but these converge to 1/2-harmonics attractors.

Thus, the coexistence of multiple steady states and the appearance of chaos may not be appeared in these period-doublings of 4/7-harmonics in this system.

The future work includes investigations of chaotic motions of other subharmonic solutions in both dissipationless and dissipative system.

Finally, it is noted that numerical calculations were performed by using **ACOS-930** as the computer center, University of Osaka Prefecture.

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