# An Extension of the Multi-dimensional Adaptive Robbins- Monro Procedure

# An Extension of the Multi-dimensional Adaptive Robbins-Monro Procedure

Yutaka MAEDA*, Yoshiaki KAWAMURA**

(Received June 16, 1990)

The multi-dimensional adaptive Robbins-Monro stochastic approximation procedure is extended. The adaptive Robbins-Monro procedure consists of two algorithms. The first algorithm estimates the optimal parameter. The second algorithm gives the gain matrix which determines the magnitude of the revision of the parameter given by the first algorithm. In this paper, an extension of the second algorithm for the gain matrix is proposed. We clarify the conditions which ensure the convergence of the gain matrix to the optimal matrix.

## 1. Introduction

This paper presents an extended procedure for the multi-dimensional adaptive Robbins-Monro procedure (ARM procedure). The ARM procedure was proposed to improve the asymptotic convergence rate of the Robbins-Monro stochastic approximation procedure[1] (RM procedure). This procedure was proposed by Venter[2], and extended to multi-dimensional case by Nevel'son and Khas'minskii[3]. Some applications of ARM procedure were reported by the authors[4,5].

Now, let $J(\boldsymbol{x})$ be a smooth real-valued evalutional function of an adjustable parameter vector $\boldsymbol{x}$. Our problem is to find the optimal value $\boldsymbol{x}_*$ which minimizes $J(\boldsymbol{x})$. Under the assumption that we only observe the gradient vector of $J(\boldsymbol{x})$ with noise, we can apply the ARM procedure to this stochastic optimization problem.

The ARM procedure consists of two recursive algorithms. The first algorithm directly revises the estimated value of the optimal parameter $\boldsymbol{x}_*$. The other algorithm gives the gain matrix which determines the magnitude of the above revision.

It is known that the asymptotic convergence rate of the first algorithm is optimal, when this gain matrix is equal to the Hessian $H(\boldsymbol{x}_*)$ of evalutional function $J(\boldsymbol{x})$ at $\boldsymbol{x}_*$ [2,3]. Therefore, the purpose of the latter is to estimate the Hessian of $J(\boldsymbol{x})$.

* Department of Electrical Engineering, Faculty of Engineering, Kansai University
** Department of Electronics, College of Engineering

Usually, the Hessian of $J(x)$ varies with respect to parameter $x$. However, in the ordinary ARM procedure, the estimated value of $H(x_*)$ is the time average of each observed value of $H(x)$ at each operating point. Since the operation point $x$ changes at each step, it is important to make much of the recent estimation for the Hessian. From this point of view, more feasible ARM procedure is required.

The authors have proposed an extended one-dimensional ARM procedure, and demonstrated its convergence for this extended procedure[6]. In this paper, we extend this procedure to multi-dimensional case.

## 2. Extended Multi-dimensional ARM Procedure

Let $J(x)$ be an evalutional function of real-valued $n$-dimensional parameter vector $x$. We consider the problem to find the optimal parameter $x_*$ which minimizes the evalutional function $J(x)$. Suppose that we can only observe the gradient vector $f(x)$ of $J(x)$ with noise. $x_t$ denotes the estimated value of $x_*$ at the $t$-th step. Let $\pm c_t$ be perturbations to each component of the vector $x_t$. We observe $f(\cdot)$ $2n$ times for each step at $x_t \pm e^i c_t$, where $e^i$ is the $n$-dimensional fundamental vector all of whose components are 0 except that the $i$-th component is 1. Let $\psi_t^{+i}$, $\psi_t^{-i}$ be the observation noises. The observation vectors $z_t^{+i}$, $z_t^{-i}$ and the observation matrices $Z_t^+$, $Z_t^-$ are defined as follows:

$$
\begin{aligned}
z_t^{+i} &= f(x_t + e^i c_t) + \Psi_t^{+i} \\
z_t^{-i} &= f(x_t - e^i c_t) + \Psi_t^{-i} \qquad (i=1,\cdots,n)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
Z_t^+ &= (z_t^{+1},\cdots,z_t^{+n}) \\
Z_t^- &= (z_t^{-1},\cdots,z_t^{-n})
\end{aligned}
\tag{2}
$$

From Eq. (2), the observation matrices $Z_t^+$ and $Z_t^-$ consist of these $2n$ observations of $f(\cdot)$.

We propose the following extended multi-dimensional ARM procedure:

$$
x_{t+1} = x_t - \alpha_t K_t^{-1} \frac{\sum_{i=1}^{n} (z_t^{+i} + z_t^{-i})}{2n} \tag{3}
$$

$$
\begin{aligned}
K_t &= \begin{cases} H_t & \text{if } H_t > K_{\min} \\ K_{\min} & \text{if else} \end{cases} \\
H_{t+1} &= b_t H_t + a_t \frac{Z_t^+ - Z_t^-}{2c_t}
\end{aligned}
\tag{4}
$$

where $\boldsymbol{H}_t$ is an $n\times n$ matrix. $\boldsymbol{K}_{\min}$ is a sufficiently small positive definite $n\times n$ matrix. For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we express $\boldsymbol{A}>\boldsymbol{B}(\boldsymbol{A}\geqq\boldsymbol{B})$ if $\boldsymbol{A}-\boldsymbol{B}$ is positive definite (positive semi-definite). $k_{\min}$ denotes the minimum eigenvalue of matrix $\boldsymbol{K}_{\min}$.

If we know a prior information about $\boldsymbol{H}(\boldsymbol{x}_*)$, we use this value for an initial value $\boldsymbol{H}_1$ of Eq. (4). We have to guarantee that the matrix $\boldsymbol{H}_1$ is positive definite in any case.

In algorithms (3) and (4), $2n$ observations of $f(\cdot)$ are made at each step. The observation vectors $z_t^{+i}$ and $z_t^{-i}$ are given by these observations. On the basis of the average of these observation vectors $z_t^{+i}$ and $z_t^{-i}$, Eq. (3) changes $\boldsymbol{x}_t$. Simultaneously, on the basis of the difference of the observation matrices, we estimate the Hessian and Eq. (4) changes the estimated value $\boldsymbol{H}_t$.

The first algorithm (3) is a kind of usual RM procedure. The magnitude of the revision at each step depends on not only the coefficient $\alpha_t$ but also the matrix $\boldsymbol{K}_t^{-1}$. If we take $a_t=t/(t+1)$, $b_t=1/(t+1)$ in the second algorithm (4), this procedure becomes the usual ARM procedure. In our procedure, under certain conditions described below, we can arbitrarily choose the coefficients $a_t$, $b_t$ and $c_t$.

The algorithms (3) and (4) are not simple extension of the algorithms in Reference 6). In Reference 6), we could not separately prove the convergence of the parameter and the convergence of the estimation of the Hessian. However, we can separately discuss the convergence of the parameter and the convergence of the estimation of the Hessian, in this procedure.

As we state below, $\boldsymbol{H}_t$ converges to $\boldsymbol{H}(\boldsymbol{x}_*)$ with probability 1 as $t\to\infty$. Therefore, if $\boldsymbol{H}(\boldsymbol{x}_*)\geqq\boldsymbol{K}_{\min}$ for large $t$, $\boldsymbol{H}_t$ are used as $\boldsymbol{K}_t$ in the algorithm (3).

## 3. The Convergence Theorem

First of all, we state the conditions which we use below. The conditions (A1)~(C1) are also required in the usual RM procedure or ARM procedure. On the other hand, the conditions (C2)~(C8) are fundamental conditions to ensure the convergence of matrix $\boldsymbol{H}_t$ in the proposed procedure. It is natural to require the condition (A2) regarding the shape of the evalutional function. If we want to guarantee the convergence of $\boldsymbol{x}_t$ only, it is allowed that the set D, which is defined in (A2), is a simply-connected compact set. If we use the algorithms (3) and (4) under this condition, we can not ensure the asymptotically optimal convergence rate of $\boldsymbol{x}_t$ but the convergence of $\boldsymbol{x}_t$ to $\boldsymbol{x}_*$.

**Condition**

(A1) The evalutional function $J(\boldsymbol{x})$ is 2 times continuously differentiable and bounded from below.

(A2) The set D= $\{\boldsymbol{x}\mid f(\boldsymbol{x})=0\}$ consists of a unique point.

(A3) There exsit positive numbers $h_{\min}$ and $h_{\max}$ such that $h_{\min} \boldsymbol{I} \leqq \boldsymbol{H}(\boldsymbol{x}) \leqq h_{\max} \boldsymbol{I}$ where $\boldsymbol{I}$ denotes the identity matrix.
$\boldsymbol{H}(\boldsymbol{x})$ satisfies the Lipschitz condition.

(A4) $J(\boldsymbol{x}) \to \infty$ as $\| \boldsymbol{x} \| \to \infty$.

(A5) $\boldsymbol{H}(\boldsymbol{x}_*) \geqq \boldsymbol{K}_{\min}$.

(B1) The observation noise has the following properties:
$E(\boldsymbol{\Psi}_t^{\pm i}) = 0$,
$E\{(\boldsymbol{\Psi}_t^{\pm i})(\boldsymbol{\Psi}_t^{\pm i})^T\} \leqq \boldsymbol{S}, \| \boldsymbol{S} \| < \infty$.
$\boldsymbol{\Psi}_t^{\pm i}$ are independent for each observation.
E( $\cdot$ ) denotes the expectation. $\boldsymbol{S}$ is a positive definite matrix. The super script $T$ means transpose. The norm of a matrix is Euclidean.

(C1) $\alpha_t = 1/t$.

(C2) $\sum_{t=1}^{\infty} \alpha_t c_t < \infty$.

(C3) $a_t,\ c_t > 0,\ 1 \geqq b_t > 0$.

(C4) $\sum_{t=1}^{\infty} (1 - b_t)^2 < \infty$.

(C5) $\sum_{t=1}^{\infty} (a_t / c_t)^2 < \infty$.

(C6) $\sum_{t=1}^{\infty} | a_t - 1 + b_t | < \infty$.

(C7) $\sum_{t=1}^{\infty} a_t = \infty$.

(C8) There exsist positive numbers $C_1,\ C_2,\ \lambda,\ \varepsilon,\ \gamma,\ k_{\max}$ and $k_{\min}$ such that $(\gamma/2) + \lambda > 1$ for
$a_t \leqq C_1 t^{-\lambda}, c_t \leqq C_2 t^{-\varepsilon}, \gamma < \min\ (2\varepsilon,\ 2(k_{\min}/k_{\max}),\ 1)$.

In the ARM procedure, we have to consider the two kind of convergence. One is the convergence of estimated value of $\boldsymbol{x}_*$. The other is the convergence of the estimated value for the matrix $\boldsymbol{H}(\boldsymbol{x}_*)$. The convergence of the usual RM procedure is studied in many papers[1),7)]. Generally, it is important that the gain matrix $\boldsymbol{K}_t$ is not only positive definite but also bounded, since we have to demonstrate the convergence of estimated value $\boldsymbol{x}_t$. In this procedure, algorithm (3) ensures that $\boldsymbol{K}_t$ is positive definite. However, it is not obvious wether $\boldsymbol{K}_t$ bounded or not. On the other hand, we need boundedness of $\boldsymbol{H}_t$ to prove the convergence of the estimated value $\boldsymbol{H}_t$ to $\boldsymbol{H}(\boldsymbol{x}_*)$. There is no restriction about the range of $\boldsymbol{H}_t$ in algorithm (4). Therefore, in order to ensure the convergences of $\boldsymbol{x}_t$ and $\boldsymbol{H}_t$, we must demonstrate the following assertion (*) for this procedure. If the assertion (*) for the estimated value $\boldsymbol{H}_t$ is satisfied, the matrix $\boldsymbol{K}_t$ also satisfies a similar condition from the algorithm (3). We prove the assertion (*) in the following lemma.

(*) There exsists a positive number $k_{\max}$ such that
$\| \boldsymbol{H}_t \| \leqq k_{\max}$, with probability 1 as $t \to \infty$.

Lemma 1

We assume the conditions (A1)~(A4), (B1), (C3), (C5) and (C6). Then (*) holds for the algorithm (4).

Proof

We obtain the following equation from the algorithm (4).

$$\begin{aligned} \boldsymbol{H}_{t+1} &= \beta_0 \boldsymbol{H}_1 + \beta_1 a_1 \boldsymbol{Z}_1 + \cdots + \beta_{t-k-1} a_{t-k-1} \boldsymbol{Z}_{t-k-1} + \cdots + \beta_t a_t \boldsymbol{Z}_t \\ &= \beta_0 \boldsymbol{H}_1 + \beta_1 a_1 \boldsymbol{Y}_1 + \cdots + \beta_{t-k-1} a_{t-k-1} \boldsymbol{Y}_{t-k-1} + \cdots + \beta_t a_t \boldsymbol{Y}_t \\ &\quad + \beta_1 a_1 \boldsymbol{\Psi}_1 / c_1 + \cdots + \beta_{t-k-1} a_{t-k-1} \boldsymbol{\Psi}_{t-k-1} / c_{t-k-1} + \cdots + \beta_t a_t \boldsymbol{\Psi}_t / c_t \end{aligned} \tag{5}$$

where

$$\beta_i \equiv \begin{cases} \prod_{j=i+1}^{t} b_j & \text{if } i+1 \leqq t \\ 1 & \text{if } i+1 > t \end{cases}$$

$$\boldsymbol{Z}_t \equiv \frac{\boldsymbol{Z}_t^{+} - \boldsymbol{Z}_t^{-}}{2c_t} \tag{6}$$

$$\boldsymbol{\Psi}_t \equiv ((\psi_t^{+1}, \cdots, \psi_t^{+n}) - (\psi_t^{-1}, \cdots, \psi_t^{-n}))/2$$

$$\boldsymbol{Y}_t \equiv \mathrm{E}(\boldsymbol{Z}_t) \tag{7}$$

Therefore, from Eq. (5), we have the following inequality.

$$\| \boldsymbol{H}_{t+1} \| \leqq \| \beta_0 \boldsymbol{H}_1 + \sum_{i=1}^{t} \beta_i a_i \boldsymbol{Y}_i \| + \| \sum_{i=1}^{t} \beta_i a_i \boldsymbol{\Psi}_i / c_i \| \tag{8}$$

We consider the first term of the right hand side of Eq. (8). We define $\gamma_t = a_t - 1 + b_t$. Then, from (C3), we have

$$\begin{aligned} \sum_{i=1}^{t} \beta_i a_i &= b_t \cdots b_2 a_1 + \cdots + b_t \cdots b_{t-k} a_{t-k-1} + \cdots + b_t a_{t-1} + a_t \\ &= b_t \cdots b_2 (1 - b_1) + \cdots + b_t \cdots b_{t-k} (1 - b_{t-k-1}) + \cdots \\ &\quad + b_t (1 - b_{t-1}) + 1 - b_t \\ &\quad + b_t \cdots b_2 \gamma_1 + \cdots + b_t \cdots b_{t-k} \gamma_{t-k-1} + \cdots + b_t \gamma_{t-1} + \gamma_t \\ &\leqq b_t \cdots b_2 - b_t \cdots b_1 + \cdots + b_t \cdots b_{t-k} - b_t \cdots b_{t-k-1} + \cdots + b_t \\ &\quad - b_t b_{t-1} + 1 - b_t + |\gamma_1| + \cdots + |\gamma_t| \\ &= - b_t \cdots b_1 + 1 + |\gamma_1| + \cdots + |\gamma_t| \end{aligned} \tag{9}$$

Therefore, using the conditions (C3) and (C6), we can obtain the following inequality.

$$0 \leqq \sum_{i=1}^{\infty} \beta_i a_i < \infty \tag{10}$$

From the definition, $Y_i$ denotes the Hessian of the evalutional function at a certain point. So, from the conditions (A1) and (A3), there is a positive number $M_1$ such that

$$\| \boldsymbol{Y}_i \| \leqq M_1 \tag{11}$$

Equations (10), (11) and the boundedness of $\boldsymbol{H}_1$ and this result guarantee the boundness of the first term of the right hand side of Eq. (8).

We consider the noise term of the right hand side of Eq. (8). The conditions (C3), (C5) and (B1) imply

$$\begin{aligned} &\| \sum_{i=1}^{\infty} \mathrm{E}((\beta_i a_i / c_i)^2 \boldsymbol{\Psi}_i \boldsymbol{\Psi}_i^T) \| \\ &< \| \sum_{i=1}^{\infty} (a_i / c_i)^2 \mathrm{E}(\boldsymbol{\Psi}_i \boldsymbol{\Psi}_i^T) \| < \infty \end{aligned} \tag{12}$$

Accordingly, the Kolmogorov's theorem guarantees the convergence of $\| \sum_{i=1}^{\infty} (\beta_i a_i \boldsymbol{\Psi}_i / c_i) \|$ with probability 1. Therefore, the assertion of this lemma is proved.

(Q.E.D)

This lemma also ensures that $\| \boldsymbol{K}_t \| \leqq k_{\max}$.

Main theorem

We assume the conditions (A1)~(A5), (B1), (C1)~(C8). Then, for the algorithms (3) and (4),

$$\boldsymbol{x}_t \rightarrow \boldsymbol{x}_*,$$

$$J(\boldsymbol{x}_t) \rightarrow J_{\min},$$

$$\boldsymbol{K}_t \rightarrow \boldsymbol{H}(\boldsymbol{x}_*) \quad \text{with probability 1 } (t \rightarrow \infty)$$

where $J_{\min}$ denotes the minimum value of $J(\boldsymbol{x})$.

Proof

According to the previous lemma, the assumptions of this theorem imply the condition (*). Hence, the matrix $\boldsymbol{K}_t$ satisfies the same condition. That is, for an arbitrary $\delta > 0$, there exist a number $t_1$ such that

$$\mathrm{P} \{ (\sup_{t > t_1} \| \boldsymbol{K}_t \| ) \leqq k_{\max} \} > 1 - \delta$$

The algorithm (3) guarantees that the matrix $\boldsymbol{K}_t$ is positive definite. From the prevdious equation, we have the boundedness of $K_t$ with probability 1. As we described in Reference 6), we can easily apply the convergence theorem of Reference 7) to our algorithm (3). Hence, the convergence theorem of Reference 7) induces the first and second assertions of the main theorem (see Reference 6) and 7) in datail).

Now, let us consider the convergence of the matrix $K_t$. This proof is basically an extension of the theorem of Reference 6) to the multi-dimensional case. However, the condition (*) is different.

Let $\tilde{H}_t \equiv H_t - H(x_*)$. Then, let us consider $\tilde{H}_t\tilde{H}_t^T$. The algorithm (4) yields

$$\tilde{H}_{t+1}\tilde{H}_{t+1}^T = \tilde{H}_t\tilde{H}_t^T + Q_t + N_t \tag{13}$$

where

$$Q_t \equiv \tilde{H}_t\{(b_t-1)H_t + a_t\frac{Z_t^+ - Z_t^-}{2c_t}\}^T + \{(b_t-1)H_t + a_t\frac{Z_t^+ - Z_t^-}{2c_t}\}\tilde{H}_t^T$$

$$N_t \equiv (b_t-1)^2 H_t H_t^T + (\frac{a_t}{2c_t})^2 (Z_t^+ - Z_t^-)(Z_t^+ - Z_t^-)^T + (b_t-1)a_t\{H_t(\frac{Z_t^+ - Z_t^-}{2c_t})^T + H_t^T(\frac{Z_t^+ - Z_t^-}{2c_t})\}$$

This equation corresponds to Eq. (8) of Reference 6). The algorithm (4) does not explicitly restrict the range of the matrix $H_t$. If we compare Eq. (13) with Eq. (8) of Reference 6), Eq. (13) does not contain the term which is derived from this restriction.

We define $Y_t^+ \equiv \mathrm{E}(Z_t^+)$ and $Y_t^- \equiv \mathrm{E}(Z_t^-)$. There exists a point $x_s$ such that $(Y_t^+ - Y_t^-)/2c_t = H(x_s)$. Accordingly, we have

$$\frac{Y_t^+ - Y_t^-}{2c_t} = H(x_*) + \{H(x_t) - H(x_*)\} + \{H(x_s) - H(x_t)\} \tag{14}$$

Let $\Theta_t$ be the second and third terms of the right hand side of Eq. (14). From the condition (A3), there exist positive numbers $M_2$ and $M_3$ such that

$$\frac{Y_t^+ - Y_t^-}{2c_t} = H(x_*) + \Theta_t \tag{15}$$

$$\| \Theta_t \| \leq M_2 \| x_t - x_* \| + M_3 c_t$$

Now, we define $\Phi_t$ as follows:

$$\Phi_t=(a_t+b_t-1)(\tilde{H}_tH_t^T+H_t\tilde{H}_t^T)+(b_t-1)^2H_tH_t^T+a_t\Theta_t\tilde{H}_t^T+a_t\Theta_t^T\tilde{H}_t$$
$$+(a_t/c_t)^2(Y_t^+-Y_t^-)(Y_t^+-Y_t^-)^T+(a_t/2c_t)^2S \tag{16}$$
$$+\frac{(b_t-1)a_t}{c_t}\{H_t(Y_t^+-Y_t^-)^T+(Y_t^+-Y_t^-)H_t^T\}$$

Let us consider the quantity $\Phi_t$. First of all, we consider the term containing $\Theta_t$ of Eq. (16). We know the following result (see APPENDIX).

$$\lim_{t\to\infty} t^{\gamma} \| x_t-x_* \|^2=0 \tag{17}$$

This result and the condition (C8) yield

$$\sum_{t=1}^{\infty} a_t \| x_t-x_* \| <\infty \tag{18}$$

We have $\varepsilon+\lambda>1$ from the condition (C8). From (C8), we have

$$\sum_{t=1}^{\infty} a_tc_t<\infty \tag{19}$$

We can obtain the relation (20) from Eqs. (15), (18) and (19).

$$\sum_{t=1}^{\infty} \| a_t\Theta_t \| <\infty \tag{20}$$

$\Phi_t$ has the following property from Eq. (20), conditions (B1), (C4)~(C6) and the boundedness of $\tilde{H}_t$, $K_t$, $Y_t^+$, $Y_t^-$:

$$\sum_{t=1}^{\infty} \| \Phi_t \| <\infty \tag{21}$$

On the other hand, using Eq. (16) and condition (B1), we have

$$\mathrm{E}(\tilde{H}_{t+1}\tilde{H}_{t+1}^T \mid H_t)\leqq \tilde{H}_t\tilde{H}_t^T-2a_t\tilde{H}_t\tilde{H}_t^T+\Phi_t \tag{22}$$

Now, let $v$ be an arbitrary vector except 0. We define $w_t=v^T\tilde{H}_t\tilde{H}_t^Tv \mid v^Tv+\sum_{t=1}^{\infty} v^T\Phi_t v/v^Tv$. Using Eq. (22) and the definition of $w_t$, we obtain

$$E(w_{t+1} \mid H_t)=w_t-2a_tv^T\tilde{H}_t\tilde{H}_t^Tv/v^Tv \tag{23}$$

Since $\tilde{H}_t\tilde{H}_t^T\geqq 0$, $w_t$ has the property of the supermartingale[7]. Using the martingale convergence theorem[7] and condition (C7), we have the following assertion.

$$H_t \to H(x_*) \text{ with probability } 1\ (t \to \infty) \tag{24}$$

All assertions of the theorem were proved. (Q.E.D)

## 4. Conclusion

An extended ARM procedure is proposed. The convergence of this procedure is proved. We clarify the conditions which guarantee the convergence of $x_t$, $J(x_t)$ and $K_t$. In this procedure, there are several coefficients which we can arbitrarily choose. Accordingly, we can choose an adequate values for these coefficients. Especially, it becomes possible that this procedure improves the convergence rate of $x_t$ in the transit period.

## Appendix

Similar results are found in References 3) and 6). However, the procedure and the conditions are different from our procedure. We briefly prove this result (See References 3) and 6) in detail).

Lemma 2

If the conditions (A1)~(A5), (B1), (C1)~(C3) and (C8) hold, we have the following result:

For $\gamma < \min(2\varepsilon, 2k_{min}/k_{max}, 1)$,

$$\lim_{t\to\infty} t^{\gamma} \| x_t - x_* \|^2 = 0. \tag{A.1}$$

Proof

Without loss of generality, we assume $x_* = 0$. Define $V_t$ and the operator $\mathscr{D}$ as follows:

$$\begin{aligned} V_t &= t^{\gamma} \| x_t \|^2 \\ \mathscr{D}\, V_t &= \mathrm{E}(V_{t+1} \mid V_t) - V_t. \end{aligned} \tag{A.2}$$

Since $\gamma < 1$, (A.2) implies that

$$\mathscr{D}\, V_t \leqq (t+1)^{\gamma} \| x_t \|^2 + \gamma t^{\gamma-1} \| x_t \|^2 \tag{A.3}$$

On the other hand, from Eqs. (1) and (A.3), there exist $x_{si}^+$ *and* $x_{si}^-$ $(i = 1, \cdots, d)$ such that

$$
\begin{aligned}
&\left\| \frac{\mathrm{E}\sum_{i=1}^{n}(z_t^{+i}+z_t^{-i})}{2n} - f(x_t) \right\| \\
&= \frac{c_t}{2n} \left\| \{H(x_{s1}^{+})-H(x_{s1}^{-})\}+\cdots+\{H(x_{sd}^{+})-H(x_{sd}^{-})\} \right\| \\
&\leqq \frac{c_t}{2n} \{L_1 \| x_{s1}^{+}-x_{s1}^{-} \| +\cdots+ L_d \| x_{sd}^{+}-x_{sd}^{-} \| \} \\
&\leqq \frac{c_t}{n} \{L_1 c_t + \cdots + L_d c_t\} \leqq M_4 c_t^2
\end{aligned}
\tag{A.4}
$$

where $L_i(i=1, \cdots, d)$ and $M_4$ denote positive numbers. Henceforth, $M_i(i=5, \cdots)$ are also positive numbers.

From the algorithms (3), (4), Eq. (A.4), the conditions (A1), (A3), (A5), (C8), the inequality $\| f(x) \| < M_5(1+\| x \|)$ and $\| x \| < 1+\| x \|^2$, we obtain the following result for sufficiently large $t$.

$$
\begin{aligned}
\mathcal{D} \| x_t \|^2 &= -\frac{2}{t} x_t^T K_t^{-1} \frac{\mathrm{E}\sum_{i=1}^{n}(z_t^{+i}+z_t^{-i})}{2n} \\
&\quad + \frac{1}{t^2} \left\| \frac{\mathrm{E}\{K_t^{-1}\sum_{i=1}^{n}(z_t^{+i}+z_t^{-i})\}}{2n} \right\|^2 \\
&\leqq -\frac{2(k_{\min}/k_{\max})}{t} \| x_t \|^2 + \frac{M_6}{t} \| x_t \|^3 + M_7 g(t)(1+\| x_t \|^2)
\end{aligned}
\tag{A.5}
$$

where $g(t)=c_t^2/t+1/t^2$

Substituting Eq. (A.5) into Eq. (A.3), we have

$$
\begin{aligned}
\mathcal{D} V_t &\leqq \frac{\left\{ \gamma - 2\frac{k_{\min}}{k_{\max}}(1+\frac{1}{t})^{\gamma} + M_8 \| x_t \| \right\}}{t} V_t \\
&\quad + M_7 g(t)(t+1)^{\gamma}(1+\| x_t \|^2)
\end{aligned}
\tag{A.6}
$$

$\gamma < 2\varepsilon$ and $\gamma < 1$ ensure the convergence of $\sum_{t=1}^{\infty} g(t)(t+1)^{\gamma}$. Moreover, since $\gamma < 2(k_{\min}/k_{\max})$, $\gamma - 2(k_{\min}/k_{\max})(1+1/t)^{\gamma}$ is less than a certain negative number. $x_t \rightarrow x_* = 0$ with probability 1, from the lemma 1. Applying these results to Eq. (A.6), there exist $T_0$ and $P \geqq 0$. The following relation hold for $t \geqq T_0$.

$$
\begin{aligned}
\mathrm{E}(V_{t+1} \mid V_t) &\leqq (1-P) V_t + \zeta_t \\
&\leqq V_t + \zeta_t
\end{aligned}
\tag{A.7}
$$

$\zeta_t$ denotes the second term of the left-hand side of Eq. (A.6).

If we define $w'_t = V_t + \sum_{i=t}^{\infty} \zeta_i$, $w'_t$ is supermartingale. Therefore, $w'_t$ converges with probability 1. From the convergence of $\sum_{i=1}^{\infty} \zeta_i$, we have (A.1).

(Q.E.D)

## References


1) H. Robbins and S. Monro, Ann. Math. Statist., **22**, (1), 400 (1951).
2) J.H. Venter, An Extension of the Robbins-Monro Procedure, Ann. Math. Statist., **38**, (1), 181 (1967).
3) M.B. Nevel'son and R.Z.Khas'minskii, Automation and Remote control, **34,** (10), 1594 (1973).
4) Y. Maeda and Y. Kawamura, Trans. IEICE, **J73-A**, (3), 626 (1990).
5) Y. Maeda and Y. Kawamura, Trans. IEICE, **J73-A**, (4), 783 (1990).
6) Y. Maeda, Y. Kawamura, S. Minamoto and Y. Kojima, Trans. SICE, **23**, (11), 1186 (1987).
7) H.J. Kushner, IEEE Trans. Automatic Control, **AC-17**, (5), 646 (1972).