Wash Analysis of A Class of Nonlinear Dynamical Systems

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Kazuo NOHARA

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1. Introduction

Walsh functions have attracted, recently, increasing interests over a wide variety of engineering applications such as function approximation, signal composition/decomposition, image processing and so on. Van *et al.*[1] used Walsh functions for the solution of linear differential equations. Subsequently, Walsh analysis of power-law memoryless systems was discussed by Maqusi[2]. Most physical systems, however, generally have memory and involve some kinds of nonlinearities. So extension of their results to nonlinear systems having memory —nonlinear dynamical systems— has been strongly desired. The aim of this paper is to extend foregoing results to nonlinear dynamical systems. To this end, we first introduce a multiplication operator by which the Walsh coefficients of a function $a(t)$ are transformed into the coefficients of the product of two functions $a(t)$ and $b(t)$. Then we use it to develop solutions for nonlinear integro-differential equations including power-law or product-type nonlinearities.

2. Walsh Functions and Walsh Transform

Walsh functions: Walsh functions[3] form a complete orthnomal set of rectangular waveforms taking only two values $+1$ and -1. The common notation for Walsh function is Wal$(i,\ t)$, where i is a nonnegative integer and t is a real number over $[0, 1)$. Three major ordering conventions are in common use. They are (1) sequency or Walsh ordering, (2) dyadic or Paley ordering, and (3) natural or Hadamard ordering. Any one of them can be used for our present purpose, but we shall use Walsh's in this paper, so that the functions are ordered in terms of the number of zero crossings. The first eight of the Walsh functions are illustrated in Fig. 1.

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* Department of Electronics, College of Engineering.

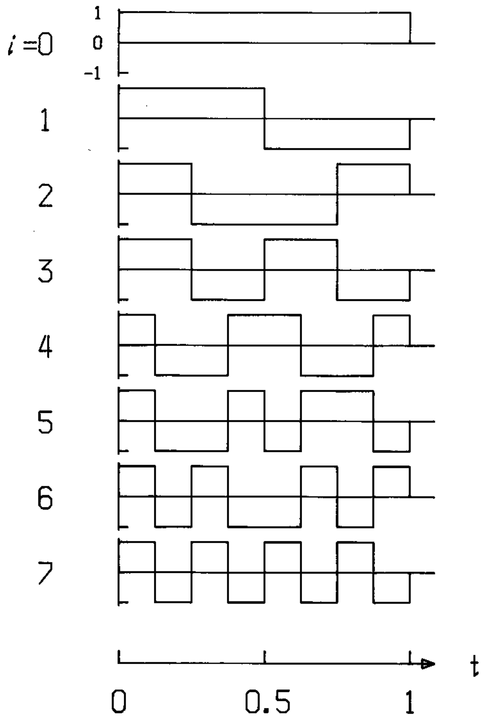

Fig. 1. First eight of Walsh functions Wal(i, t).

$$
\mathrm{Wal}(i,j) = \begin{array}{c c}
\begin{array}{cccccccc} j=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} & i= \\
\left[\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{array}\right] & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\end{array}
$$

Fig. 2. Walsh Matrix of size 8×8.

Sampling of the Walsh functions of order m, $m=0, 1, \cdots, n\text{-}1$, at n equidistant points results in a set of n-length discrete Walsh functions[4]; n is here an integral power of 2. The discrete Walsh functions of length eight are shown collectively in a matrix form in Fig. 2. We call, hereafter, this matrix as "Walsh matrix".

Useful properties[5] of the Walsh functions are their addition relationship,

$$\mathrm{Wal}(r,j)\mathrm{Wal}(s,j)=\mathrm{Wal}(r \oplus s,j), \tag{1}$$

and symmetry relationship,

$$\mathrm{Wal}(i,j)=\mathrm{Wal}(j,i), \tag{2}$$

in which $\mathrm{Wal}(i, j)$ denotes the jth argument of the ith discrete Walsh function and $\oplus$ indicates dyadic addition, i.e., addition modulo 2.

Walsh series and Walsh transform: For a 2^N-length real sequence $f(j)$, $j=0, 1, \cdots, 2^N-1$, the finite Walsh transform is defined as

$$F(i)=\frac{1}{N}\sum_{j=0}^{2^N-1} f(j)\mathrm{Wal}(i,j), \qquad i=0,1,\cdots,2^N-1. \tag{3}$$

Similarly, we can express $f(j)$ as the inverse Walsh transform of $F(i)$

$$f(j)=\sum_{i=0}^{2^N-1} F(i)\mathrm{Wal}(i,j), \qquad j=0,1,\cdots,2^N-1. \tag{4}$$

Equation (4) is a finite Walsh series representation for the function $f(t)$. Equations (3) and (4) can also be expressed in matrix forms as

$$\left.\begin{aligned} \boldsymbol{F} &= \boldsymbol{W}\boldsymbol{f} \\ \boldsymbol{f} &= \boldsymbol{W}^{-1}\boldsymbol{F} \end{aligned}\right\} \tag{5}$$

where

$$\boldsymbol{f}= [f(0),f(1),\cdots,f(2^N-1)]^T,$$
$$\boldsymbol{F}= [F(0),F(1),\cdots,F(2^N-1)]^T,$$

and $\boldsymbol{W}$ and $\boldsymbol{W}^{-1}$ are the Walsh matrix and its inverse, respectively.

3. Integral and Differential Operators

Assume a function $f(t)$ integrable on $[0, 1]$ and its integral $g(t)$,

$$g(t)=\int_0^t f(x)\mathrm{d}t+g(0), \quad 0\leqq t\leqq 1, \tag{6}$$

where $g(0)$ is the integration constant. Using Eq. (4) we can approximate $f(t)$ in the above equation in terms of Walsh functions, then we obtain

$$g(t)=\sum_{i=0}^{2^N-1} F(i)\int_0^t \mathrm{Wal}(i,x)\mathrm{d}x+g(0). \tag{7}$$

Again approximate $\int_0^t \mathrm{Wal}(i,x)\mathrm{d}x$ in the above equation in terms of the Walsh functions,

$$\int_0^t \mathrm{Wal}(i,x)\mathrm{d}x=\sum_{j=0}^{2^N-1} E_{ji}\mathrm{Wal}(j,t). \tag{8}$$

Substitution of Eq. (8) into Eq. (7) leads to

$$g(t)=\sum_{j=0}^{2^N-1}\left[\sum_{i=0}^{2^N-1} E_{ji}F(i)\right]\mathrm{Wal}(j,t)+g(0), \tag{9}$$

Thus the Walsh transform of $g(t)$ is

$$G(j)=\sum_{i=0}^{2^N-1} E_{ji}F(i)+g(0)\delta_{j0}, \tag{10}$$

where δ_{j0} designates the Kronecker delta function.

Equation (10) can be written in a matrix form as

$$\boldsymbol{G}=\boldsymbol{E}\boldsymbol{F}+\boldsymbol{C}, \tag{11}$$

in which

$$\boldsymbol{G}=\left[G(0),G(1),\cdots,G(2^N-1)\right]^T,$$
$$\boldsymbol{F}=\left[F(0),F(1),\cdots,F(2^N-1)\right]^T,$$
$$\boldsymbol{C}=\left[g(0),0,0,\cdots,0\right]^T,$$
$$\boldsymbol{E}=\left[E_{ji}\right], \quad j,i=0,1,2,\cdots,2^N-1,$$

where

$$E_{ji}=\int_0^1\left[\int_0^t \mathrm{Wal}(i,x)\mathrm{d}x\right]\mathrm{Wal}(j,t)\mathrm{d}t. \tag{12}$$

The matrix $\boldsymbol{E}$ in Eq. (11) is referred to as an integral operator in the sense that it converts the Walsh transform of a function $f(t)$ into the transform of its integral within a integration constant $g(0)$. The integral operator has been calculated for the practical purpose using Eq. (12) and is illustrated in Fig. 3 for the case N=3.

The integral matrix E has an inverse, then we obtain from Eq. (11)

$$\boldsymbol{F}=\boldsymbol{E}^{-1}(\boldsymbol{G}-\boldsymbol{C}). \tag{13}$$

$$
E = \begin{bmatrix}
1/2 & 1/4 & 0 & 1/8 & 0 & 0 & 0 & 1/16 \\
-1/4 & 0 & 1/8 & 0 & 0 & 0 & 1/16 & 0 \\
0 & -1/8 & 0 & 0 & 0 & 1/16 & 0 & 0 \\
-1/8 & 0 & 0 & 0 & 1/16 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/16 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/16 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/16 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/16 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Fig. 3. Integral operator (N=3).

$$
E^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 \\
0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 \\
0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 \\
0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 \\
0 & 0 & 0 & 16 & 0 & 0 & 0 & -32 \\
0 & 0 & 16 & 0 & 0 & 0 & -32 & 0 \\
0 & 16 & 0 & 0 & 0 & 32 & 0 & -64 \\
16 & 0 & 0 & 0 & 32 & 0 & 64 & 128
\end{bmatrix}
$$

Fig. 4. Differential operator (N=3).

The matrix $\boldsymbol{E}^{-1}$ is a differential operator in the Walsh transform domain in the sense that it permits one to calculate the transform coefficient $\boldsymbol{F}$, given $\boldsymbol{G}$ and $\boldsymbol{C}$. The differential operator is shown in Fig. 4 for the case N=3.

Transforms of higher order derivatives can be derived by the use of Eq. (13). Let $y^{(i)}(t)$ be the ith derivative of $y(t)$ and $\boldsymbol{Y}^{(i)}$ be its Walsh transform. Then we have, from Eq. (13)

$$\boldsymbol{Y}^{(i+1)} = \boldsymbol{E}^{-1}(\boldsymbol{Y}^{(i)} - \boldsymbol{C}_i), \tag{14}$$

where C_i is the Walsh transform of the initial value of $y^{(i)}(t)$,

$$C_i = [y^{(i)}(0),0,\cdots,0]^T.$$

Equation (14) leads to

$$\begin{aligned}
&Y^{(1)} = E^{-1}(Y - C_0),\\
&Y^{(2)} = E^{-1}(Y^{(1)} - C_1) = E^{-1}\{E^{-1}(Y - C_0) - C_1\} = E^{-2}Y - (E^{-2}C_0 + E^{-1}C_1),\\
&\vdots\\
&Y^{(i)} = E^{-i}Y - \sum_{j=1}^{i} E^{-j}C_{i-j}.
\end{aligned} \tag{15}$$

4. Multiplication Operator

Consider two functions $a(t)$ and $b(t)$ square integrable on $[0, 1]$ and their product $c(t)=a(t)\cdot b(t)$. Approximate $a(t)$ and $b(t)$ by 2^N-length Walsh series as

$$\begin{aligned}
a(t) &= \sum_{i=0}^{2^N-1} A(i)\mathrm{Wal}(i,t),\\
b(t) &= \sum_{j=0}^{2^N-1} B(j)\mathrm{Wal}(j,t).
\end{aligned} \tag{16}$$

Then the product $c(t)$ is

$$c(t) = \sum_{i=0}^{2^N-1}\sum_{j=0}^{2^N-1} A(i)B(j)\mathrm{Wal}(i,t)\mathrm{Wal}(j,t). \tag{17}$$

Using the addition relationship of Walsh functions given in Eq. (1), we can rewrite Eq. (17) as

$$c(t) = \sum_{i=0}^{2^N-1}\sum_{j=0}^{2^N-1} A(i)B(j)\mathrm{Wal}(i\oplus j,t). \tag{18}$$

The function $c(t)$ can also be represented in a series of the form

$$c(t) = \sum_{k=0}^{2^N-1} C(k)\mathrm{Wal}(k,t), \tag{19}$$

in which the coefficients $C(k)$'s are given by

$$C(k) = \int_0^1 c(t)\mathrm{Wal}(k,t)\mathrm{d}t. \tag{20}$$

Substituting Eq. (18) into Eq. (20) and interchanging the order of the integration and the summation, we obtain

$$C(k)=\sum_{i=0}^{2^N-1}\sum_{j=0}^{2^N-1} A(i)B(j)\int_0^1 \mathrm{Wal}(i\oplus j\oplus k,t)dt. \tag{21}$$

Remembering the property of the Walsh functions that

$$\int_0^1 \mathrm{Wal}(i\oplus j\oplus k,t)dt=\begin{cases}1, & i\oplus j=k,\\ 0, & \text{elsewhere,}\end{cases} \tag{22}$$

we finally obtain

$$C(k)=\sum_{i,j} A(i)B(j), \tag{23}$$

in which the summation is over all possible pairs of i and j satisfying the relation $i\oplus j=k$, $(i,j,k=0,1,\cdots,2^N-1)$. Equation (23) can also be written in a matrix form as

$$\boldsymbol{C}=\boldsymbol{M}(\boldsymbol{A})\boldsymbol{B}, \tag{24}$$

where

$$\boldsymbol{C}=[C(0),C(1),\cdots,C(2^N-1)]^T,$$
$$\boldsymbol{B}=[B(0),B(1),\cdots,B(2^N-1)]^T,$$
$$\boldsymbol{M}(\boldsymbol{A})=[m_{ij}],$$

and

$$m_{ij}=A(i\oplus j),\ i,j=0,1,\cdots,2^N-1.$$

$$\mathrm{M(A)}=\begin{bmatrix}
A(0) & A(1) & A(2) & A(3) & A(4) & A(5) & A(6) & A(7)\\
A(1) & A(0) & A(3) & A(2) & A(5) & A(4) & A(7) & A(6)\\
A(2) & A(3) & A(0) & A(1) & A(6) & A(7) & A(4) & A(5)\\
A(3) & A(2) & A(1) & A(0) & A(7) & A(6) & A(5) & A(4)\\
A(4) & A(5) & A(6) & A(7) & A(0) & A(1) & A(2) & A(3)\\
A(5) & A(4) & A(7) & A(6) & A(1) & A(0) & A(3) & A(2)\\
A(6) & A(7) & A(4) & A(5) & A(2) & A(3) & A(0) & A(1)\\
A(7) & A(6) & A(5) & A(4) & A(3) & A(2) & A(1) & A(0)
\end{bmatrix}$$

Fig. 5. Multiplication operator (N=3).

The matrix $M(A)$ in Eq. (24) can be considered as a multiplication operator in the sense that it operates into the Walsh transform of $b(t)$ and gives the transform of the product $c(t)=a(t)\cdot b(t)$. Figure 5 illustrates the multiplication operator for the case N=3.

5. Analysis of Nonlinear Dynamical Systems

A good variety of nonlinear devices have been used over a spreading area of practical systems. So we have a very difficulty to cover all possible cases. Therefore, we only give an example to show the solution.

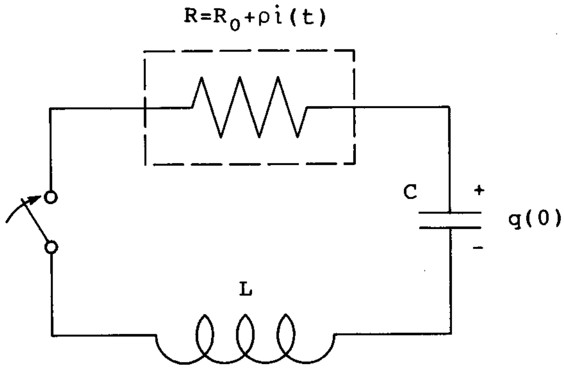

Fig. 6. Series L-C-R circuit with power-law nonlinearity.

Consider the nonlinear series L-C-R circuit shown in Fig. 6. The inductor and the capacitor are linear; while the resistor is nonlinear, varying in proportional to the current through it,

$$R=R_0+\rho i(t). \tag{25}$$

The nonlinear integro-differential equation for this circuit is

$$L\frac{\mathrm{d}}{\mathrm{d}t}\, i(t)+R_0 i(t)+\rho i^2(t)+\frac{1}{C}\int i(t)\mathrm{d}t+\frac{1}{C}\, q(0)=0, \tag{26}$$

where $q(0)$ is the initial charge of the capacitor. Equation (26) is a class of nonlinear integro-differential equations having power-law nonlinearity.

Taking the Walsh transform of the both sides of Eq. (26), we obtain

$$\boldsymbol{L}\boldsymbol{D}(\boldsymbol{J}-\boldsymbol{J}(0))+\boldsymbol{R}_0\boldsymbol{J}+\boldsymbol{\rho}\boldsymbol{M}(\boldsymbol{J})\boldsymbol{J}+\frac{1}{\boldsymbol{C}}\,\boldsymbol{E}\boldsymbol{J}=-\frac{1}{\boldsymbol{C}}\,\boldsymbol{Q}(0), \tag{27}$$

where $\boldsymbol{D}$ designates $\boldsymbol{E}^{-1}$. The symbols $\boldsymbol{J}$ and $\boldsymbol{J}(0)$ are the Walsh transform of $i(t)$ and its initial value, respectively, and $\boldsymbol{Q}(0)$ is the Walsh transform of initial charge $q(0)$ of the capacitor C,

$$\boldsymbol{J}=[J(0)J(1),\cdots,J(2^N-1)]^T,$$
$$\boldsymbol{J}(0)=[i(0),0,0,\cdots,0]^T,$$
$$\boldsymbol{Q}(0)=[q(0),0,0,\cdots,0]^T,$$

and

$$\boldsymbol{R}_0=\mathrm{diag}[R(0),R(0),\cdots,R(0)]\ ,$$
$$\boldsymbol{\rho}=\mathrm{diag}[\rho,\rho,\cdots,\rho]\ ,$$
$$\frac{1}{\boldsymbol{C}}=\mathrm{diag}[\frac{1}{C},\frac{1}{C},\cdots,\frac{1}{C}]\ ,$$

Equation (27) leads to

$$\boldsymbol{J}=-[\boldsymbol{LD}+\boldsymbol{R}_0+\boldsymbol{\rho M}(\boldsymbol{J})+\frac{1}{\boldsymbol{C}}\boldsymbol{E}]^{-1}\cdot[\frac{1}{\boldsymbol{C}}\boldsymbol{Q}(0)-\boldsymbol{LDJ}(0)]\ . \tag{28}$$

To solve Eq. (28), let

$$\boldsymbol{J}_1=-[\boldsymbol{LD}+\boldsymbol{R}_0+\boldsymbol{\rho M}(\boldsymbol{J}_0)+\frac{1}{\boldsymbol{C}}\boldsymbol{E}]^{-1}\cdot[\frac{1}{\boldsymbol{C}}\boldsymbol{Q}(0)-\boldsymbol{LDJ}(0)]\ . \tag{29}$$

and assume the first approximation to the result $\boldsymbol{J}_0$. Substitute $\boldsymbol{J}_0$ into Eq. (29) and

Table 1. Comparison of 8-term Walsh solution with Runge-Kutta solution for the circuit shown in Fig. 6.

Range	This Method	Runge-Kutta Solution (mid-interval)
0 - 1/8	0.1109	0.1178
1/8 - 2/8	0.3117	0.3178
2/8 - 3/8	0.4748	0.4808
3/8 - 4/8	0.6056	0.6119
4/8 - 5/8	0.7042	0.7110
5/8 - 6/8	0.7669	0.7740
6/8 - 7/8	0.7874	0.7945
7/8 - 8/8	0.7589	0.7650

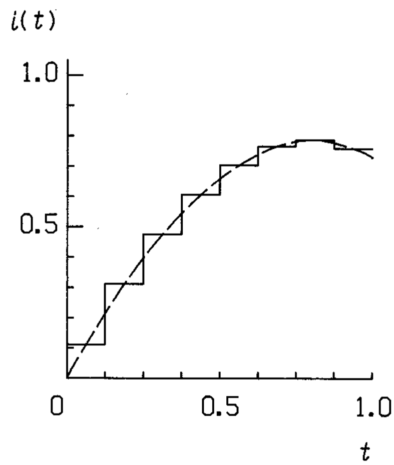

Fig. 7. Solution for $i(t)$ for the circuit shown in Fig. 6.

find the second approximation J_1. Substitute J_1 so obtained into Eq. (29) again and find a new result. This process is repeated until the result is as accurate as desired.

Figure 7 shows the computed solution via 8-term Walsh transform with parameters $R_0=1.0$, $\rho=1.0$, L=0.5, C=0.5, $i(0)=0$ and $q(0)=-0.5$. The first approxiomation used was $J_0=[0,0,0,0,0,0,0,0]^T$, and computation was repeated 7 times until the computed result J_n becames equal, down to the 4th decimal point, to the last result J_{n-1}. If a smooth curve is drawn through the mid-point of each horizontal step, it will be very close to the correct curve. Table 1 compares the 8-term Walsh solution to Runge-Kutta solution at each mid-interval. The Runge-Kutta solution was calculated using 4th order formula with step size 1/100. Both results agree within an error less than 1%. The accuracy is very good considering that the series was truncated after the 8th term.

6. Conclusions

Walsh analysis was applied to the dynamical systems having power-law or product-type nonlinearity. Nonlinear integro-differential equation representing the system dynamics was solved as an algebraic equation by the use of Walsh transform with the aid of integral, differential, and multiplication operators through iterative calculations. Example showed that a good accuracy was obtained via small number of iterations using a limited length Walsh transform.

References

1) T.L. Van, L.D.C. Tam, and N.V. Houtte, IEEE Trans. Circuits and Systems, **CAS-22**, (5) 419 (1975).
2) M. Maqusi, IEEE Trans. Information Theory, **IT-23**, (1) 144 (1977)
3) J.L. Walsh, Amer. J. Math., **45**, (1), 5 (1923).
4) N. Ahmed, K.R. Rao, "Orthogonal Transforms for Digital Signal Processing", p. 90, Springer-Verlag, Berlin (1975).
5) K.G. Beauchamp, "Application of Walsh and Related Functions", p. 53, Academic Press, London (1984).